The unaccomplished perfection of Kepler's world

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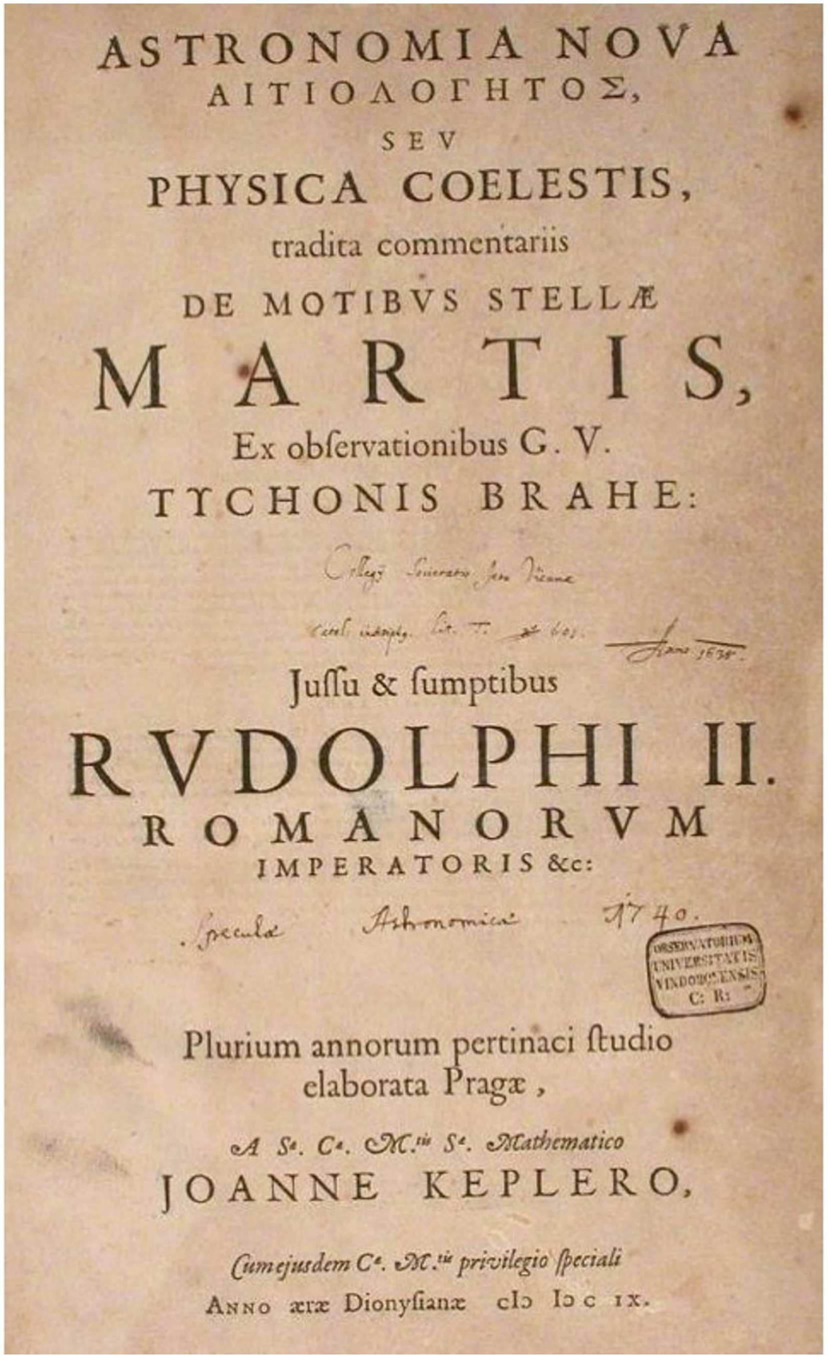
Johannes Kepler (1571-1630): the program



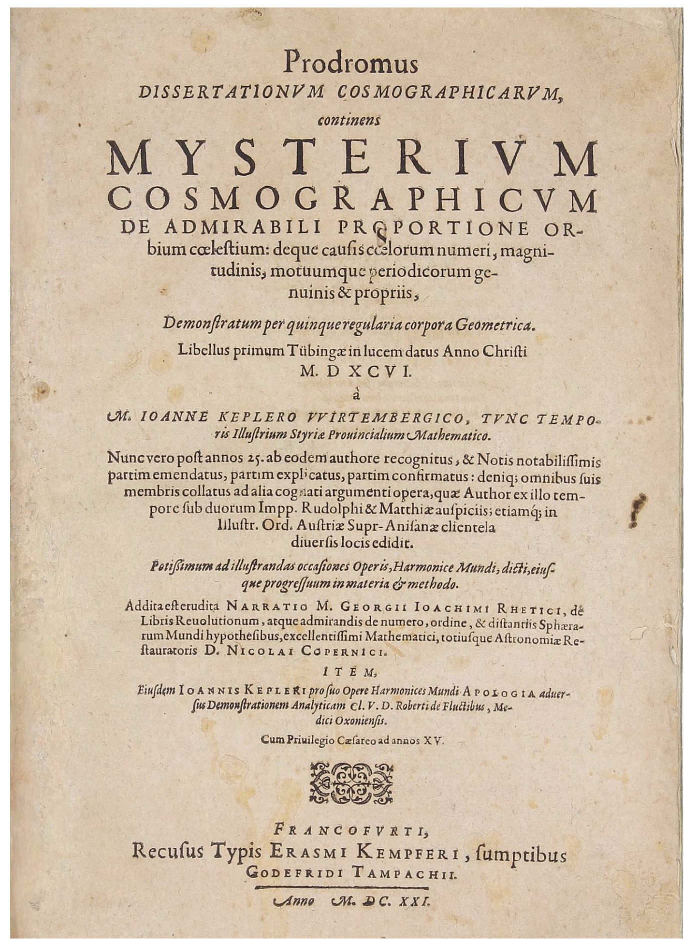
ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

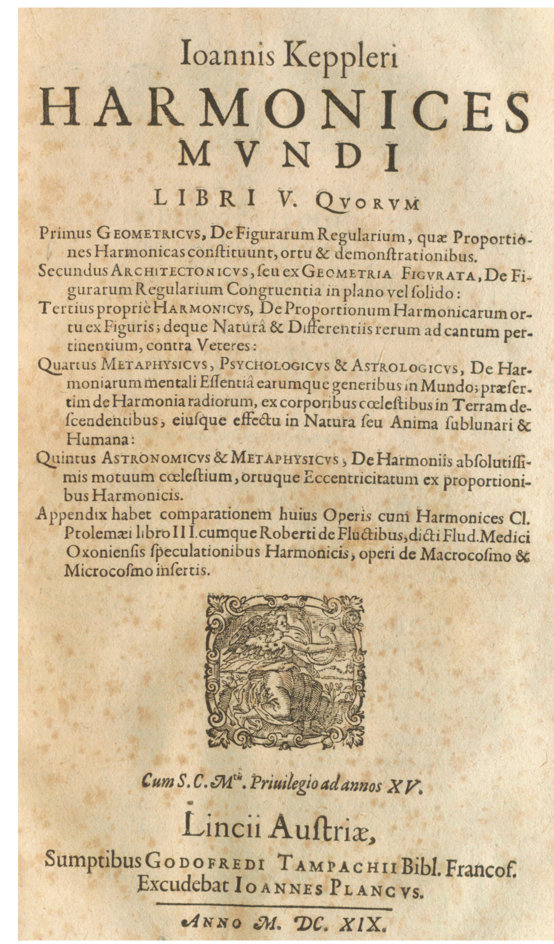
Specula Astronomica 1740.

Plurium annorum pertinaci ſtudio
elaborata Pragæ,
A S[a]. C[æ]. M.[tis] S[æ]. Mathematico
JOANNE KEPLERO,
Cum ejusdem C[æ]. M.[tis] privilegio ſpeciali
ANNO æræ Dionyſianæ cIɔ Iɔ c IX.

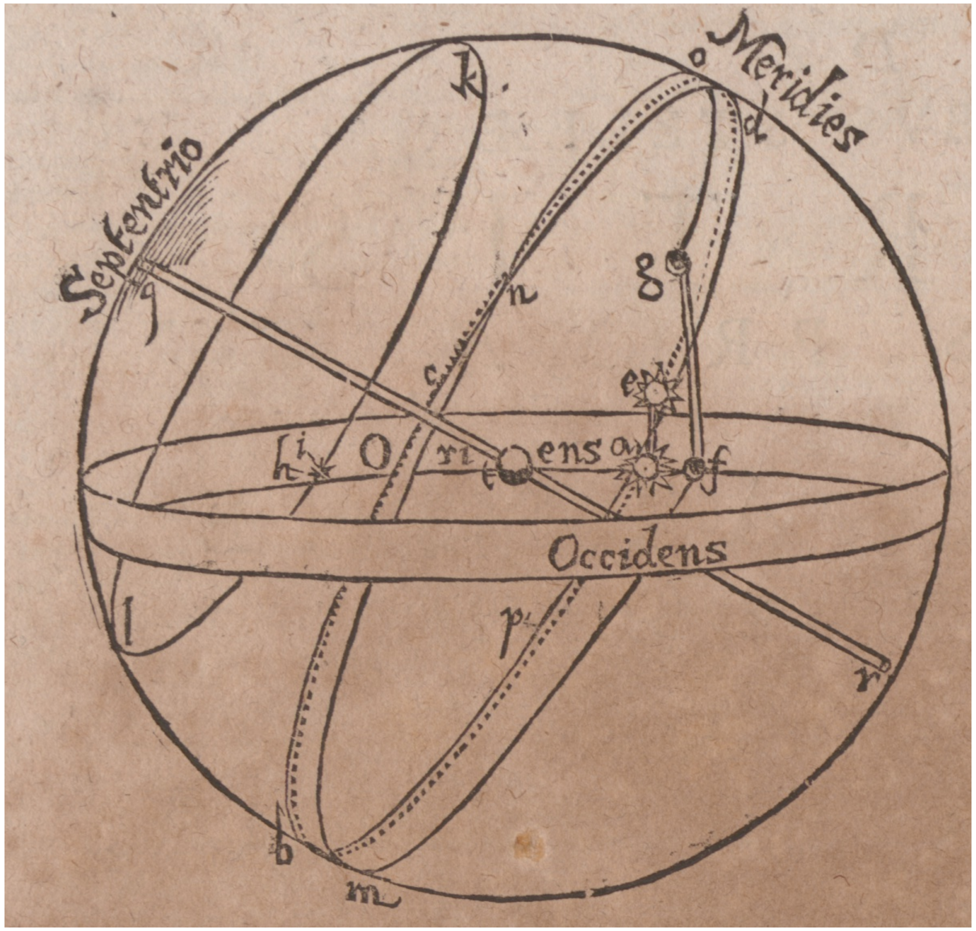
- To discover the plan of God when He created the universe:
  - Mysterium Cosmographicum, 1596;
  - Harmonices Mundi Libri V, 1619.
- To investigate the physical causes of the orbits and the motions of the planets:
  - Astronomia Nova, 1609.

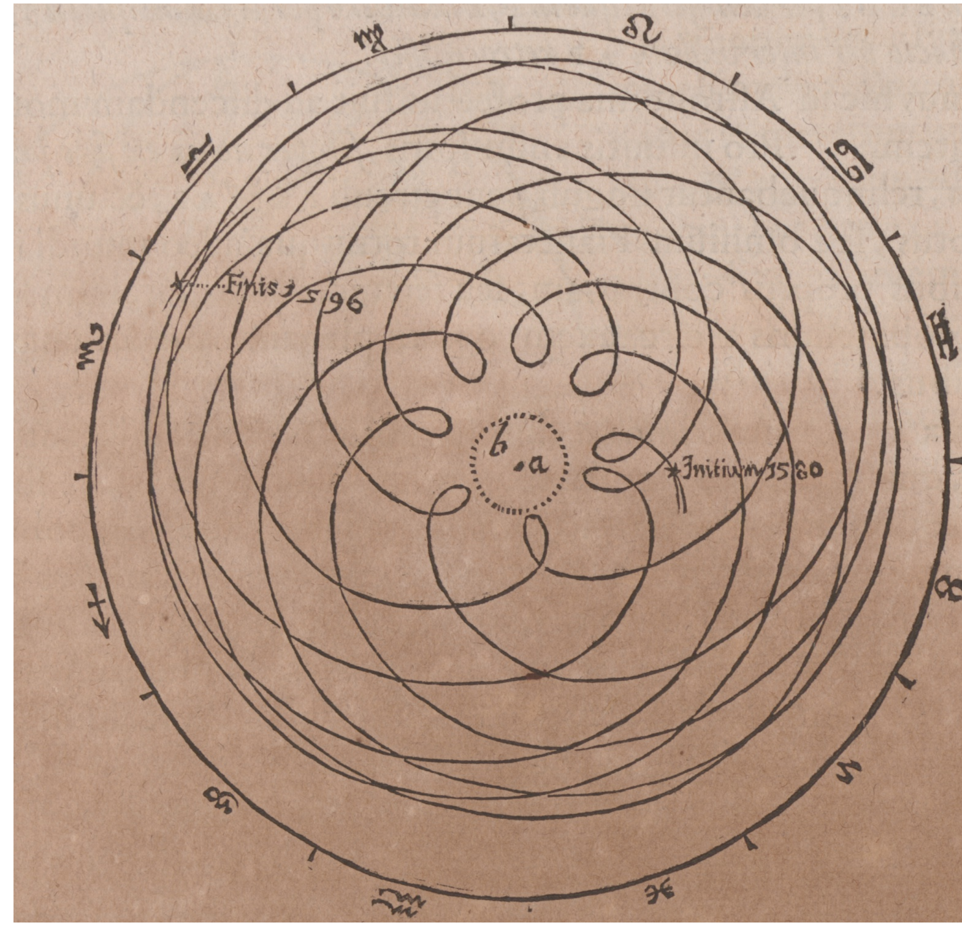


Prodromus
DISSERTATIONVM COSMOGRAPHICARVM,
continens
MYSTERIVM
COSMOGRAPHICVM
DE ADMIRABILI PROPORTIONE ORbium cœleſtium: deque cauſis cœlorum numeri, magnitudinis, motuumque periodicorum genuinis & propriis,
Demonſtratum per quinque regularia corpora Geometrica.
Libellus primum Tübingæ in lucem datus Anno Chriſti
M. D XCVI.
à
M. IOANNE KEPLERO VVIRTEMBERGICO, TVNC TEMPOris Illuſtrium Styriæ Prouincialium Mathematico.
Nunc vero poſt annos 25. ab eodem authore recognitus, & Notis notabiliſſimis partim emendatus, partim explicatus, partim confirmatus: deniq; omnibus ſuis membris collatus ad alia cognati argumenti opera, quæ Author ex illo tempore ſub duorum Impp. Rudolphi & Matthiæ auſpiciis; etiamq; in Illuſtr. Ord. Auſtriæ Supr-Aniſanæ clientela diuerſis locis edidit.
Potiſſimum ad illuſtrandas occaſiones Operis, Harmonice Mundi, dicti, eiuſque progreſſuum in materia & methodo.
Addita eſt erudita NARRATIO M. GEORGII IOACHIMI RHETICI, de Libris Reuolutionum, atque admirandis de numero, ordine, & diſtantiis Sphærarum Mundi hypotheſibus, excellentiſſimi Mathematici, totiuſque Aſtronomiæ Reſtauratoris D. NICOLAI COPERNICI.
ITEM,
Eiuſdem IOANNIS KEPLERI pro ſuo Opere Harmonices Mundi APOLOGIA aduerſus Demonſtrationem Analyticam Cl. V. D. Roberti de Fluctibus, Medici Oxonienſis.
Cum Priuilegio Cæſareo ad annos XV.
FRANCOFVRTI,
Recuſus Typis ERASMI KEMPFERI, ſumptibus
GODEFRIDI TAMPACHII.
Anno M. DC. XXI.



