

April 27, 2020

3. Ruled and rational surfaces

S surface. $\mathbb{B}(S)$ contains a single or several minimal models, e.g. $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{P}^2 are both minimal & birational.

Def. S is ruled (or birationally ruled) if

$S \xrightarrow{\text{birat}} B \times \mathbb{P}^1$, where B is a smooth projective curve.
 S is rational if $S \xrightarrow{\text{birat}} \mathbb{P}^2$ (or $S \xrightarrow{\text{birat}} \mathbb{P}^1 \times \mathbb{P}^1$).

Hence: rational \Rightarrow ruled.

Theorem (Noether-Ernest) Let $p: S \rightarrow C$ (sm-proj. curve) be a surjective morphism s.t. $\exists z \in C$ with $p^{-1}(z) \cong \mathbb{P}^1$. Then \exists a rational map $\varphi: S \dashrightarrow \mathbb{P}^1$ giving rise to a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi = (p, \varphi)} & C \times \mathbb{P}^1 \\ p \searrow & & \swarrow \mathbb{P}^1 \\ & C & \end{array}$$

where, on an open subset $U = C \setminus \{z_1, \dots, z_n\}$, φ induces an isomorphism

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\sim} & U \times \mathbb{P}^1 \\ p \searrow & & \swarrow \mathbb{P}^1 \\ & U & \end{array}$$

In particular, φ is birational, hence S is ruled.

Rmk. Let $p: S \rightarrow C$ be as in the Thm. Then $p^{-1}(x) \cong \mathbb{P}^1$ for the general fiber of p ($\forall x \in U$). We say that p is a fibration in rational curves⁽¹⁾.

A partial converse of the NE theorem is
Proposition. Every ruled irrational surface admits a fibration $p: S \rightarrow B$ irrational curves.

Pf. Suppose $S \xrightarrow{\text{birat}} B \times \mathbb{P}^1$ with $g(B) > 0$.

Resolve the indeterminacies of $p \circ \varphi: S \dashrightarrow B$ via blow-ups, getting a

morphism $f: \tilde{S} \rightarrow B$. $E \subset \tilde{S}$

The exceptional curve $E \subset \tilde{S}$ $\tilde{S} \dashrightarrow B \times \mathbb{P}^1$

introduced by the last blow-up, is in a fiber

of f , by Riemann-Hurwitz. Thus the last blow-up was not necessary, iterate this argument and get that $p = p \circ \varphi$ itself is a morphism. \square

Rmk. Clearly this argument does not work for rational surfaces, i.e. $S \xrightarrow{\text{birat}} \mathbb{P}^1 \times \mathbb{P}^1$.

However, the same conclusion is true (a posteriori) provided that $S \neq \mathbb{P}^2$

Theorem (Enriques) S non-ruled \Rightarrow

$\mathbb{B}(S)$ contains a unique minimal model.

The proof uses two lemmas

Key Lemma (the core of Enriques theory. See next lecture) Let S be a non-ruled minimal surface. Then K_S is nef.

The second is

Lemma 2 (let $f: S \rightarrow S'$ be a birat. morphism and let $C \subset S$ be an irreducible curve. $f(C) = C' \subset S'$ is a curve. Then

$$K_{S'} \cdot C' \leq K_S \cdot C$$

Pf. By the structure thm. can assume that $f \circ S \dashrightarrow S'$ is a blow-up. So:

$$\begin{array}{c} S \xrightarrow{\quad} E \xrightarrow{\quad} C' \\ \downarrow \quad \quad \quad \downarrow \\ S' \xrightarrow{\quad} E \xrightarrow{\quad} C \end{array}$$

We know

$$E^2 = C + rE \text{ where } r = \text{mult}_p(C) \geq 0$$

Then (properties of blow-ups)

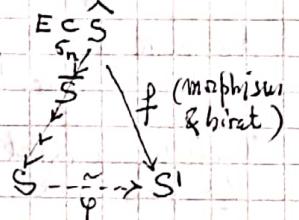
$$K_{S'} \cdot C' = (E^2 K_S + E) (E^2 C - rE) = K_S \cdot C + r \geq K_S \cdot C \quad \square$$

Pf of the E. theorem. By contrad., suppose S, S' are two distinct minimal models in $\mathbb{B}(S)$. Then \exists birat map $\varphi: S \dashrightarrow S'$. Set $n := \min \# \text{ of blow-ups}$

we need to resolve the indeterminacies of φ .

getting a birat. morphism

$$f: \tilde{S} \rightarrow S'$$



We show that $n=0$, so that φ is a birat. morphism. But S is minimal, so φ is an isomorphism, contrad.