Ioannis Keppleri
HARMONICES
MVNDI
LIBRI V. QVORVM
Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas conſtituunt, ortu & demonſtrationibus.
Secundus ARCHITECTONICVS, ſeu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel ſolido:
Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:
Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Eſſentiâ earumque generibus in Mundo; præſertim de Harmonia radiorum, ex corporibus cœleſtibus in Terram deſcendentibus, eiuſque effectu in Natura ſeu Anima ſublunari & Humana:
Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis abſolutiſſimis motuum cœleſtium, ortuque Eccentricitatum ex proportionibus Harmonicis.
Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxonienſis ſpeculationibus Harmonicis, operi de Macrocoſmo & Microcoſmo inſertis.
Cum S. C. M.[tis] Priuilegio ad annos XV.
Lincii Auſtriæ,
Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.
ANNO M. DC. XIX.



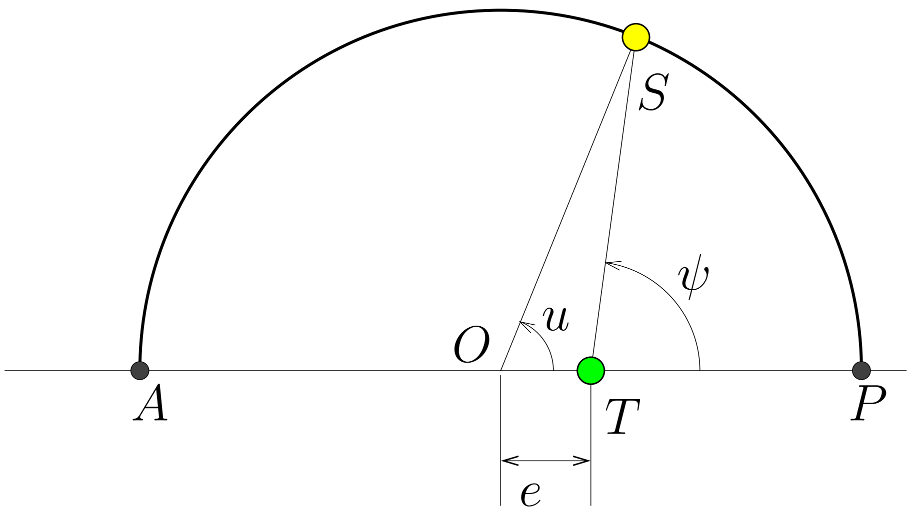


Durissima est hodie conditio scribendi libros Mathematicos, præcipue Astronomicos. Nisi enim servaveris genuinam subtilitatem propositionum, instructionum, demonstrationum, conclusionum, liber non erit Mathematicus: sin autem servaveris, lectio efficitur morosissima, præsertim in Latina lingua, quæ caret articulis, & illa gratia quam habet græca, cum per signa literaria loquitur.

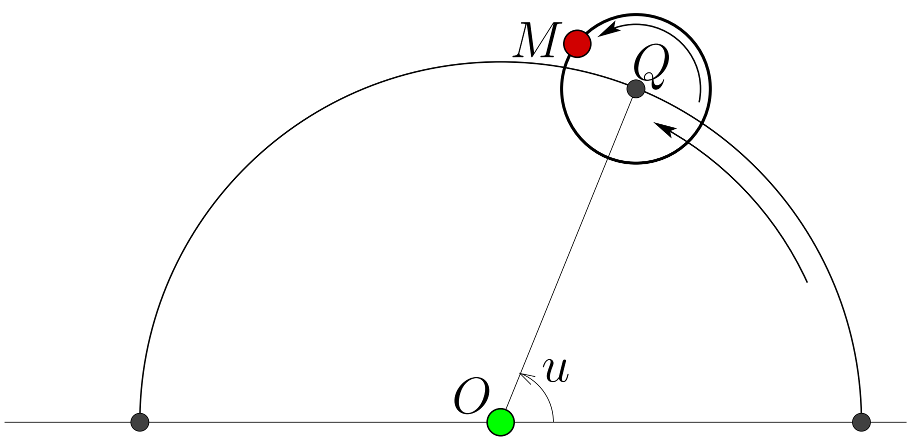
(*Astronomia Nova,* Introduction.)

It is extremely painful nowadays to write mathematical books, especially astronomical ones. For unless one maintains the innate exactness of propositions, constructions, demonstrations and conclusions the book will not be mathematical; but if you respect that sequence it will be most laboriuos to communicate through written symbols, especially in Latin, which lacks the articles and that gracefulness possessed by Greek.

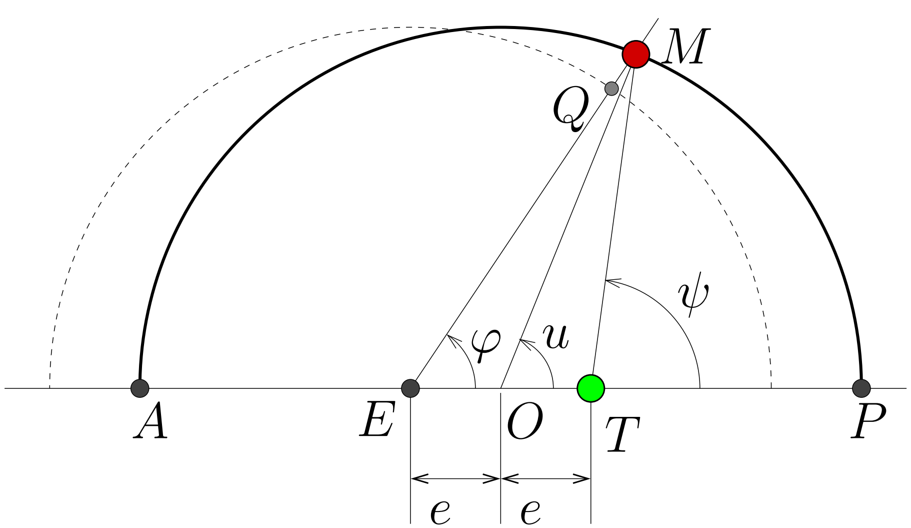
Geometric tools of Greek Astronomy



- *Eccentric:* Apollonius (262–190 bC).
  - Explains the lack of uniformity in the annual motion of the Sun and the Planets.
  - The Earth T is in eccentric position with respect to the center O of the circular orbit.
  - P: perigee (for the Sun S); A: apogee.
  - The *eccentric anomaly* u evolves uniformly.
  - The *true anomaly* ψ increases faster at the perigee than at the apogee.

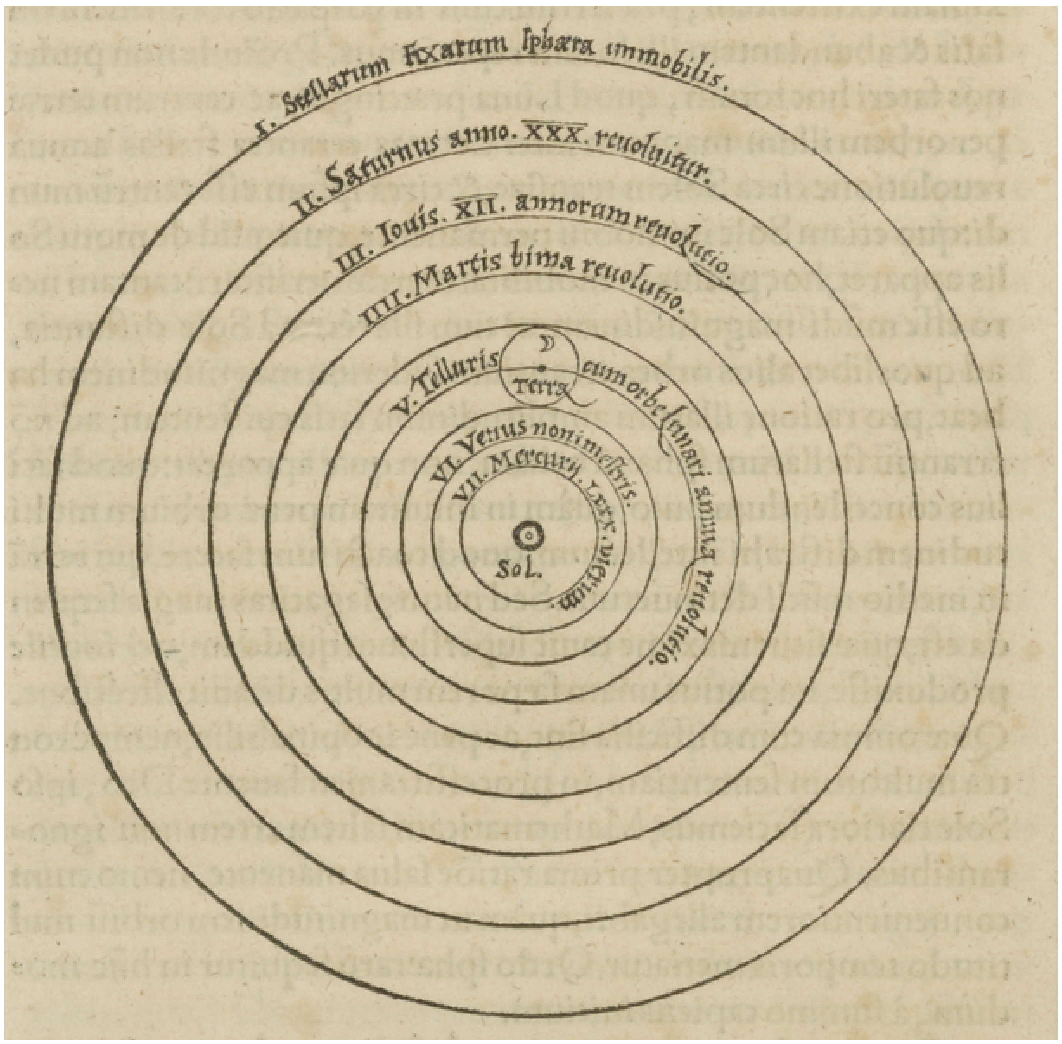


- *Epicycle:* Apollonius or Hipparchus (190–120 bC).
  - Explains the retrograde motion of Planets.
  - The Planet (e.g., M for Mars) rotates on a circle with center Q (epicycle); the center Q rotates on another circle (deferent).
  - Compare with our expansion in Fourier series.



- *Equant:* Ptolemæus (90–168 aC), or may be Hipparchus.
  - An improvement for the non uniformity introduced by the eccentric.
  - The point Q rotates uniformly with respect to the equant point E.
  - The *mean anomaly* φ evolves uniformly.
  - The position of the Planet M is the intersection of the line EQ with the orbit.

The universe of Nicolaus Copernicus (1473-1543)
compared to Claudius Ptolemæus ($\sim$ 100-170)



- Ptolemy:
  - Geocentric model, fixed Earth;
  - eccentric orbit (non uniform revolution);
  - epicycles (retrograde motion);
  - uniform revolution around the *equant point*;
  - the Earth has no equant point, only eccentricity.

- Copernicus:
  - Heliocentric model, Earth one of the planets;
  - center of the planetary orbit in the *mean Sun*;
  - replaces the motion around the equant point with epicycles.

- Both:
  - planets moving on material spheres made of *æther*.

The "petitiones" (postulates) of Copernicus

Prima petitio:
Omnium orbium caelestium sive sphaerarum unum centrum non esse.

There is no unique common center of all the celestial orbs or spheres.

Secunda petitio:
Centrum terrae non esse centrum mundi, sed tantum gravitatis et orbis Lunaris.

The center of the Earth is not the center of the universe, but only the center of gravity and of the sphere of the moon.

Tertia petitio:
Omnes orbes ambire Solem, tanquam in medio omnium existentem, ideoque circa Solem esse centrum mundi.

All the spheres encircle the Sun, which appears to be in the middle of them all, so that the center of the universe is near the Sun.

Quarta petitio:
Minorem esse comparationem distantiarum Solis et terrae ad altitudinem firmamenti, quam semidimetientis terrae ad distantiam Solis, adeo ut sit ad summitatem firmamenti insensibilis.

The ratio of the Earth–Sun distance to the height of the firmament is smaller than the ratio of the Earth's radius to the Earth–Sun distance, to such a degree that the Earth-Sun distance is imperceptible compared with the height of the firmament.

Quinta petitio:

Quicquid ex motu apparet in firmamento, non esse ex parte ipsius, sed terrae. Terra igitur cum proximis elementis motu diurno tota convertitur in polis suis invariabilibus firmamento immobili permanente ac ultimo caelo.

Whatever motion we see in the firmament is due, not to it, but to the Earth. Thus the Earth together with the elements close to it revolves around its fixed poles, while the firmament remains fixed, being the highest heaven.

Sexta petitio:

Quicquid nobis ex motibus circa Solem apparet, non esse occasione ipsius, sed telluris et nostri orbis, cum quo circa Solem volvimur ceu aliquo alio sidere, sicque terram pluribus motibus ferri.

Whatever motion we observe in the Sun is due, not to its motion, but to the motion of the Earth and of our sphere, with which we revolve about the Sun, as any other planet, and so the Earth undergoes many motions.

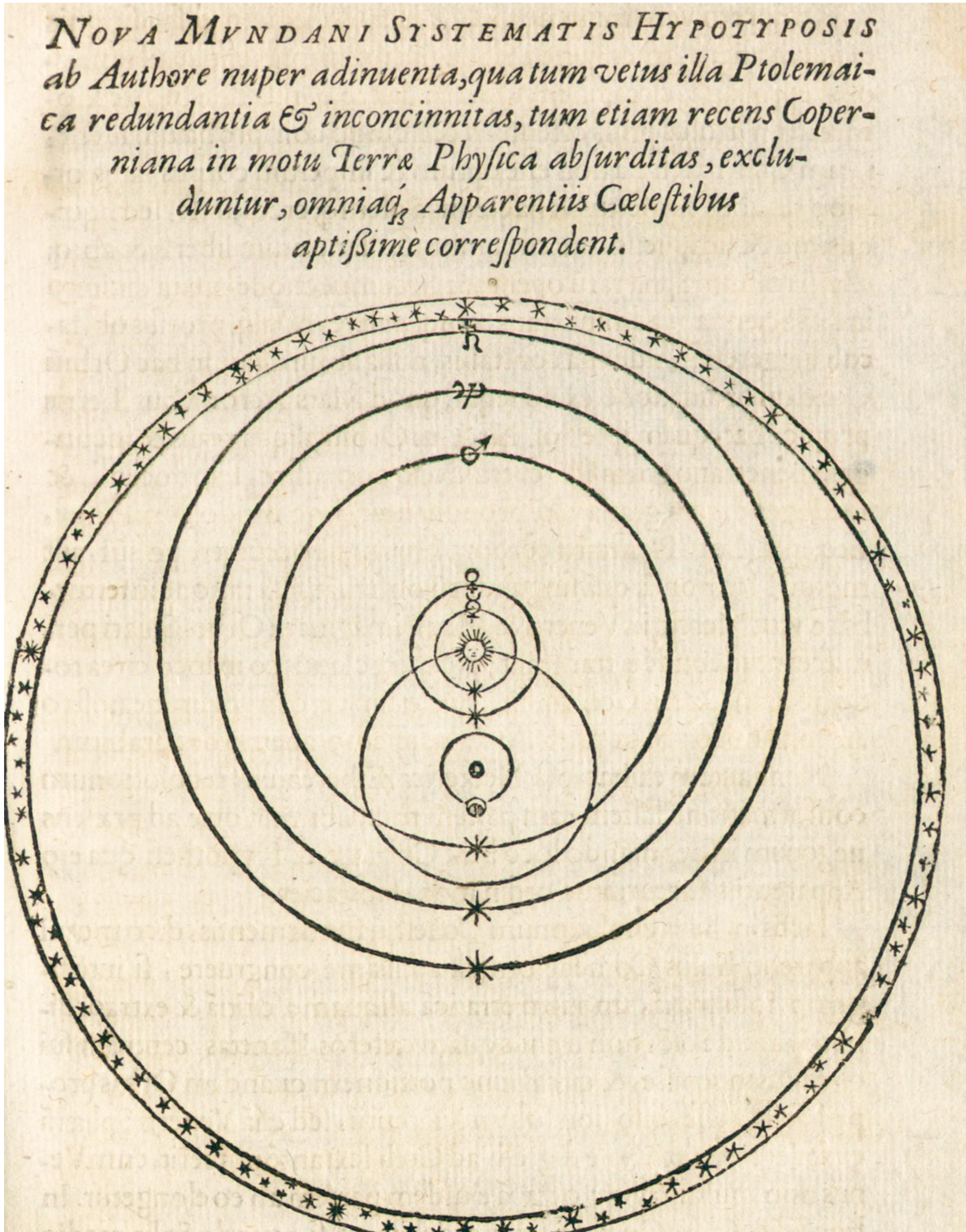
Septima petitio:

Quod apparet in erraticis retrocessio ac progressus, non esse ex parte ipsarum sed telluris. Huius igitur solius motus tot apparentibus in caelo diversitatibus sufficit.

What appears in the planets as retrograde and direct motion is due, not to their motion, but to the Earth's. Thus the motion of the Earth alone suffices to explain all apparent irregularities in the heaven.

(Copernicus: *Commentariolus*).

The universe of Tycho Brahe (1546-1601)



- The Earth is fixed at the center of the Universe:
  - The Moon and the Sun revolve around the Earth;
  - The other planets revolve around the Sun.

- Removes the material spheres:
  - a great comet observed in 1577 was beyond the sphere of the Moon, and its orbit did cross the sphere of Venus;
  - Mars in opposition is closer to the Earth than the Sun, as observed in 1582 (the sphere of Mars should intersect the sphere of the Sun).

Figure from: *De mundi ætherei recentioribus phænomenis.*

The first part of "Astronomia Nova"

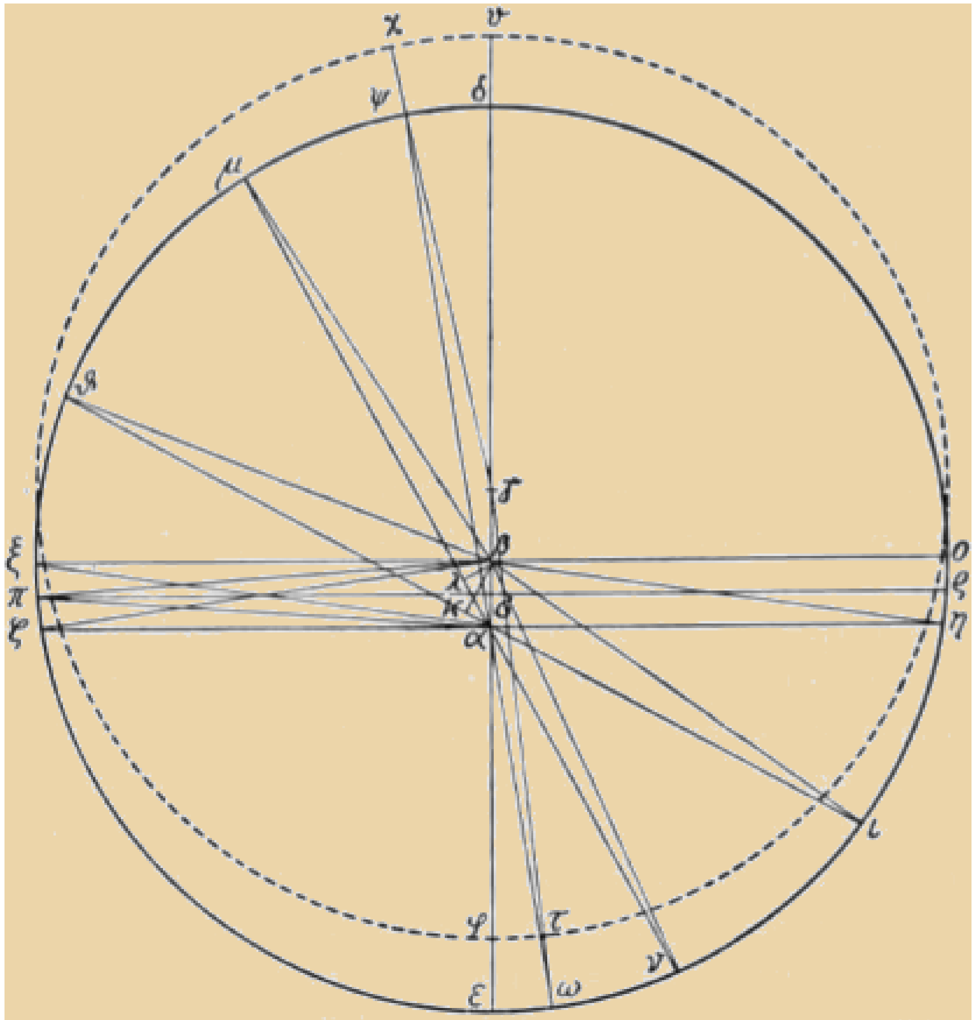
in very short terms

- Comparison between the models of Ptolemy, Copernicus and Tycho; they are geometrically equivalent.
- For *physical reasons*, the heliocentric model of Copernicus in preferred, but:
  - there are no solid spheres (in agreement with Tycho Brahe);
  - the reference point is located in the *true* Sun (not the mean Sun) for all planets;
  - *this leads to conclude that the orbits of every planet lies in a plane through the Sun.*
- The Earth has an equant point, like all planets, but:
  - *bisection* of the eccentricity (as Ptolemy did): putting the center of the orbit halfway between E (equant) and S (Sun) makes the orbit of the Earth plane;
  - the *vicarious hypothesis*: setting the distances of E and S form the center in the ratio $8 : 5$ produces the correct angles, but *wrong distances.*
- *Ad imitationem veterum.* Why the equant?
  - using the *equant point* in place of the double epicycle of Copernicus is more convenient anyway, as a geometrical tool.
  - however, the equant is just an useful geometric artifice: there is nothing there.

References:

- B. Stephenson: *Kepler's physical astronomy*, Springer–Verlag (1987).
- N.M. Swerdlow: *Astronomy in the Renaissance*, in *Astronomy before the telescope*, C. Walker ed., British Museum Press (1996).

Towards the "law of areas"

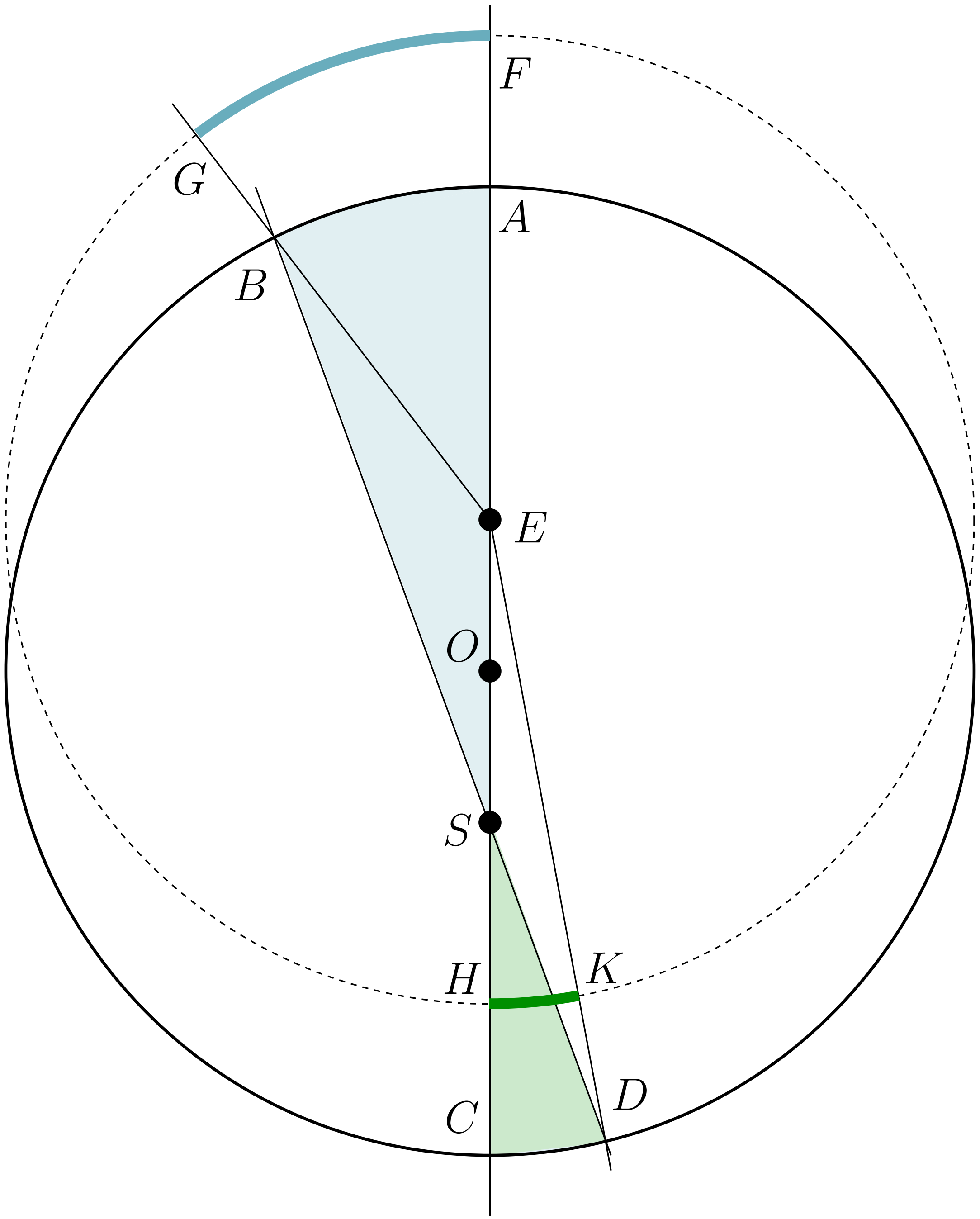


- Concentrate attention of the orbit of the Earth, assumed circular with bisected eccentricity.
  - The velocity at perihelion is larger than at aphelion (true for all planets);
  - make this remark quantitative.
- Two equivalent claims, that hold true *quam proxime*:
  - the time spent on equal arcs (*mora*) is proportional to the distance from the Sun;
  - the velocity in aphelion and perihelion is inversely proportional to the distance from the Sun.
- Remark: *Kepler does not mention the areas here.*

Primum sciat in omni hypothesi Ptolemaica hac forma instructa, quantacunque eccentricitas fuerit, celeritatem in perihelio & tarditatem in aphelio proportionari quam proxime lineis ex centro mundi eductis in Planetam.

(*Astronomia Nova, ch. XXXII*)

First, [the reader] should understand that in all hypotheses constructed according to this Ptolemaic form, no matter of the value of eccentricity, the rapidity at perihelion and the slowness at aphelion exhibit very close a proportion with the lines drawn from the centre of the world to the planet.



The proof of Kepler

- Measuring the time (*mora*)
  - $|FG|$ measures the *mora* for the arc AB ;
  - $|HK|$ measures the *mora* for the arc CD .
- Geometric relations:
  - sector SAB similar to SCD;
  - sector EAB similar to EFG;
  - sector EHK similar to ECD;
- Want to prove

$$\frac{|HK|}{|FG|} = \frac{|SC|}{|SA|} \ .$$

  - true if

$$|OA|^2 \simeq |EA|\,|SA| \ , \quad |OC|^2 \simeq |EC|\,|SC| \ ;$$

  - have instead

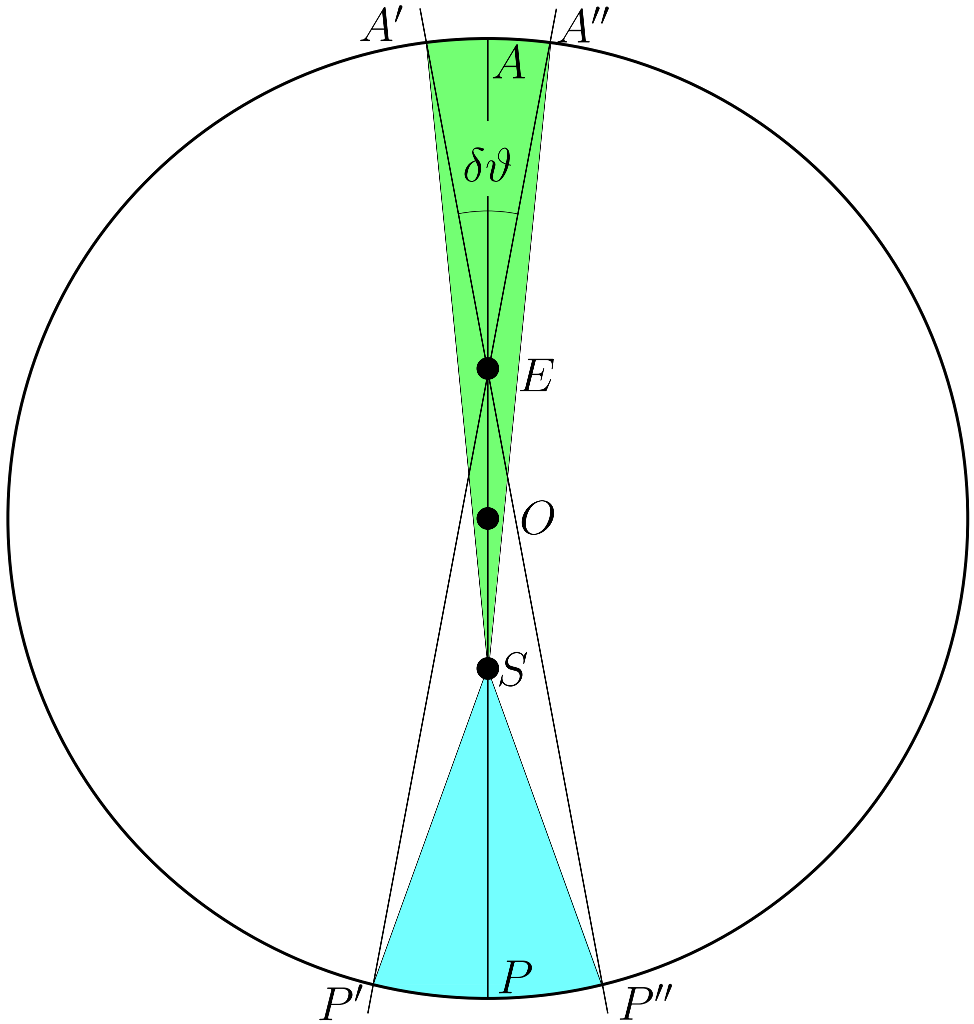
$$|OA| = \frac{|EA| + |SA|}{2} = \frac{|EC| + |SC|}{2} = |OC| \ .$$

  - Must use $r\sqrt{(1+e)(1-e)} \simeq r(1-\frac{e^2}{2})$, which is equal to r *quam proxime*.

Let us do it in simpler terms (for us...) using areas.

S : Sun ; E : Equant ; O : center of the eccentric orbit ; CF : line of absides.

$|OA| = |OC| = |EF| = |EH| = r$, $|EA| = |OH| = |SC| = r(1-e)$, $|SA| = |EC| = |OF| = r(1+e)$.



- S : Sun;
- O : center of the orbit;
- E : equant point;
- P : perihelion;
- A : aphelion.