Let $E \subset \tilde{S}$ be the exc. curve introduced by the last blow-up \tilde{S}_n . We have two possibilities:

$$f(E) = \begin{cases} p^1, & \text{pt. or} \\ \text{curve.} & \end{cases}$$

If $f(E) = p^1 \subset S'$, then the indeterminacies of φ are eliminated at the level of \tilde{S} ; actually we can define $\tilde{f}: \tilde{S} \rightarrow S'$ by

$$\tilde{f}(x) = \begin{cases} f(\tilde{S}_n^{-1}(x)) & \text{if } x \neq \tilde{S}_n(E) \\ f(E) = p^1, & \text{if } x = \tilde{S}_n(E) \end{cases}$$

But this contradicts the definition of n .

Therefore, $f(E) = E' \subset S'$ is a curve. Then lemma 2 $\Rightarrow K_{S'} \cdot E' \leq K_S \cdot E = -1$

Thus S' which is minimal, has $K_{S'}$ not nef, and then $\Rightarrow S'$ is ruled; contradiction \square

Numerical characters

$q, P_n (P_1 = P_2)$ are birational invariants

Those of S (ruled), are the same as those of $\Sigma := B \times \mathbb{P}^1$

(1) We can also assume that all fibers of p are connected in view of general results, e.g. the Stein factorization theorem, or the Zariski connectedness theorem.

$$H^1(\Sigma, \mathbb{Z}) = H^1(B, \mathbb{Z}) \oplus H^1(P^1, \mathbb{Z}) \Rightarrow q(\Sigma) = \frac{1}{2} b_1(\Sigma) = g(B) \quad \text{Theorem (Enriques ruledness crit.)}$$

$$H^2(\Sigma, \mathbb{Z}) = H^2(B, \mathbb{Z}) \oplus H^2(P^1, \mathbb{Z}) = \mathbb{Z}^2 \text{ generated by } \begin{matrix} \Sigma \\ \Sigma \end{matrix} \text{ the classes } B \text{ & } f = P^1 \text{ of the factors.}$$

Note: $B^2 = f^2 = 0$. Moreover $K_{\Sigma} = aB + bf$ for some $a, b \in \mathbb{Z}$ and genus formula applied to B and to f gives $K_{\Sigma} = -2B + (2g-2)f$

Useful remark. S surface, $T \in S$ med. class. $T^2 \geq 0$. Then for any effective divisor D , $D \cdot T \geq 0$.

$$\begin{aligned} D &= \sum n_j T_j + n \Gamma, \quad T_j \text{ indep. } \neq \Gamma, \quad n_j, n \geq 0; \text{ then} \\ D \cdot \Gamma &= \sum n_j T_j \cdot \Gamma + n \Gamma^2 \geq 0 \end{aligned}$$

Consequence. $P_n(\Sigma) = 0 \quad \forall n \geq 1$.

Pf. By contrad., let $D \in \ln K_S$. Then $0 \leq D \cdot f = n(-2B + (2g-2)f) \cdot f = -2n$, contrad.

Therefore we have

Theorem. S ruled $\Rightarrow P_n(S) = 0 \quad \forall n \geq 1$.

S rational $\Rightarrow P_n(S) = q(S) = 0$

(because $q(S) = q(B)$).

Are there also sufficient conditions?