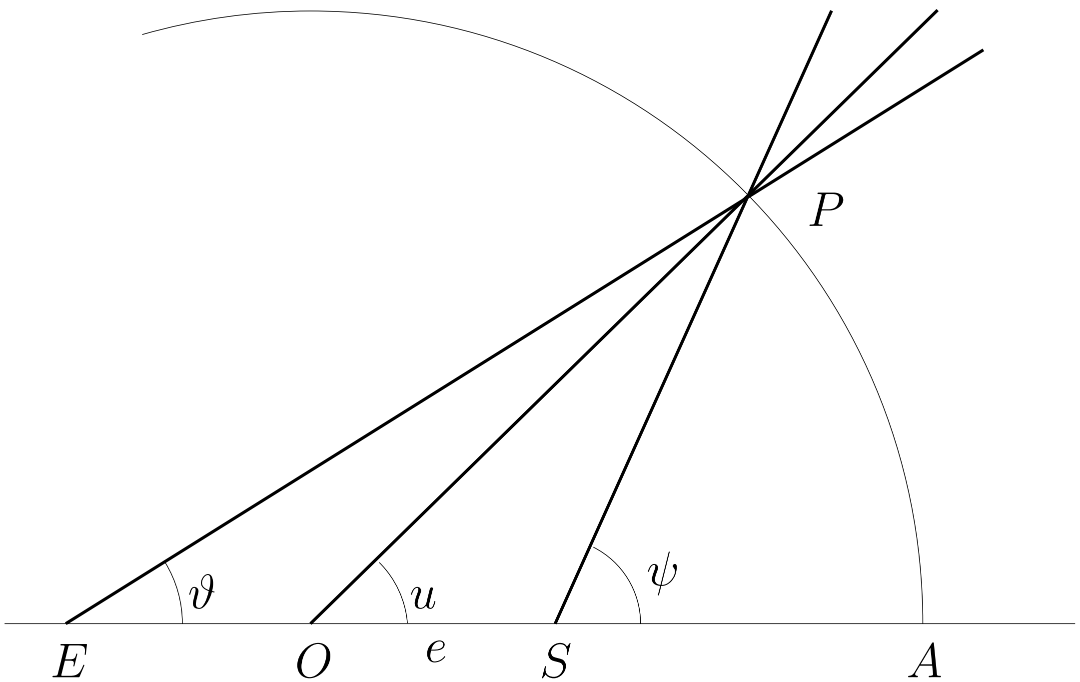
$$\begin{aligned}
|OA| &= |OP| = a \ , \\
|OS| &= |OE| = ea \ , \\
|SP| &= |EA| = (1-e)a \ , \\
|SA| &= |EP| = (1+e)a \ , \\
|A'A''| &= (1-e)a\delta\vartheta \ , \\
|P'P''| &= (1+e)a\delta\vartheta \ .
\end{aligned}$$

$$\left.\begin{aligned}
\mathrm{area}(SP'P'') &= \frac{1}{2}|P'P''|\cdot|SP| \\
\mathrm{area}(SA'A'') &= \frac{1}{2}|A'A''|\cdot|SA|
\end{aligned}\right\} = \frac{1}{2}(1-e)(1+e)a^2\delta\vartheta \ .$$

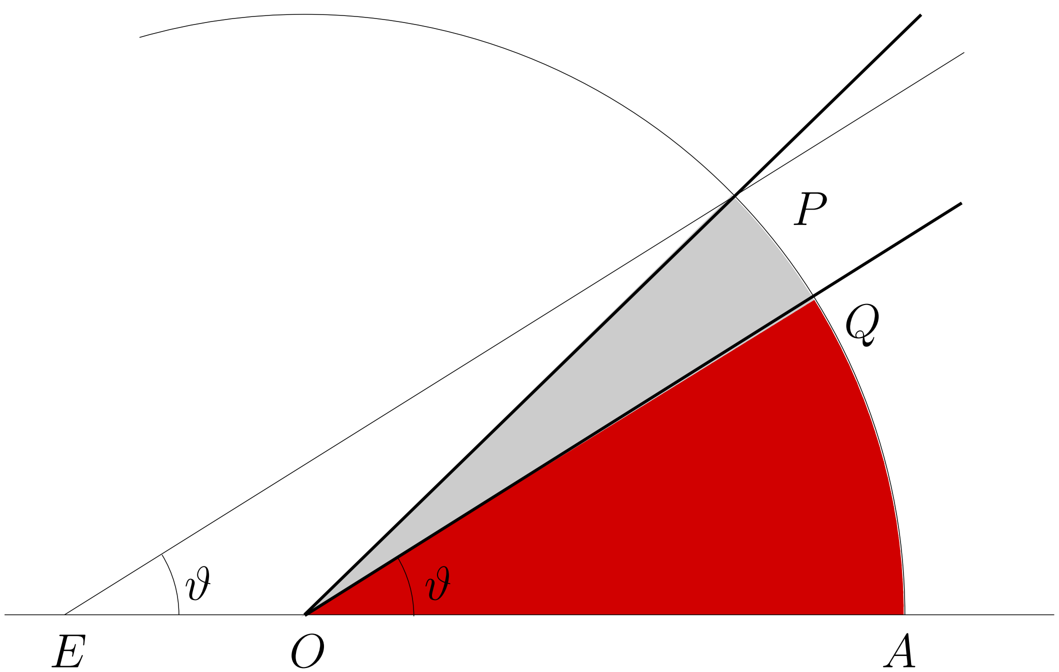
- The argument is perfect for the apsides; only approximate for all other points.
- Write $\delta s = r\,\delta\varphi$; recover the claim of Kepler in either form

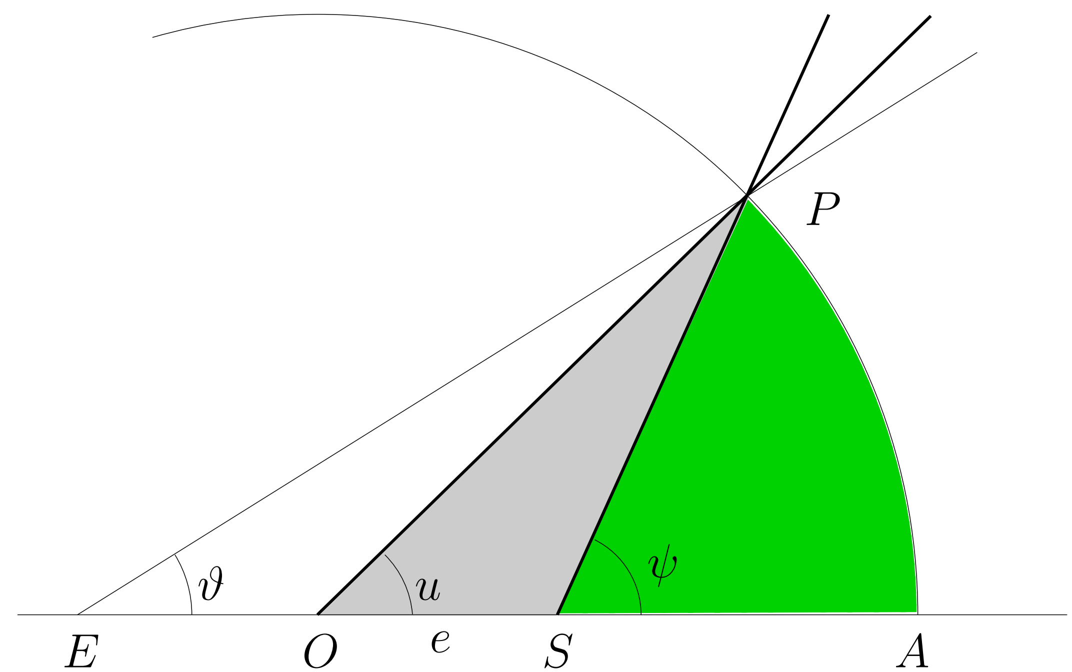
$$\frac{\delta s}{\delta t} = \frac{1}{r} \quad \text{or} \quad \delta t = r\,\delta s \ .$$

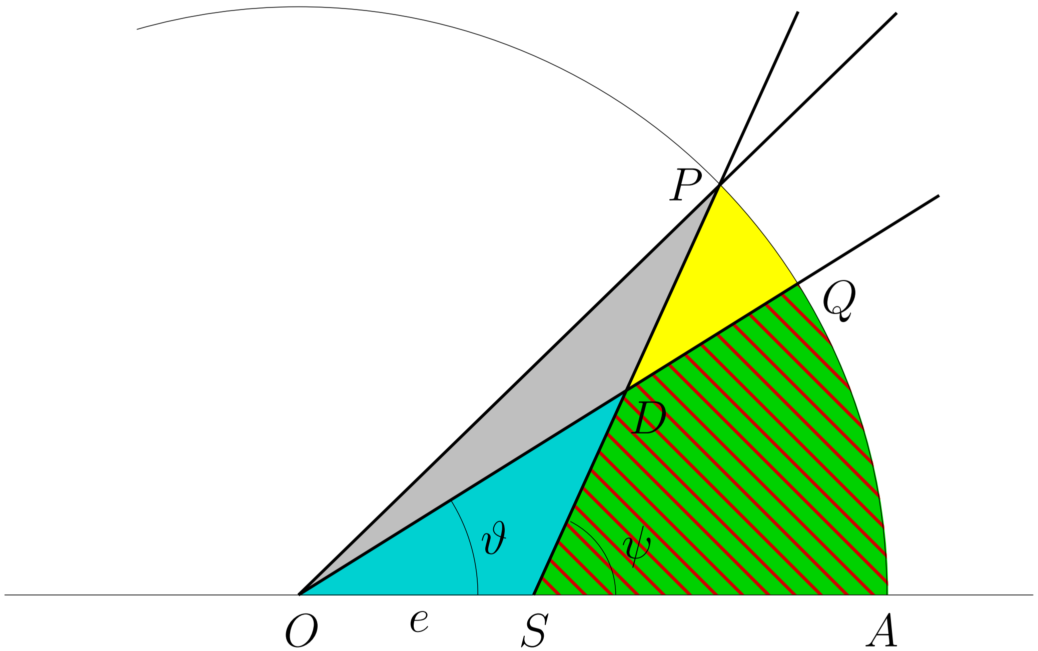
- Kepler calculates the sum of small areas in order to calculate the *mean anomaly*.
  - A long discussion: *Astronomia Nova*, ch. L.

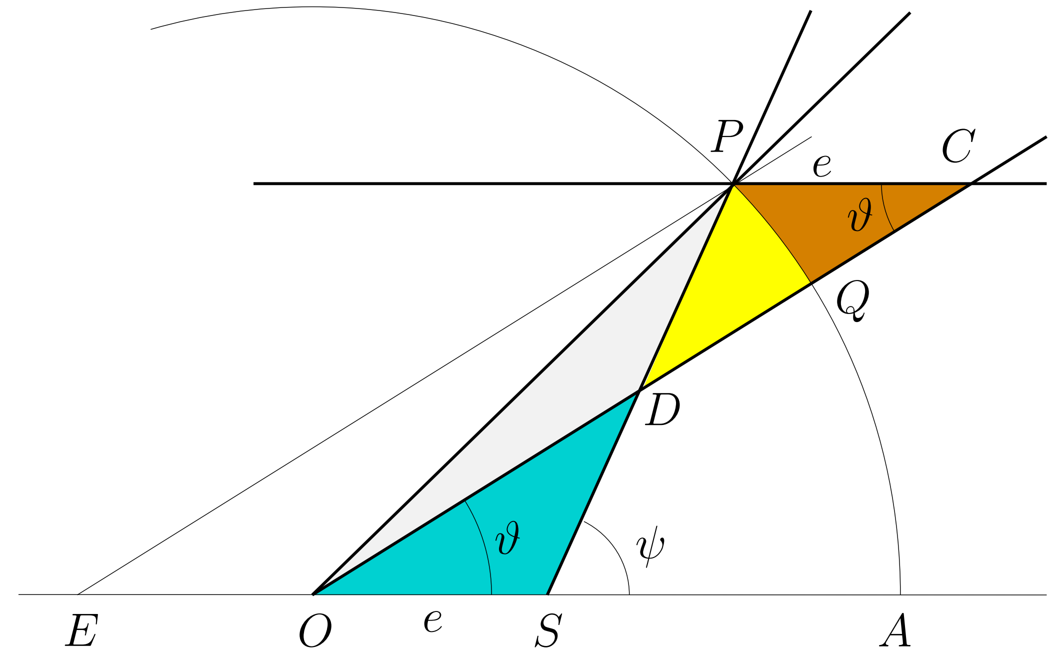


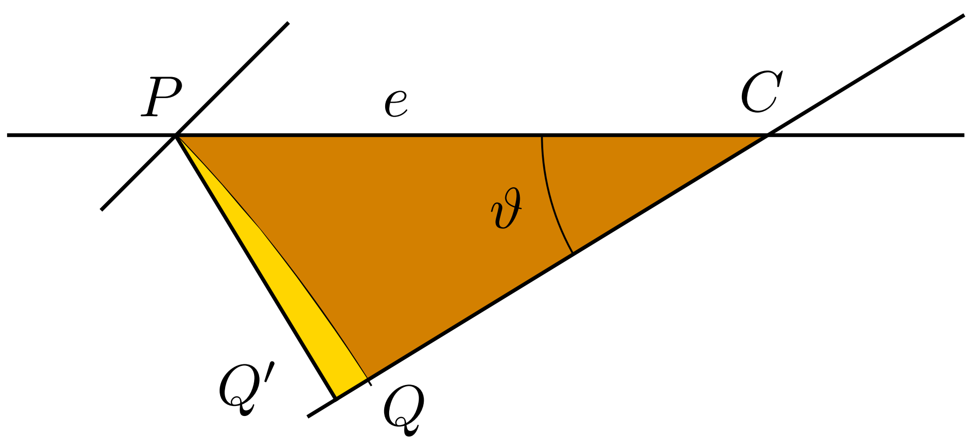
- S : Sun;
- O : center of the orbit;
- E : equant point;
- P : planet;
- A : perihelion;
- ψ : true anomaly;
- u : eccentric anomaly;
- ϑ : equant's anomaly, measures the time.
- Translate EP to OQ;
- area(OAQ) evolves uniformly in time (red);
- area(SAP) : the area swept by the radius from the Sun (green);
- want to compare the red and the green area.







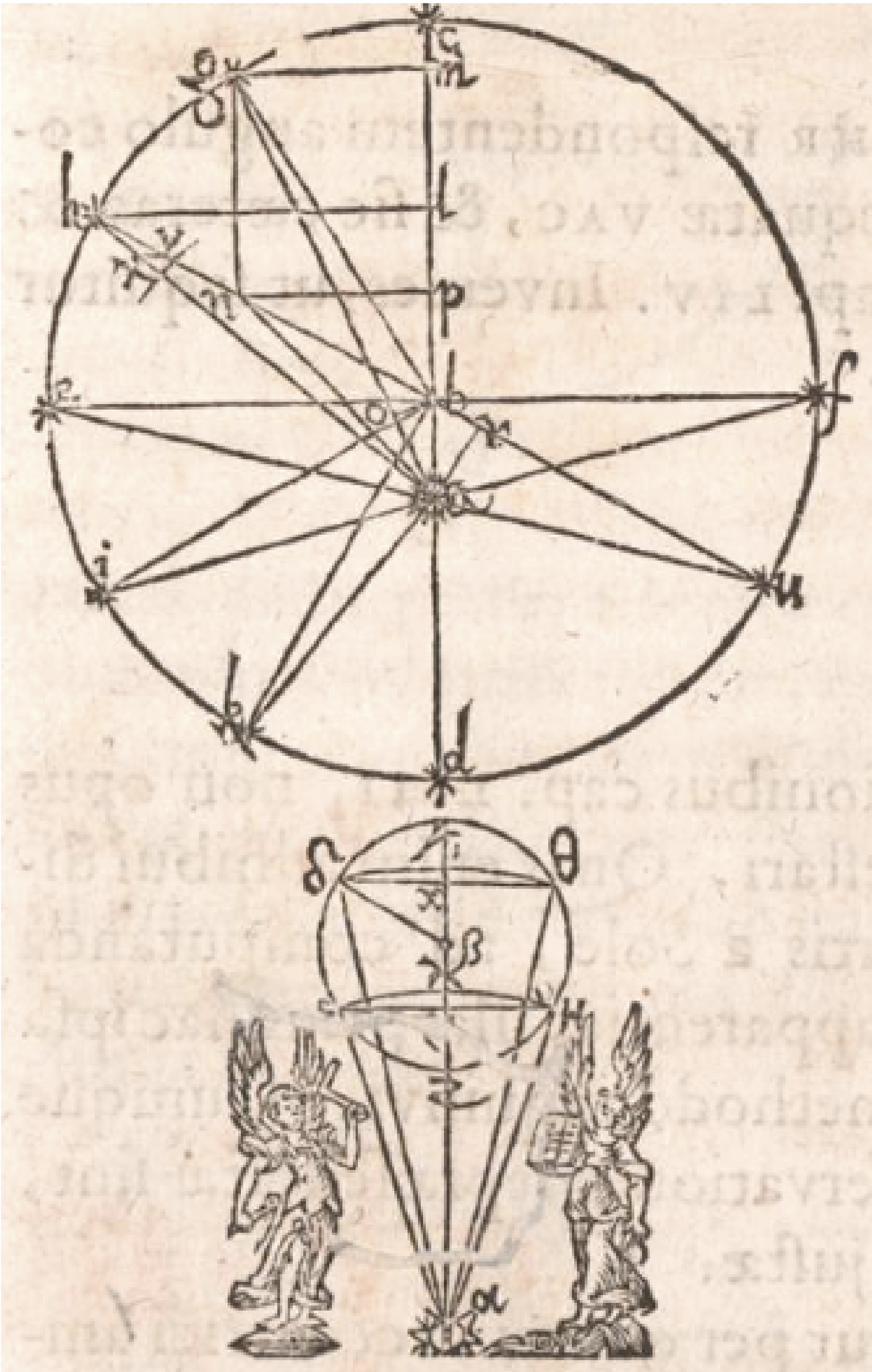




- equivalent to comparing area(DSO) (light blue) with area(DPQ) (yellow);
- draw the straight line $PC \parallel OA$;
- $|PC| = |OS| = e$;
- get area$(DSO) =$ area(DPC);
- draw $PQ' \perp CQ$;
- get area$(PQC) =$ area$(PQ'C) + \mathscr{O}(e^3)$;
- area$(DSO) -$ area$(DPQ) \sim \frac{e^2 \sin 2\vartheta}{4}$.
- conclude:

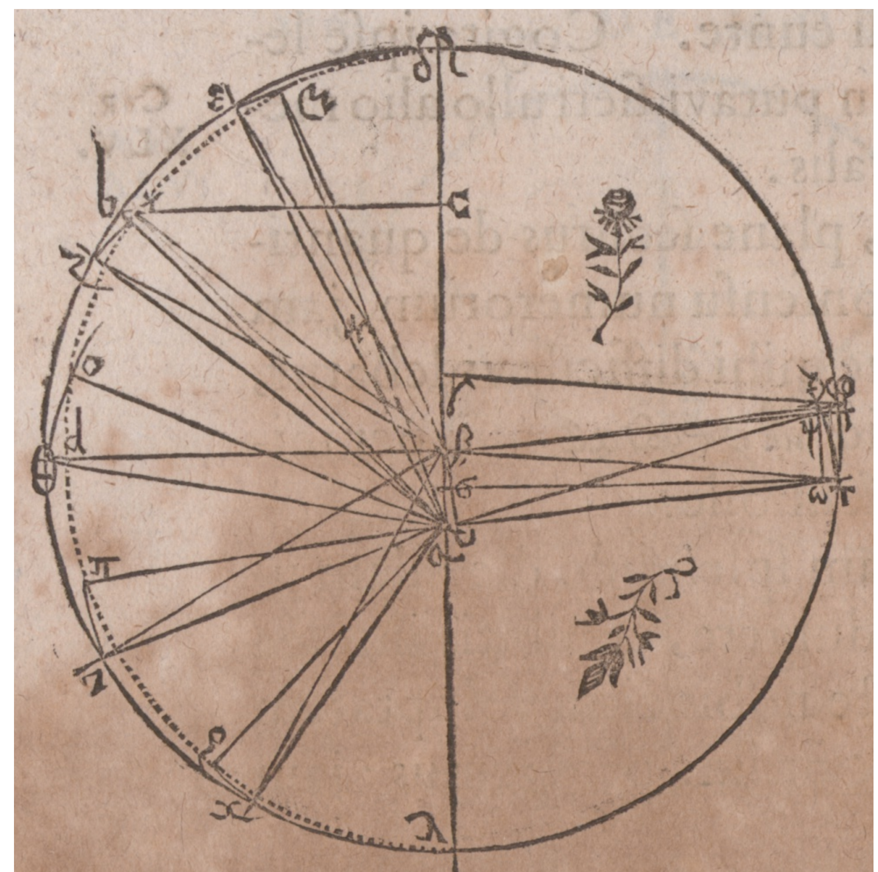
$$\big|\mathrm{area}(SAP) - \mathrm{area}(OAQ)\big| \sim \frac{e^2 \sin 2\vartheta}{4} \ .$$

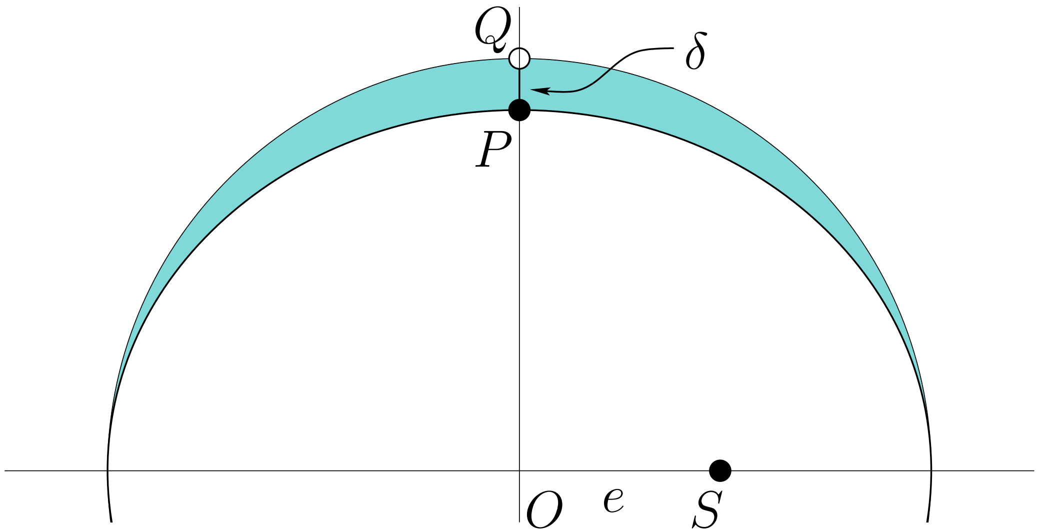
Second law in terms of areas: Newton, *Principia*, sect. II, Prop. I theorema I.

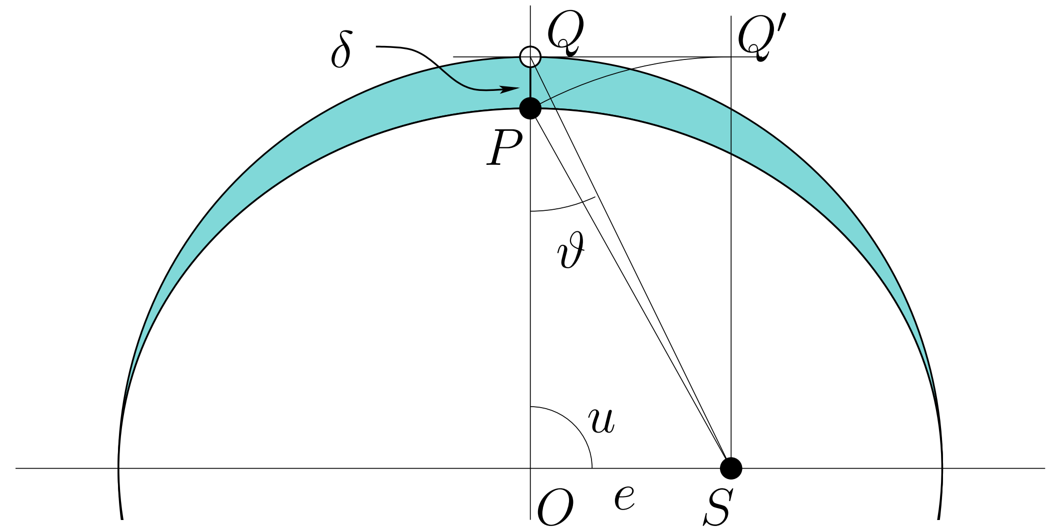


The elliptic orbit of Mars

- The orbit of Mars, the *inobservabile sidus*:
  - the method of equants gives the correct angles;
  - but the *distances* from the true Sun do not correspond to observations of Tycho Brahe.
- The orbit exhibits an oval form:
  - the displacement of the planet from the circle resembles an oscillation along a diameter of an epicycle,
  - which is hardly intepreted as due to a physical cause!
  - the circle and the orbit are separated by a *lunula*.







Cum igitur duobus argumentis (...) non obscure colligerem, lunulæ illius latitudinem dimidiam tantum assumendam, scilicet 429, *correctius* 432, *(...); cœpi de causis & modo cogitare, quibus tantæ latitudinis lunula rescinderetur.*

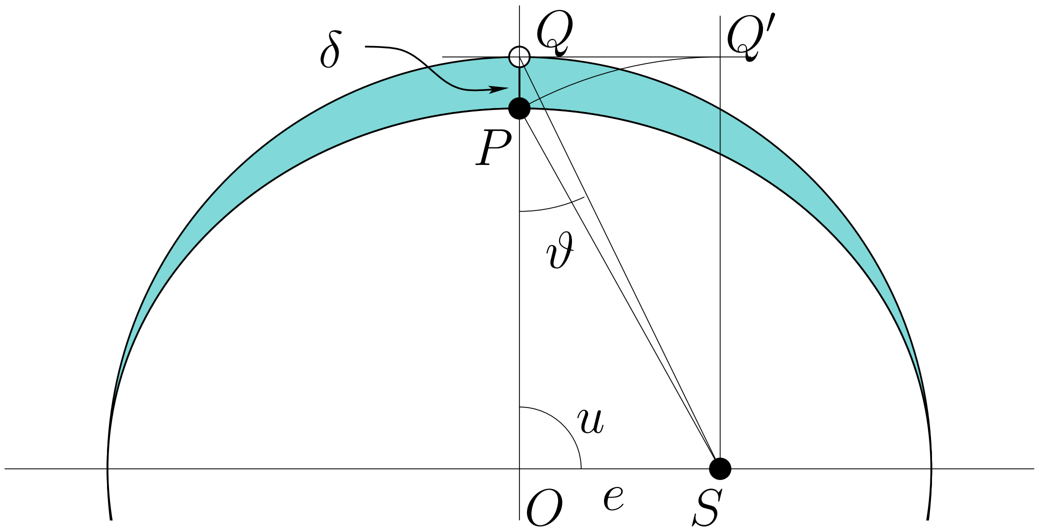
Qua in cogitatione dum versor anxie, ... forte fortuito incido in secantem anguli $5^\circ 18'$ *quæ est mensura æquationis Opticæ maximæ. Quem cum viderem esse* 100429, *hic quasi e somno expergefactus, & novam lucem intuitus, sic cœpi ratiocinari.*

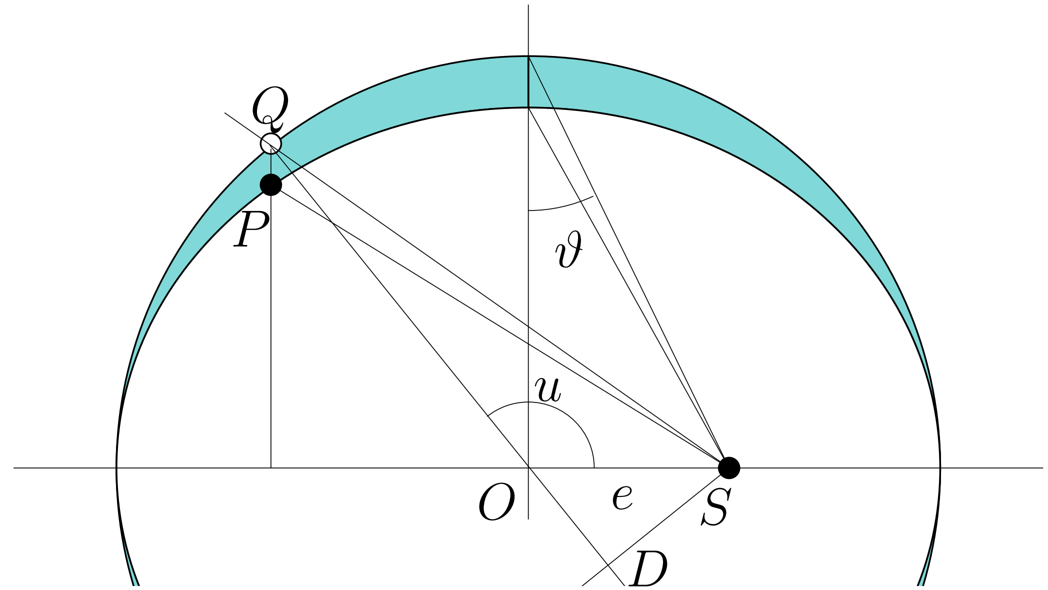
Thus, having clearly concluded from two different arguments (...) that the width of that lunula should be halved, i.e., 429, or more correct 432, (...); I began to investigate how and why such a big lunula should be subtracted.

While I am plunged anxiously into these reflections by pure chance I fall on the secant of the angle $5^\circ 18'$, which is the maximal amplitude of the optical equation. When I saw it to be 100429, it was like being suddenly awakened from sleep, and seeing a new light. Then I began to argue as follows.

For Mars: Optical equation $\vartheta \simeq 5^\circ 18'$; width of the lunula $\delta \simeq 0.00429$).

(*Astronomia Nova*, ch. LVI)



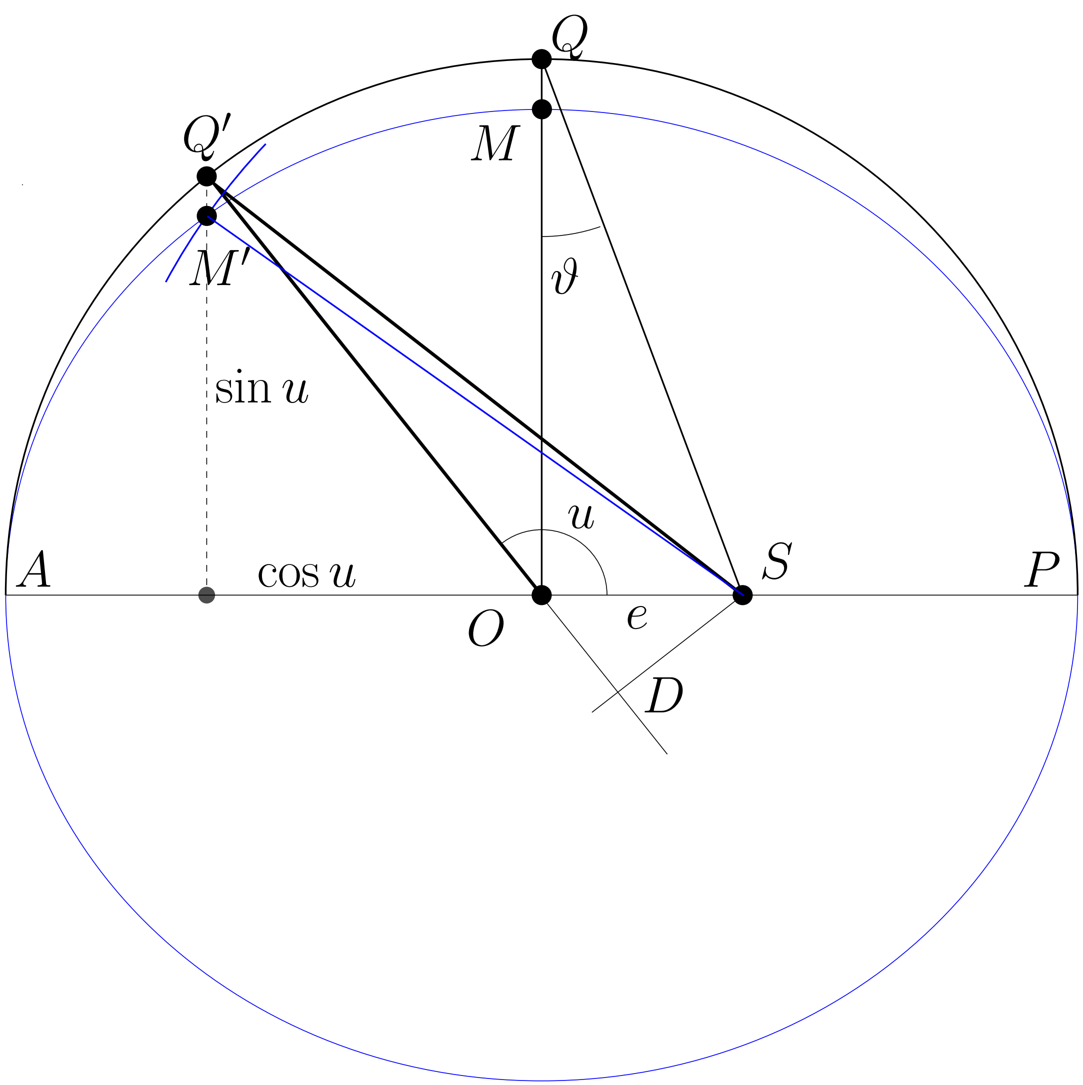


In longitudinibus mediis, æquationis pars Optica fit maxima. In longitudinibus mediis lunula seu curtatio distantiarum est maxima, etque tanta, quantus est excessus secantis æquationis opticæ maximæ 100429 *supra radium* 100000. *Ergo si pro secante usurpetur radius in longitudine media, efficitur id, quod suadent observationes. Et in schemate cap. XL conclusi generaliter, si pro HA usurpes HR, pro VA vero VR, & pro EA substituas EB, & sic in omnibus, fiet idem in locis cæteris eccentrici, quod hic factum est in longitudinibus mediis.*

In correspondence with the average value of the longitude the optical equation is close to a maximum. The amplitude of the lunula takes a maximum there, and is the same as the excess of the secant of the optical equation, namely 100429 over the radius 100000. According to ch. XL, I thus concluded, in general, that if you replace HA with HR, VA with VR and EA with EB, and similarly for all points, the same will happen at the other points of the eccentric circle that occurred here for average longitude.