(Noether's conjecture "if $p_g = q = 0 \Rightarrow S$ minimal" is false)

Theorem (Castelnuovo's rationality crit., 1894)

$$P_2(S) = q(S) = 0 \Rightarrow S \text{ rational}$$

Pf. It uses the Key Lemma again (" S min with K_S not nef \Rightarrow S ruled").

i). Can assume: S minimal (P_2, q birat. invar.)

ii). Enough to prove: S ruled ($q = 0 \Rightarrow S$ rational)

look at K_S^2

If $K_S^2 < 0 \Rightarrow K_S$ not nef \Rightarrow S ruled

So let $K_S^2 \geq 0$. Apply RR to $2K_S$

$$\begin{aligned} h^0(2K_S) + h^2(2K_S) &\geq X(0_S) + \underbrace{\frac{1}{2}(4K_S^2 - 2K_S^2)}_{\substack{\text{"Serre"} \\ \text{"P}_2 \\ \text{assumption}}} \\ h^0(-K_S) &\geq 1 + \underbrace{K_S^2}_{0} \end{aligned}$$

So

$$h^0(-K_S) \geq 1 + K_S^2 \geq 1. \quad \text{Two possibilities:}$$

1) $-K_S \sim 0$ i.e. $\mathcal{O}(-K_S) = \mathcal{O}_S$: then $K_S \sim 0$ and $2K_S \sim 0$. So $P_2 = h^0(2K_S) = 1$, contrad.

2) $\exists D \in \{-K_S\}$. Let H be an ample divisor; then $D \cdot H > 0$, but $D \cdot H = -K_S \cdot H$ hence $K_S \cdot H < 0$

$\Rightarrow K_S$ not nef \Rightarrow S ruled.

II

$P_{12} = 0 \Rightarrow S$ ruled

(in part. $P_{12} = 0 \Rightarrow P_n = 0 \quad \forall n \geq 1$)

Lemma of Pf.

$P_{12} = 0 \Rightarrow p_g = P_2 = P_3 = P_4 = P_6 = 0$:

If $\exists S \in H^0(mK_S)$ where $m \cdot m' = 12$, then $0 \leq S^{m'} \in H^0(12K_S)$, contrad.

Consequence S minimal and $q > 0$, otherwise S is rational, by Castelnuovo, hence ruled.

If K_S is not nef \Rightarrow S ruled

So let K_S be nef. Then $K_S^2 \geq 0$ (Clebsch)

Use Noether's formula

$$\begin{aligned} 12X(0_S) &= K_S^2 + e(S) \\ 12(1-q+p_2) &= 2-4q+b_2 \end{aligned}$$

hence

$$(X) \quad 10-8q = K_S^2 + b_2 \geq b_2 \geq p > 0$$

$\Rightarrow q \leq 1$ and therefore $q=1$

Fact: $p_g = 0 \wedge q = 1 \Rightarrow \exists$ fibration (Albanese)

$\alpha: S \rightarrow A$ (elliptic curve)

Then $b_2 \geq 2$ (in $H^2(S, \mathbb{Z})$ there are the class of a fiber of α and the class of a hyperplane section; they are lin. indep.)

therefore $b_2 = 2$ and so $K_S^2 = 0$ from (X)

The role of P_{12} arises from a detailed study of such surfaces (S min, $p_g = 0, q = 1, b_2 = 2$ and $K_S^2 = 0$). They are named bielliptic; their classification goes back to Bagnera & de Franchis

Minimal models

S ruled and $\neq \mathbb{P}^2 \Rightarrow S$ admits a

fibration in rational curves, so

$\exists \alpha: S \rightarrow B$ (sm. proj. cre) surjective morphism with connected fibers and s.t. $\alpha^{-1}(b) \subseteq \mathbb{P}^1$ for $b \in B$ general.

What about the other fibers?

Facts. Let $F_0 = \bigcup_{i=1}^m m_i C_i$ be a fiber of α , where $m_i \geq 1$ and C_i is an irreducible curve $i=1, \dots, t$ and denote by F the generic fiber of α . We suppose F_0 not general.

a) $\forall D \in \text{Div}(S)$ we have $D \cdot F_0 = D \cdot F$.