(The text refers to Kepler's figure. In the present figure: replace SQ with SP, with $|SP| = |QB|$).

The equation of the orbit

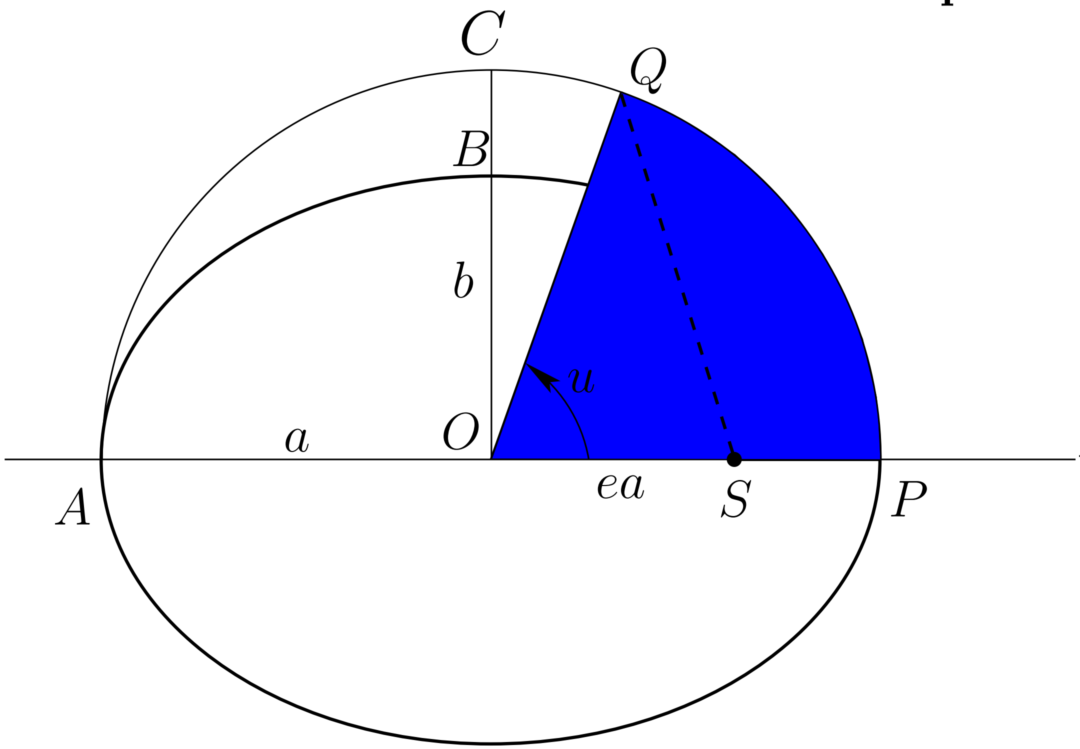


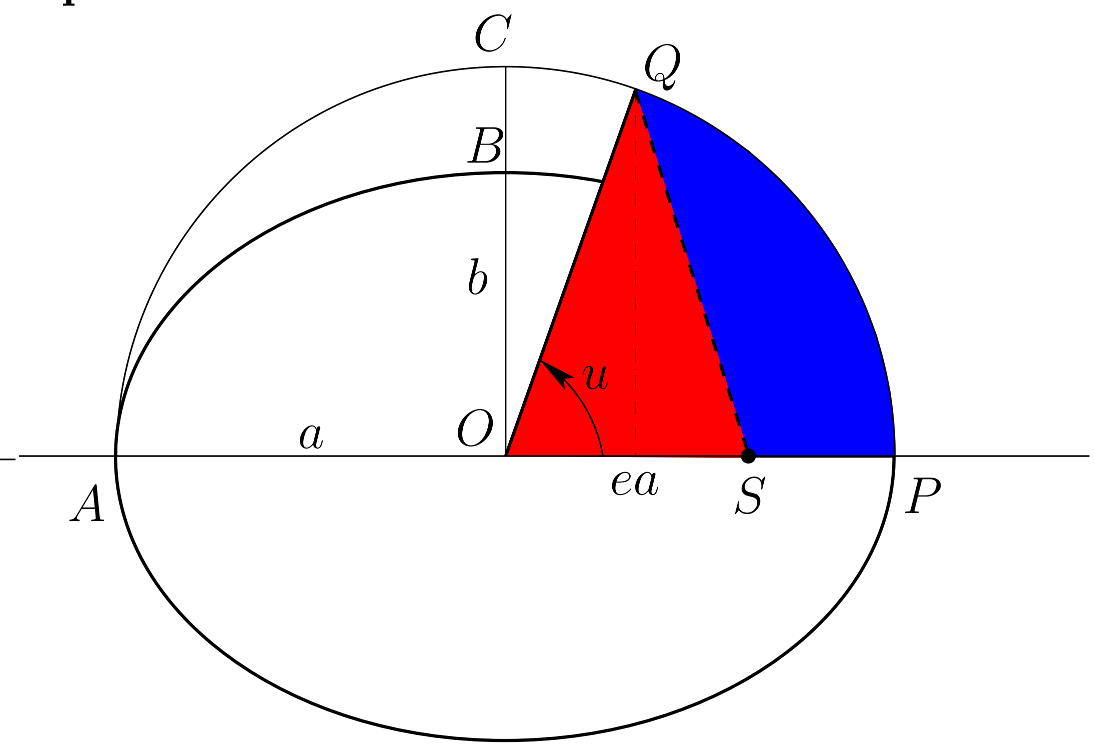
- Cartesian coordinates:
  - O the origin; line of apsides AP the x axis;
  - S : Sun; Q, Q' : fictitious positions of Mars on the circle;
  - SM, M' : true positions of Mars;
  - u : eccentric anomaly;
- Calculate:
  - $SQ' = (\cos u - e, \sin u)$;
  - $OQ' = (\cos u, \sin u)$;
  - $|DQ'| = 1 - e \cos u$ (scalar product between OQ' and SQ');
- With Kepler:
  - let $a = |OP|$ be the semimajor axis;
  - let u be given (eccentric anomaly);
  - draw the arc of radius $r = |DQ'|$;
  - find M' by intersecting the arc with the line from Q' orthogonal to AP;
  - get

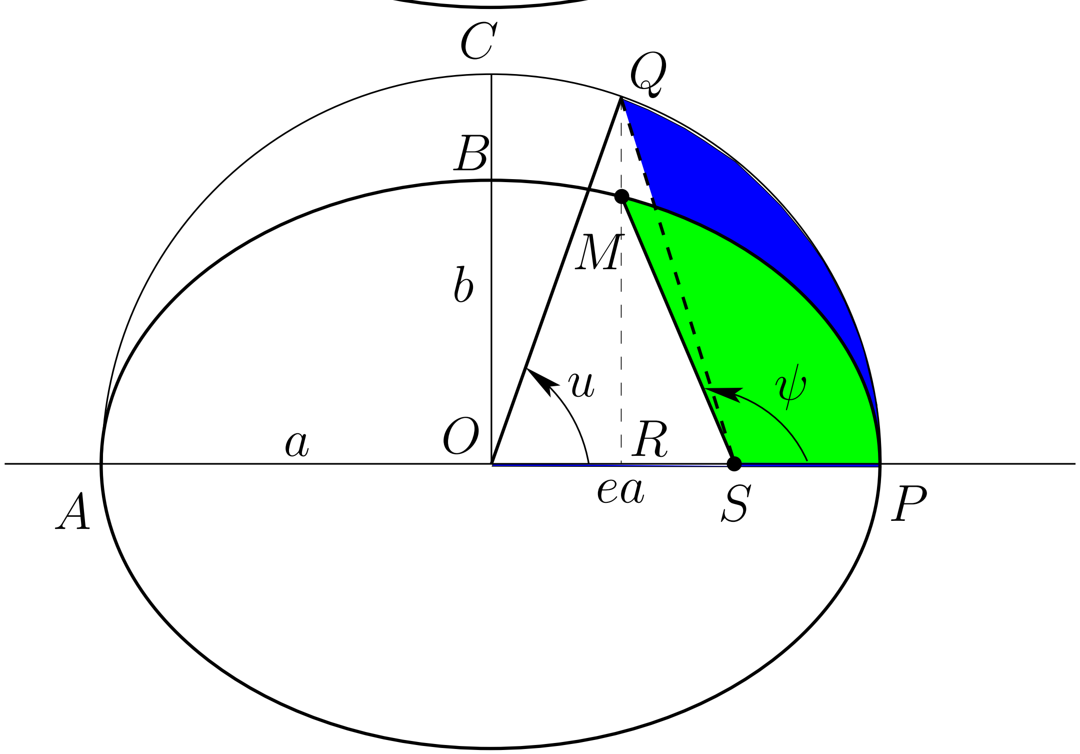
$$r = a(1 - e \cos u) \ ,$$

the equation of an ellipse.

Kepler's equation







$$\mathrm{area}(OQP) = \frac{1}{2}a^2 u \ ;$$

$$\mathrm{area}(SPQ) = \frac{1}{2}a^2 u - \frac{1}{2}a^2 e \sin u \ ;$$

$$\mathrm{area}(SPM) = \frac{b}{a}\left(\frac{1}{2}a^2 u - \frac{1}{2}a^2 e \sin u\right) = \frac{\pi ab}{T} t \ ;$$

Kepler's equation:

$$u - e \sin u = \ell \ , \quad \ell = \frac{2\pi}{T} t \ : \quad \text{mean anomaly} \ .$$

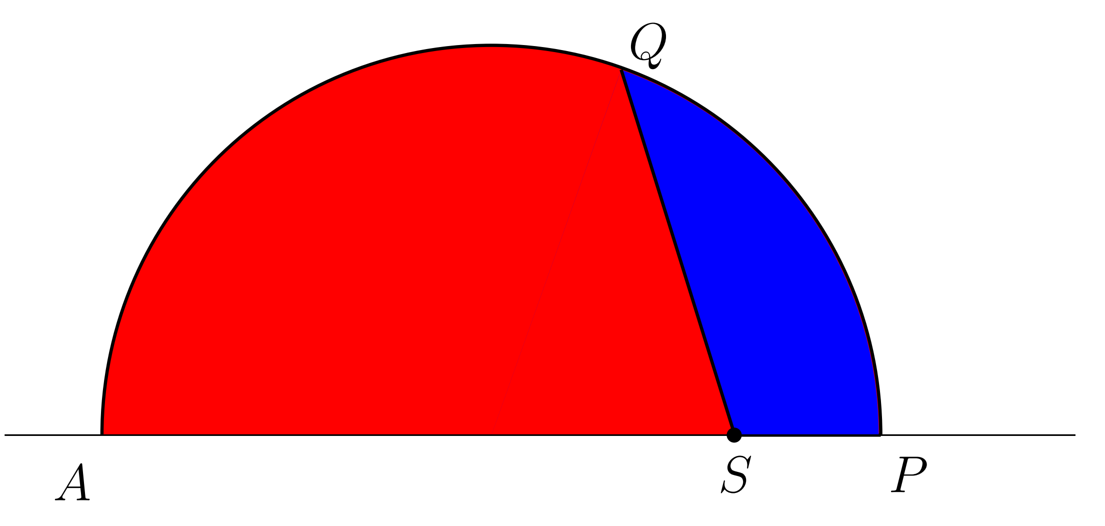
- Kepler could not find a solution of his equation:
  - suggests the problem to geometers:

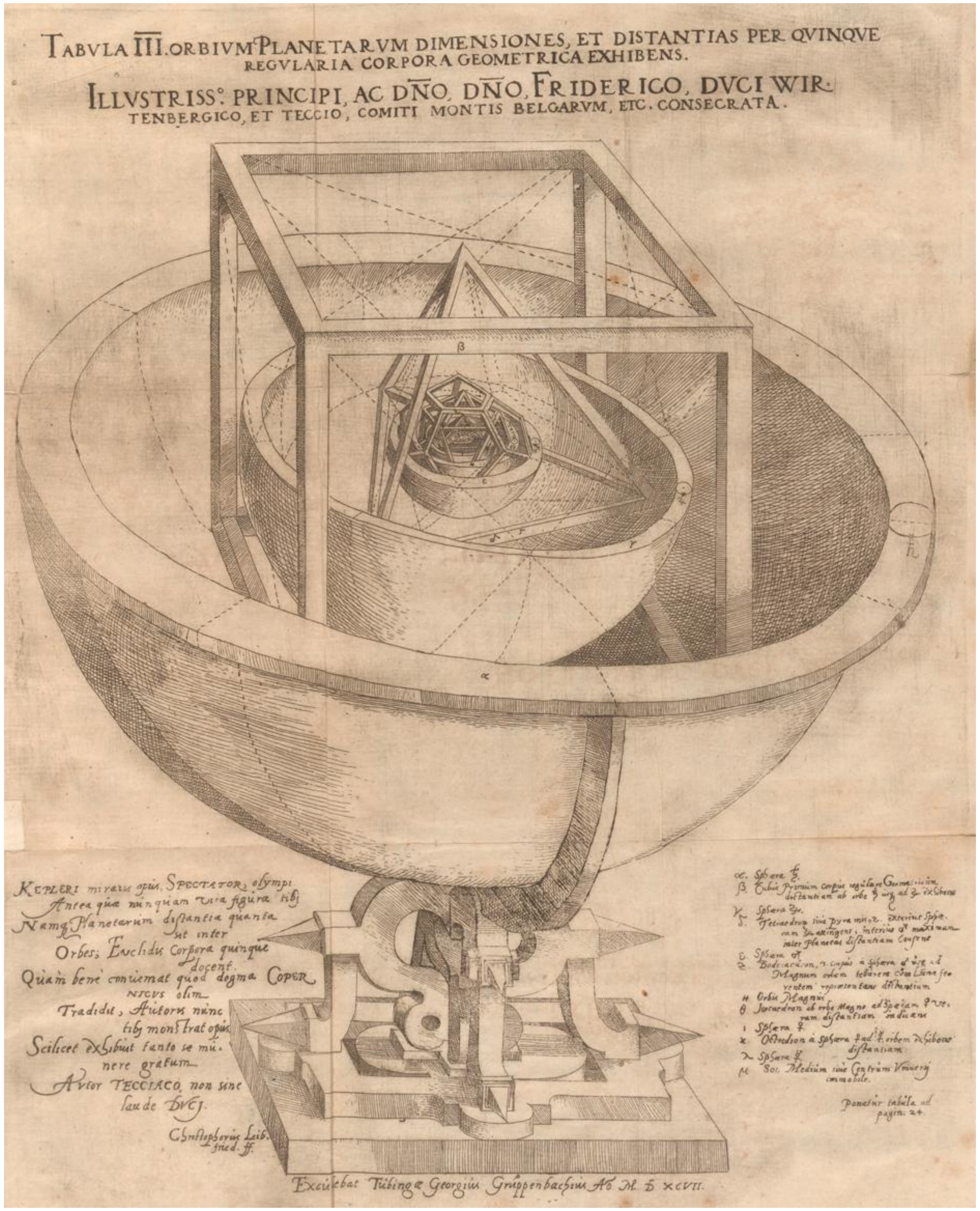
Data area partis semicirculi, datoque puncto diametri, invenire arcum, & angulum ad illud punctum: cujus anguli cruribus, & quo arcu, data area comprehenditur. Vel: Aream semicirculi ex quocunque puncto diametri in data ratione secare.
Mihi sufficit credere, solvi a priori non posse propter arcus & sinus ἑτερογένειαν. *Erranti mihi, quicunque viam monstraverit, is erit mihi magnus Apollonius.*

(Astronomia Nova, ch. LX.)

Let the area of a part of a semicircle and a point on the diameter be given; to find the arc and the angle at that point, such that the sides of that angle and that arc enclose the given area. Or, similarly: to divide the area of a semicircle in a given ratio from any given point on the diameter.
I just think a priori that this can not be solved, for the arc and the sine are *heterogeneous* quantities. I'm wandering here, and if anyone will show me the way, he will be for me a great Apollonius.

- A solution has been found by Lagrange (1771).



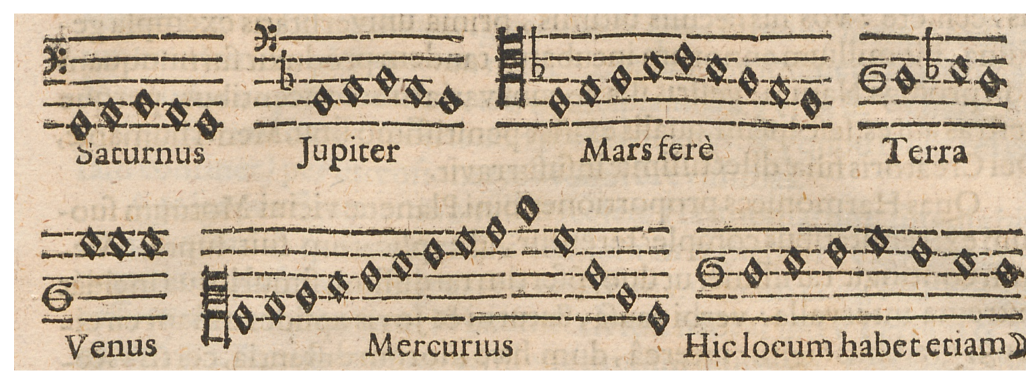


The perfection of the world

- Mysterium Cosmographicum (1596): the 5 regular solids inside the orbits of the planets (left).

. . . so many years . . .

- Harmonice Mundi (1619): the music played by planets in honor of the Creator (below).



The third law of Kepler

Hactenus egimus de diversis moris vel arcubus unius et eiusdem Planetæ. Jam etiam de binorum Planetarum motibus inter se comparatis agendum. (...)

Rursum igitur hic aliqua pars mei Mysterii Cosmographici, suspensa ante 22 annos, quia nondum liquebat, absolvenda et huc inferenda est. Inventis enim veris orbium intervallis per observationes Brahei, plurimi temporis labore continuo, tandem, tandem genuina proportio temporum periodicorum at proportionem orbium —

sera quidem respexit inertem,
Respexit tamen et longo post tempore venit;(*)

eaquem si temporis articulos petis, 8 Mart. hujus anni millesimi sexcentesimi decimi octavi animo concepta, sed infeliciter ad calculos vocata, eoque pro falsa rejecta, denique 15 Maji reversa, novo capto impetu expugnavit mentis meae tenebras, tanta comprobatione et laboris mei septendecennalis in observationibus Braheanis et meditationis hujus in unum conspirantibus, ut somniare me et præsumere quæsitum inter principia primo crederem.

So far we have considered either time intervals or arcs of one and the same Planet. Now we should consider the motions of pairs of planets, compared with each other. (...)

Therefore we should now achieve another part of my *Cosmographical Mystery*, suspended twenty-two years ago, because it was not yet clear enough. For after I had discovered the true distances of the orbits, thanks to Brahe's observations, and had spent a great amount of time working hard, at last!, at last the true proportion of the periodic times to the orbits —

late she saw me, lying helpless,
yet she came and gazed at me after a long time;

and if you want to know the precise moment, the first idea came across me on the 8th March of this year 1618; but it was rejected as false, due to an unappropriate reduction to calculation. But later it fell again upon me on the 15th May, and conquered with outstanding power the darkness of my mind, thanks to the agreement between this idea and my seventeen years labour on Brahe's observations, so that I thought I was dreaming, and took my result for granted in my first assumptions.

(*) Publius Vergilius Maro: *Bucolica*, Ecloga I.

Sed res est certissima exactissimaque, quod **proportio quæ est inter binorum quorumcunque Planetarum tempora periodica, sit præcise sesquialtera proportionis mediarum distantiarum, id est orbium ipsorum**; *attento tamen hoc, quod medium arithmeticum inter utramque diametrum ellipticæ orbitæ sit paulo minus longiore diametro. Itaque si quis ex periodo, verbi causa Telluris, quæ est annus unus, et ex periodo Saturni triginta annorum, sumserit tertiam proportionis partem, id est, radices cubicas, et huius proportionis duplum fecerit, radicibus quadrate multiplicatis, is habet in prodeuntibus numeris intervallorum Terræ et Saturni a Sole mediorum proportionem justissimam. Nam cubica radix de 1 est 1, ejus quadratum 1. Et cubica radix de 30 est major quam 3, eius igitur quadratum majus quam 9. Et Saturnus mediocriter distans a Sole, paulo altior est noneuplo mediocris distantiæ Telluris a Sole.*

(Harmonices Mundi Libri V, Liber V, Caput III.)

For it is a definitely certain, an absolutely exact truth that *the actual proportion between the periodic times of any two planets is exactly the sesquialtera proportion of the mean distances of the orbits, i.e., of the orbits themselves;* this taking into account that the arithmetic mean between the two diameters of the elliptic orbit is a little less than the longer diameter. Thus if one considers, e.g., the period of the Earth, which is 1 year, and the thirty years period of Saturn, and takes one third of the proportions, that is the cubic roots, and doubles that proportion, making the square of the roots, he will get from the resulting numbers the correct proportion of the mean distances of the Earth and of Saturn from the Sun. For the cubic root of 1 is 1, and its square is 1; and the cubic root of 30 is greater than 3, and its square greater than 9. Indeed the average distance of Saturn from the Sun is a little bigger than nine times the average distance of the Earth from the Sun.

The plan of God has been discovered!

but . . .

The “Tabulæ Rudolphinæ

- The dates:
  - conceived by Tycho Brahe (he was 17 years old) in 1564;
  - work undertaken in 1572;
  - Kepler’s collaboration begins in 1600;
  - Tycho Brahe dies in 1601;
  - compilation completed by Kepler in 1623;
  - published in 1627.

Et de certitudine quidem calculi testabuntur observationes præsentium temporum, imprimis Braheanæ; de futuris vero temporibus plura præsumere non possumus, quam vel observationes veterum, quibus usus sum, vel ipsa motuum mediorum conditio, nondum penitus explorata, concursusque causarum physicarum præstare possunt, cum observationes Regiomontani et Waltheri testentur, omnino de æquationibus secularibus esse cogitandum, ut singulari libello reddam demonstratum suo tempore; quæ tamen æquationes quales et quantæ sint, ante plurimum sæculorum decursum observationesque eorum, a gente humana definiri nequaquam possunt.

And the observations made in our epoch, especially by Brahe, will prove the reliability of our calculations. However, concerning the future we can not expect so much. The validity may be questioned by ancient observations, that I’m well aware of, by the knowledge of the mean motions, that have not yet been fully explored, and by the concurrence of physical actions. The observations of Regiomontanus and Walther do indeed show that we should definitely think about secular equations, as I will explain in a specially devoted booklet. Which and how many equations we need, however, humanity will be unable to decide it before many centuries of observations have been accumulated.

(Tabulæ Rudolphinæ, preface)

Observations of Regiomontanus and Walther (sample)

Anno 1478, 22 Aug. h. 3 post medium noctis fuerant in una linea ♃ et duo oculi ♉, et erat ♃ occidentalior, distans per medietatem distantiæ, qua duo oculi distant, ab oculo occidentaliori; sic visui apparuit.

In 1478, august 22, tree hours past midnight, Jupiter and the two eyes of Taurus were on the same line, and Jupiter was toward west, the distance from the west eye of Taurus being half the distance between the two eyes; so it visually appeared.

1478, 24 Sept, $40'$ *ante ortum solis vidi lunam circa ♄, quasi coniunctos; distabat Luna modicum ad septentrionem, ita ut inter circunferentiam eius et ♄ videretur mediare spatium unius palmæ.*

In 1478, September 24, 40 minutes before sunrise I saw Saturn and the Moon approximately in conjunction; Moon's position was scanty on the north direction, and it appeared that between her circle and Saturn's one could insert the space of a palm.

(...) *stella ♃ videbatur inter duas Virginis, quarum lucidior est circa medietatem alæ sinistræ Virginis, alia obscurior circa oculum eius versus Leonem* (...).
Magna cum perplexitate diu conflictatus sum, quænam essent hæc duæ stellæ.

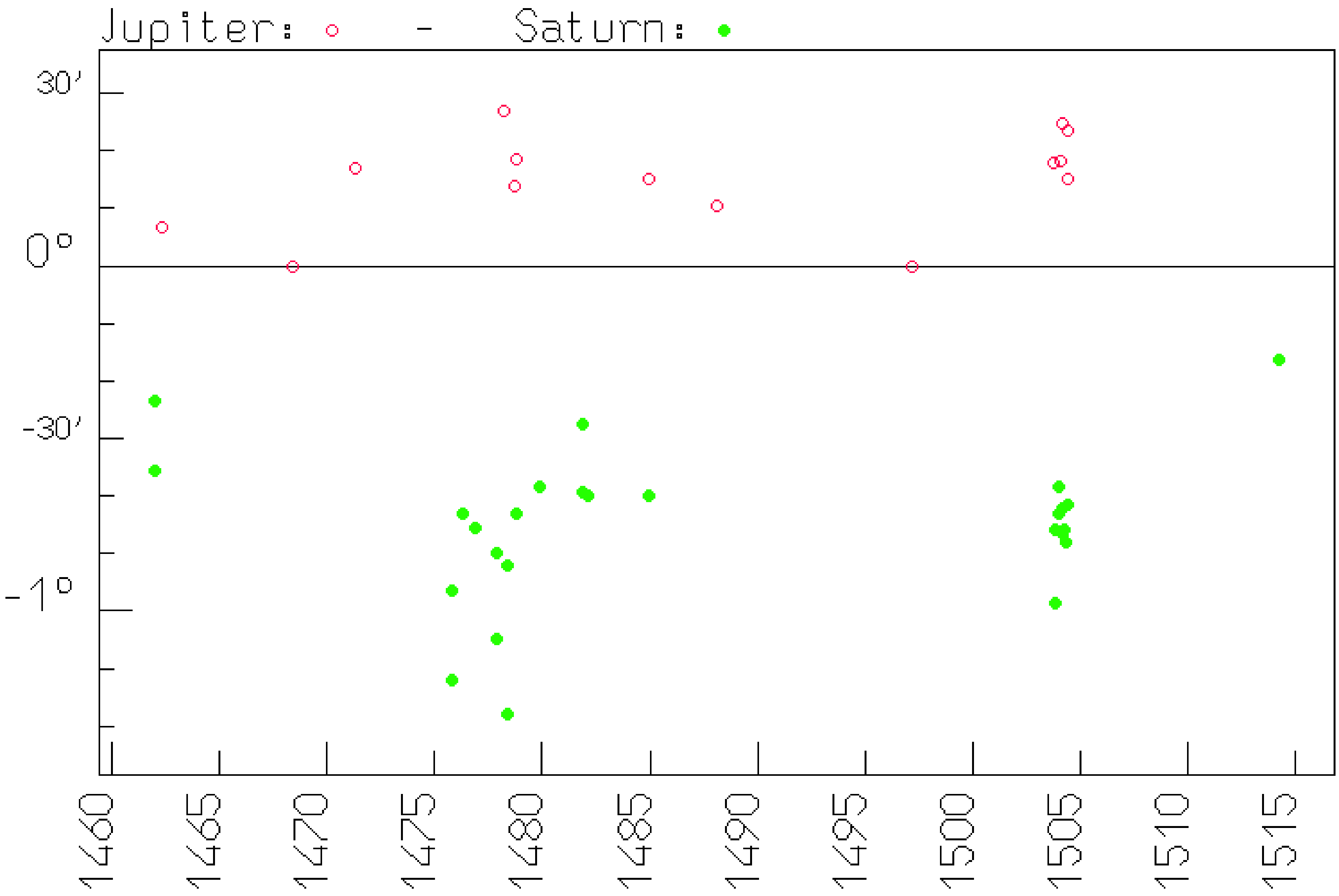
(...) the star Jupiter was seen between two stars of Virgo. The brighter one is close to the center of the left wing of Virgo, the other one, less bright, is close to her eye, towards Leo (...).
With great perplexity and for a long I felt vexed trying to figure out which these two stars are.

Observation reported by Ptolemy

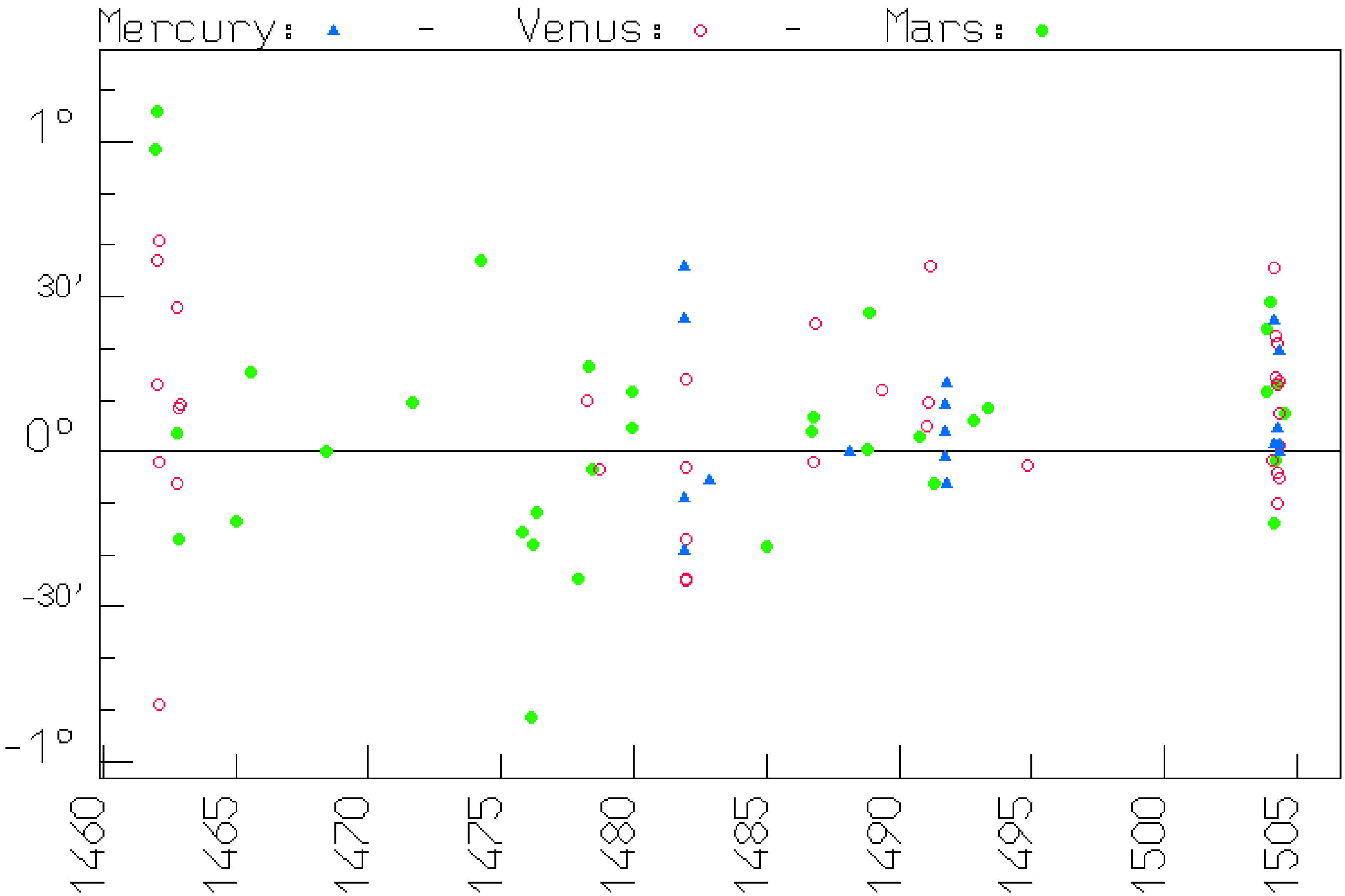
(...) a. 82 die 2 Xantichi vesperi, quod ex fide Ptolemæi interpretis fuit ante Chr. anno 229 d. 1 Mart. Tunc ♄ sub australi humero ♍ visus est 2 digitos.