$F_0 = \alpha^{-1}(a)$ for some $a \in B$. Then $F_0 = \alpha^*(a)$ as divisor. For $m \gg 0$ ma is very ample on B , hence

$ma \sim x_1 + \dots + x_m$ where $x_i \neq a \wedge x_i$ general. Set $F_i = \alpha^{-1}(x_i)$; then $mF_0 \sim \sum F_i$ hence

$mF_0 \cdot D = \sum F_i \cdot D = mF \cdot D$ since all F_i are general.

b) It cannot be $F_0 = m_1 C_1$ with $m_1 \geq 2$.

Otherwise,

$$0 = C \cdot F = C \cdot F_0 = m_1 C_1 \Rightarrow C_1^2 = 0 \quad \text{genus formula}$$

$$\begin{cases} -2 = K_S \cdot F = K_S \cdot F_0 = m_1 K_S \cdot C_1 \Rightarrow C_1 \cdot K_S < 0 \\ \text{even} \end{cases}$$

Thus $F_0 = \sum_{i=1}^t m_i C_i$ with $t \geq 2$

c) $C_i^2 < 0 \quad \forall i=1, \dots, t$

from $0 = C \cdot F = C_1 \cdot F_0 = C_1 \left(\sum_{i=1}^t m_i C_i \right) = m_1^2 C_1^2 + \sum_{i>1} m_i C_1 C_i$
because $C_i \cdot C_1 > 0$ for some i due to the connectedness
of the fibers. Thus $C_1^2 < 0$, but this can be repeated
for all $i=1, \dots, t$.

d) For some i , C_i is a (-1)-curve

Actually $C_i^2 < 0 \quad \forall i$, and

$-2 = F \cdot K_S = F_0 \cdot K_S = \sum_i m_i C_i \cdot K_S \Rightarrow C_i \cdot K_S < 0$ for some i .
For this i , C_i is a (-1)-curve.

This is not true if $g(B) = 0$, because
 S_0 could contain some (-1)-curve which
is transverse to the fibers.

In particular we get:

Theorem (Severi): let $g > 0$. The minimal
models of ruled surfaces over B , of genus g ,
are exactly the \mathbb{P}^1 -bundles over B .

As to the rational case, the geometrically
ruled surfaces over \mathbb{P}^1 are the so-called
Segre-Hirzebruch surfaces F_n , $n \geq 0$,
where

$$F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$$

(up to a twist by a line bundle, every rank
2 vector bundle over \mathbb{P}^1 is of this form)

Projective models of F_n are the rational
scrolls Σ_n described as follows:

$$\begin{cases} \mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^{n+1} \\ \text{rat. normal} \\ j(t) \text{ curve of degree } n+1 \\ t \in \mathbb{P}^1 \end{cases} \quad j: \mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^{n+1}$$

$$\Sigma_n = \bigcup_{t \in \mathbb{P}^1} \langle t, j(t) \rangle$$

Note: $\Sigma_0 \subset \mathbb{P}^3$ is the quadric surface
 $F_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^1$

$\Sigma_1 \subset \mathbb{P}^4$ is the cubic scroll already described

$F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$
is induced on S' : F_0 in S is replaced in S' by
a fiber F'_0 for which we can repeat the argument. contraction gives \mathbb{P}^2 . So $\Sigma_1 = \mathbb{B}_P(\mathbb{P}^2)$

We can iterate the procedure until F_0 is
replaced by a fiber $F_0^{(t)} = m C$, $m \geq 1$, C irreducible
curve.

Moreover, $m=1$, i.e. $F_0 \stackrel{(6)}{\sim} \mathbb{P}^1$, by b)

Facts: i) $\text{Pic}(\Sigma_n) \cong \mathbb{Z}^2$, generated by the
classes of l and f (resp. section and fiber of
the fiber bundle $\pi: \Sigma_n \rightarrow \mathbb{P}^1$, $\pi(x) = t$ if $x \in \langle t, j(t) \rangle$)
ii) For $n \geq 1$ there is just one irreducible curve
 $C \subset \Sigma_n$ s.t. $C^2 < 0$; it is $C = l$, for which
 $l^2 = -n$.

iii) $K_{\Sigma_n} = -2l - (2+n)f$.

iv) A divisor $D = al + bf$ on Σ_n is
ample iff $a > 0$ & $b > an$.

Theorem (Vaccaro, Kodaira, Andreotti, Nagata):
The minimal models of rational
surfaces are \mathbb{P}^2 and the F_n , $n \neq 1$.

More on \mathbb{P}^1 -bundles

There is a short unified treatment of
geometrically ruled surfaces (\mathbb{P}^1 -bundles)
over a rational or irrational curve.

First let us define the general concept
of projective bundle.

Note: If $g(B) > 0$, S_0 cannot contain

(-1)-curves: they would be contained in
fibers, by Riemann-Hurwitz, but this is
impossible since for all fibers F_i of S_0 ,

$$F_i \cong \mathbb{P}^1 \quad \& \quad F_i^2 = 0$$

let X be a smooth projective variety and let E be a rank- r holomorphic vector bundle over X .
def. The projective bundle $\mathbb{P}(E) \xrightarrow{\pi} X$ is

def. as follows:

$\forall x \in X$ set $(\mathbb{P}(E))_x = \overline{\pi^{-1}(E_x)} =$ the \mathbb{P}^{r-1} parametrizing the r -subspaces of $\dim_{\mathbb{C}} r-1$ of $E_x \subseteq \mathbb{C}^r$.

So $(\mathbb{P}(E))_x \cong \mathbb{P}^{r-1} \forall x \in X$, hence $\mathbb{P}(E)$ is a \mathbb{P}^{r-1} -bundle over X .

Note: $\mathbb{P}(E)$ is a smooth proj.-variety of dimension $= \dim X + r - 1$.

We define $\bar{\pi}: \mathbb{P}(E) \rightarrow X$ as follows:

if $p \in (\mathbb{P}(E))_x$, then p corresponds to an $(r-1)$ -dimensional subspace of E_x ; we put $\bar{\pi}(p) = x$

$$\mathbb{P}(E) \xrightarrow{\bar{\pi}}$$

$$P \downarrow \pi$$

$$x \quad X$$

Consider the vector bundle on $\mathbb{P}(E)$ given by the pull back of E via π , $\pi^* E$.

$\pi^* E$ contains a rank $(r-1)$ subbundle \mathcal{W} , where

$$\mathcal{W}_p = p \subset (\pi^* E)_p \text{ as a vector subspace of } E_x$$

The tautological line bundle ξ of E on $\mathbb{P}(E)$ is defined by

$$\xi = \pi^* E / \mathcal{W}$$

$$(rk \xi = rk \pi^* E - rk \mathcal{W} = r - (r-1) = 1)$$

For every fiber of $\mathbb{P}(E)$, $\mathbb{P}(E_x) \cong \mathbb{P}^{r-1}$, one has

$$\xi \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$$

(the hyperplane bundle of the fiber)

Facts i) $\pi_* \xi = E$ (more generally, $\pi_* \xi^{\otimes t} \cong S^t E$,
t symmetric power of E)

ii) given two \mathbb{P} -vect. bundles E, E' on X , we have

$$\mathbb{P}(E) = \mathbb{P}(E') \iff E' = E \otimes L$$

for some line bundle L on X .

Now, suppose that $X = B$ is a curve, and let $r=2$.

Then the surface $S = \mathbb{P}(E)$ is a \mathbb{P}^1 -bundle over B .

Proposition (cf. Beauville's book, Prop III.18).

Let f be a fiber of $\pi: S \rightarrow B$. Then

$$i) \quad \text{Pic}(S) \cong \pi^* \text{Pic}(B) \oplus \mathbb{Z} \cdot \xi$$

$$ii) \quad H^2(S, \mathbb{Z}) \cong \mathbb{Z}^2 \text{ generated by the classes of } \xi \text{ and } f,$$

$$iii) \quad \xi^2 = \deg E \quad (= \deg \lambda^2 E)$$

$$iv) \quad K_S = -2\xi + \pi^*(K_B + \det E)$$