(...) in the evening of the second day of the Xantic month of the year 82, which according to the interpreters of Ptolemy is March 1, 229 BC. Then Saturn was seen two fingers below the austral shoulder of Virgo.

The difference between observations and calculation for Jupiter and Saturn



The difference between observations and calculation for Mercury, Venus and Mars



The last observation of Walther

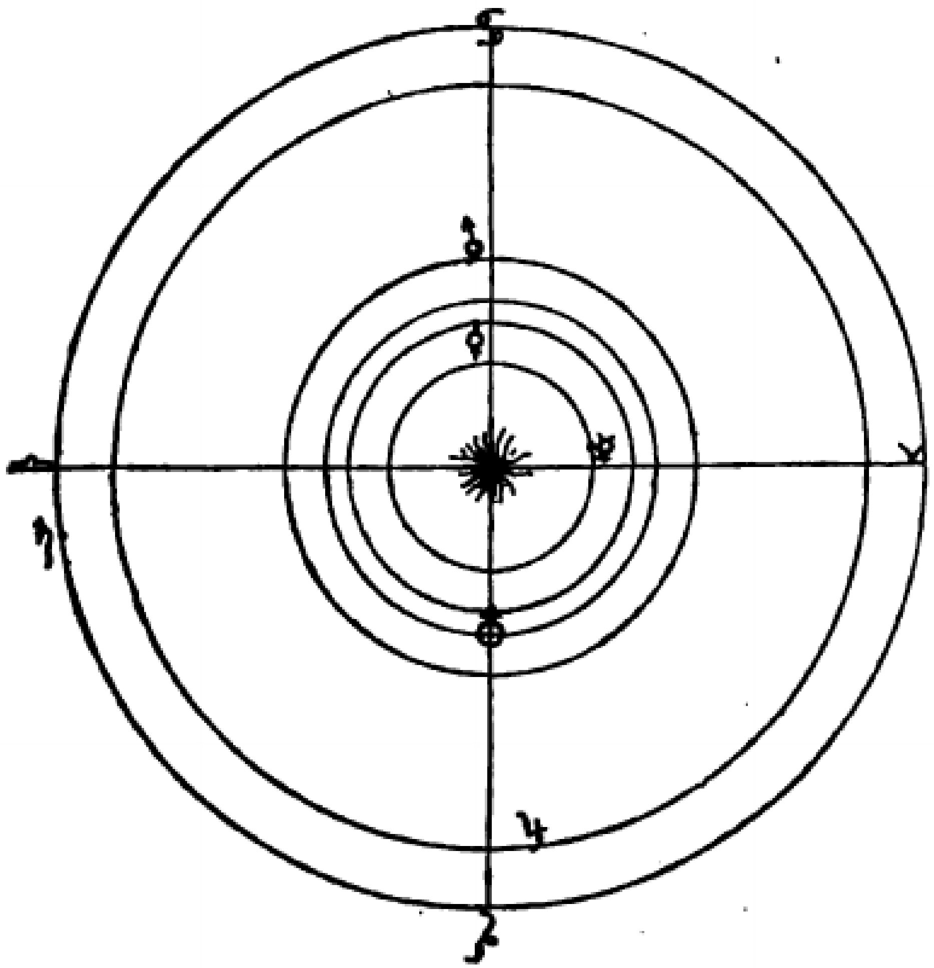
- May 24, 1504:
  - Walther reports a conjunction between Jupiter and Saturn;
  - the calculation results in a distance of $58'$

Et hic dissensus calculi in ♃ and ♄, excurrens ad integrum gradum, est remora illa, quæ me, plurima perplexitate circumventum, per solidos quinque menses in observationibus Waltherianis exercuit tandemque ad nova consilia circa motuum mediorum speculationem adegit, deprehensa manifesta inææqualitate motuum seculari. (Absolvi hucusque 18 Junii 1624.)

And this discrepancy in the calculation for Jupiter and Saturn, which amounts to a whole degree, is such an hindrance that it caused me to be assailed by many perplexities, and for five solid months I have been troubled until I eventually changed my advice concerning average motions, having accepted the manifest secular inequality of the motions. (I came to this conclusion on June 18, 1624.)

- Kepler tries to introduce *secular equations*, with great difficulty:
  - assumes that the Earth's motion is uniform over all centuries;
  - keeps the eccentricites constants, as measured by Tycho Brahe;
  - attempts to modify the aphelion;
- Successful (partially) only for the observations reported by Ptolemy.

A theological constraint



- The configuration of the planets at creation time:
  - Common opinion at Kepler's time: the world was created around 4000 BC (based on *Genesis*);
  - Kepler's hypotesis: the planets must have been created in a privileged configuration;
  - Kepler's calculation: the creation date is july 24, 3993;
  - at that time the planets were very close to the cardinal points of the orbit of the Earth.

Certe non temere Deus instituit motus, sed ab uno quodam certo principio et illustri stellarum conjunctione, et in initio zodiaci, quod creator per inclinationem Telluris domicilii nostri effinxit, quia omnia propter hominem.

It is certain that God did not establish the motions inconsiderately, but from one well definite beginning and a privileged configuration of stars, and at the beginning of the zodiac, which has been moulded by means of the inclination of the Earth, our house, because everything has been created for the human beings.

(*Mysterium Cosmographicum*, ch. XXIII)

The conclusions of Kepler

At cum ex his epochis computarem postea Waltherianas et Regiomontani observationes exque iis appareret clarissime, ♄ motus indigere æquatione seculari, eoque frustra nos medium affectare inter longe distantes, si inter se pugnent, nec in unam certis vicinis observationibus confirmatam commensurationem se cogi patiantur; (...)

Moreover, using their epochs I have calculated the observations of Walther and Regiomontanus, and from them undoubtedly appears that the motion of Saturn is affected by a secular equation. Thus it is vain that we try to find an average between very distant observations, if they fight together and do not accept to be represented by a definite proportion confirmed by reliable and close observations; (...)

Nam quod Tychonicum attinet, videor ex oppositionibus acronychiis per totam triacontaëderis periodum jam sentiscere effectum æquationis secularis. Id autem fieri solet non in æquatione maxima, tunc enim quantitas consistit, insensibili existente varietate, sed in æquatione prope nulla, tunc enim desinente adjectoria, incipiente subtratoria, vel e contrario, quantum potest maxima sentitur.

Concerning Tycho's time indeed it seems to me that considering achronichous oppositions over a complete period of thirty years the effect of a secular equation begins to be perceived. This however usually does not happen for a maximal equation, since in that case the quantity remains almost constant due to an insensible change, but it rather happens to be close to a null value, when the quantities to be added before and to be subtracted after, or the contrary, reach the maximum attainable.

(*Consideratio observationum Regiomontani et Waltheri.*)

- Kepler's opinion: *we should accumulate observations over many centuries*, so as to determine the secular equations to be added.

There is still a lot of work for the future.

After Kepler

- The question regarding the *great inequality:*
  - investigated by Newton, Halley, D'Alembert, Lalande, Euler, Boskovich, Lagrange, Laplace (among others);
  - the solution of Laplace (1785): a perturbation with period $\sim$ 900 years, due to closeness to $5:2$ resonance.
- The development of perturbation theory:
  - too many people ... (we know).
- The hurricane Poincaré:
  - the discovery of chaos ($\sim$ 1890),
  - essentially forgotten for some 70 years.
- The rediscovery of chaos in Astronomy:
  - Contopoulos ($\sim$ 1960), Hénon and Heiles (1964), ...
  - ... well, we all know the story ... it's matter of the last 60 years ...

What about the search of Kepler for a perfect world?

May try many answers possibly located between two extrema ...

Our theory is perfect;

the world is wrong.

Then I would feel sorry for the dear Lord. The theory is correct anyway. (A. Einstein)

Our theory is perfect;

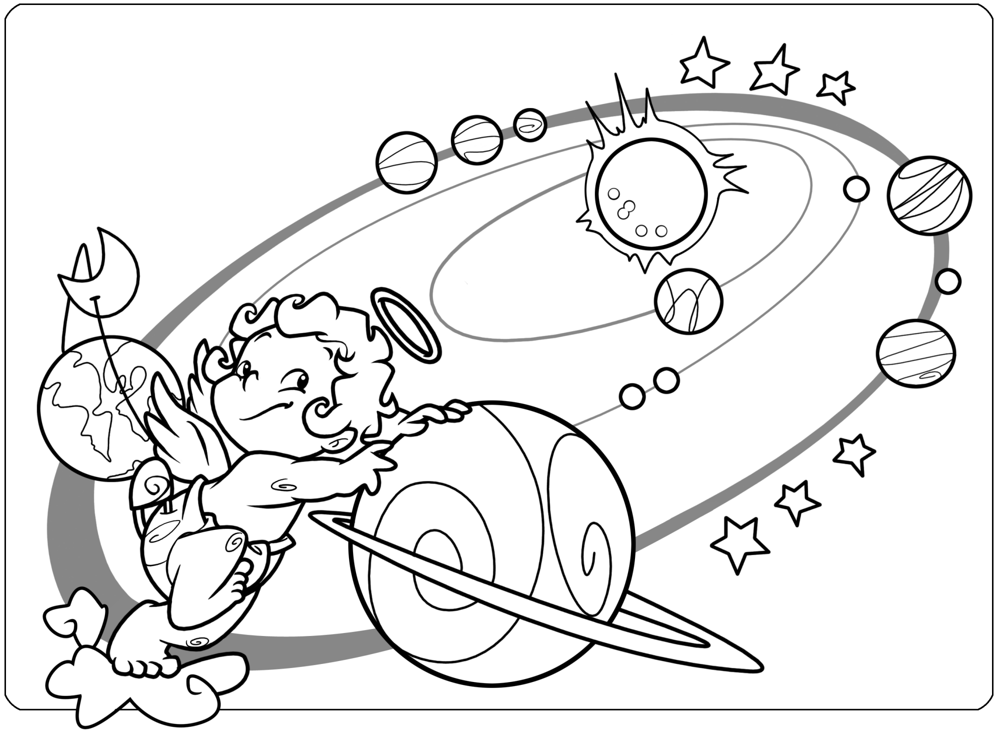
the world is wrong.

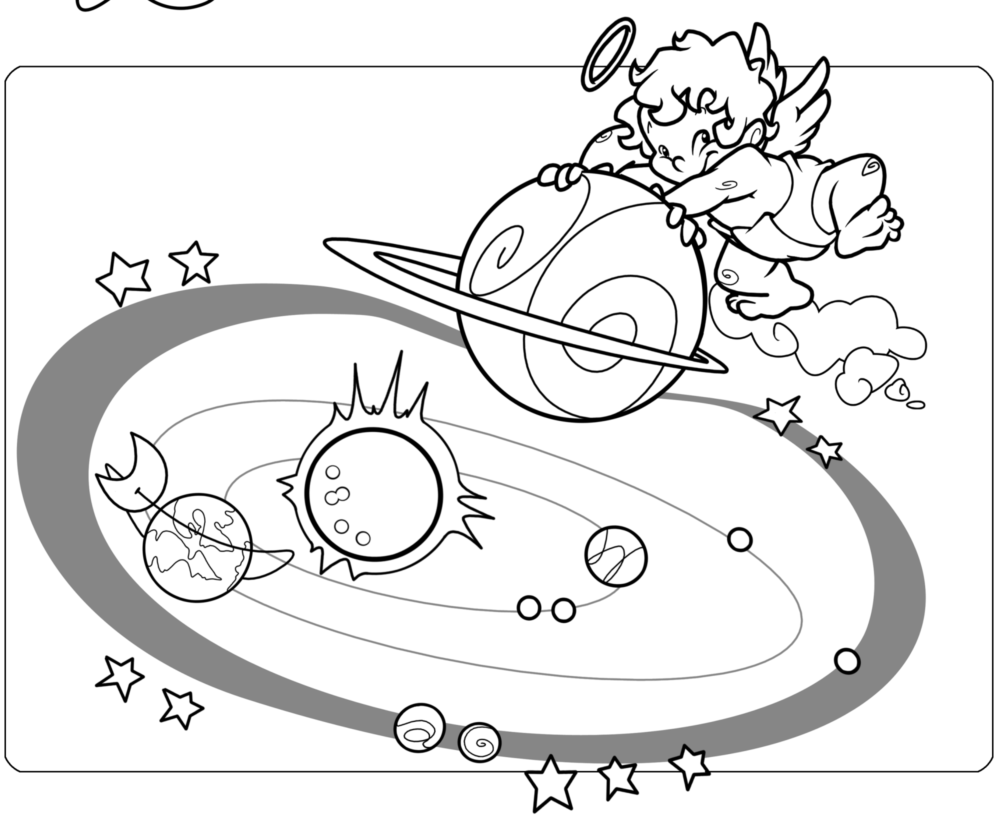
Then I would feel sorry for the dear Lord. The theory is correct anyway. (A. Einstein)

or

The world is perfect;

what is wrong is our idea of perfection.

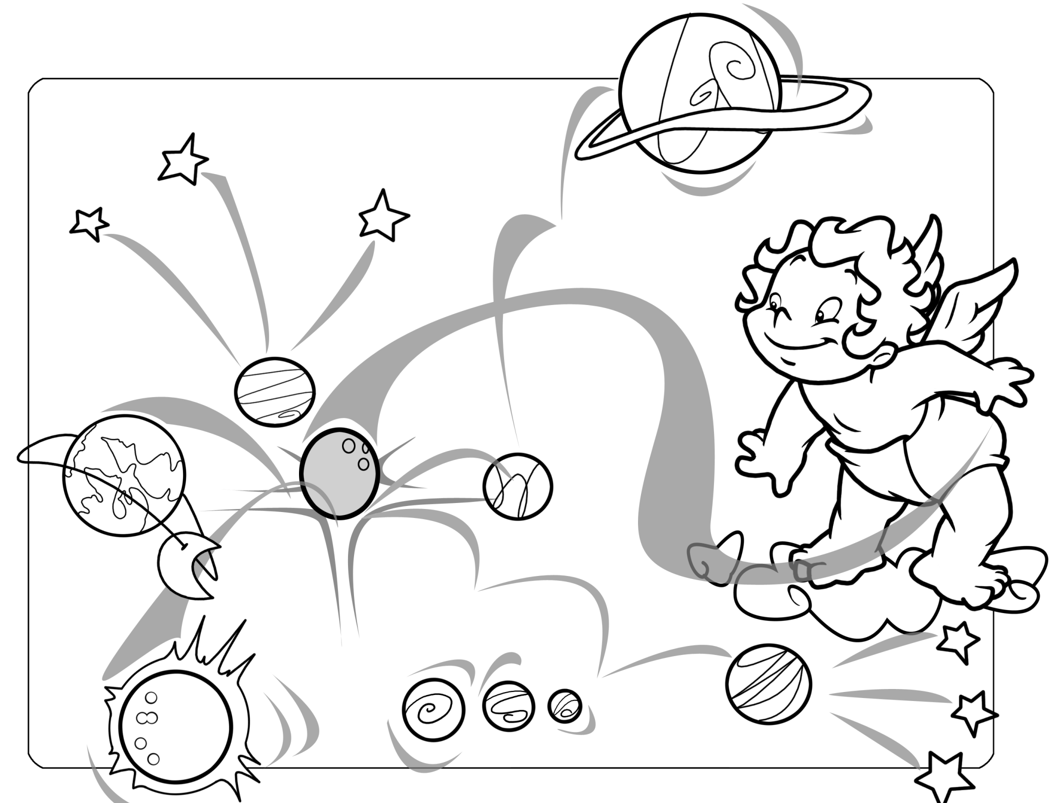




Thanks for

your attention!

(With my apologies)



Drawings by Cristina Giorgilli