

April 30, 2020

We use the KRT to prove the Key Lemma.

### 4. Nef threshold and the Key Lemma

First recall the Kodaira vanishing theorem

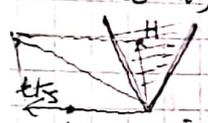
**Theorem (KVT)**  $X = \text{complex proj. mfd of dim } X = n$   
 $A \in \text{Pic}(X)$  ample. Then  $h^i(K_X + A) = 0$  for  $i = 1, 2, \dots, n$   
 (Equivalently, by Serre duality,  $h^i(-A) = 0$  for  $q = 0, 1, \dots, n-1$ .)

Now come back to surfaces:  $S = \text{surface}$ ,  $N(S) = \mathbb{R}^p$ .  
 Recall: the nef cone  $\overline{\text{Amp}}(S)$  is the closure of the ample cone  $\text{Amp}(S)$ , which is open and convex.

**def.** a polarized surface is a pair  $(S, H)$ , where  $S = \text{surface}$  and  $H \in \text{Pic}(S)$  (or  $\text{Div}(S)$ ) is ample.

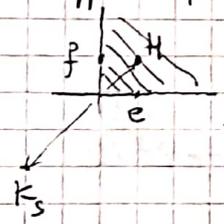
Assume that  $K_S$  is not nef.  
**def.** nef threshold of  $(S, H)$  is  $t_0 = t_0(S, H)_{\text{def}} = \text{Sup}\{t \in \mathbb{R} \mid H + tK_S \in \text{Amp}(S)\}$

Note: i)  $t_0 < +\infty$ .  
 Actually, for  $t \gg 0$   
 $H + tK_S \notin \text{Amp}(S)$



ii)  $t_0 > 0$ ;  
 clearly  $H + 0K_S = H \in \text{Amp}(S)$  (=int. of  $\overline{\text{Amp}}(S)$ ).

**Examples** 1)  $S = \mathbb{P}^1 \times \mathbb{P}^1$ ,  $e, f$  = classes of the factors  
 $H = e + f$  is ample (in fact v. ample) being the hyperplane section of  $S \subset \mathbb{P}^3$  as a quadric.  
 $K_S = -2H = -2e - 2f$  not nef (in fact, the opposite of an ample l. b.)



$\overline{\text{Amp}}(S) = 1^{\text{st}} \text{quadrant}$   
 $H + tK_S = (1-2t)H$   
 $\begin{cases} \notin \text{Amp}(S) & \text{if } t > \frac{1}{2} \\ \in \partial \text{Amp}(S) & \text{if } t = \frac{1}{2} \\ \in \text{Amp}(S) & \text{if } t < \frac{1}{2} \end{cases}$

Hence  $t_0 = \frac{1}{2}$

2) Similarly for  $S = \Sigma_1 \subset \mathbb{P}^4$ , the cubic scroll, with  $H = l + 2f$  (the hyperplane section), we know that  $K_S = -2l - 3f$  not nef, and then

$H + tK_S = (1-2t)l + (2-3t)f$ ,  
 and so, as in 1) we get  $t_0 = \frac{1}{2}$

A crucial result is the Kawagata rationality theorem.

**Theorem (KRT)** For every  $(S, H)$  we have that  $t_0 \in \mathbb{Q}$ .

**Theorem (Key Lemma)**.  $S = \text{minimal surface}$  with  $K_S$  not nef. Then  $S$  is ruled.

**Pf**  $\exists H \in \text{Div}(S)$  ample: consider  $t_0(S, H)$ .  
 By the KRT  $t_0 = \frac{u}{v}$  with  $u, v$  integers, both  $> 0$   
 Set  $L := vH + uK_S$   
 $L$  is nef but not ample, because  $L = v(H + t_0 K_S) \in \partial \text{Amp}(S)$

$\forall m > 0$  consider  
 $mL - K_S = m(vH + (\frac{u}{v} - \frac{1}{m})K_S)$   
 $= mv(H + (\frac{u}{v} - \frac{1}{mv})K_S)$

therefore, by def of  $t_0$ ,  $\underbrace{(\frac{u}{v} - \frac{1}{mv})}_{< t_0} \in \text{Amp}(S)$ .

So  $mL - K_S$  is ample. Then  $h^i(mL) = h^i(K_S + \underbrace{(mL - K_S)}_{\text{ample}}) = 0$  for  $i = 1, 2$ .  
 KVT

hence  
 $h^0(mL) = \chi(mL) = \chi(O_S) + \frac{1}{2}(m^2 L^2 - mL \cdot K_S)$  (\*)

Now,  $L$  nef  $\xrightarrow{\text{Kleinman}} L^2 \geq 0$ .

Suppose  $L^2 > 0$ .  
 $L$  not ample  $\xrightarrow{\text{NM}} \exists C \subset S$  s.t.  $CL = 0$   
 ined. cve

$L^2 > 0$  of HIT (strong form)  $\Rightarrow C^2 < 0 \Rightarrow C$  is a (-1)-curve on  $S$ , contradict. because  $S$  is minimal.  
 $0 = L \cdot C = vH \cdot C + uK_S \cdot C \Rightarrow K_S \cdot C < 0$

Therefore,  $L^2 = 0$  (unpleasant in view of (\*))

look at  $H \cdot L \geq 0 \Rightarrow$   
 ample nef  $\begin{cases} \textcircled{1} HL > 0, \text{ or} \\ \textcircled{2} HL = 0 \end{cases}$

Case  $\textcircled{1}$ .  $0 = L^2 = vHL + uK_S L \Rightarrow K_S L < 0$   
 So (\*)  $\Rightarrow$

$h^0(mL)$  grows linearly with  $m$ ,  
 hence, for  $m \gg 0$ ,  $mL$  is effective & nontrivial.  
 Write

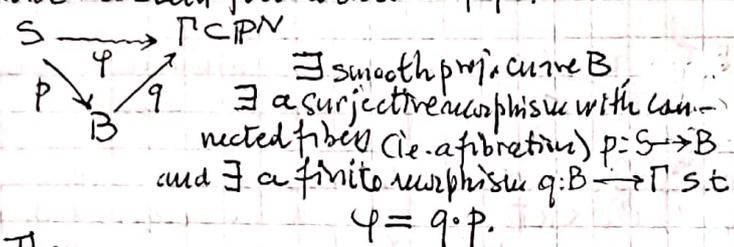
$|mL| = Z + |M|$   
 fixed part moving part

$0 = L^2 = mL \cdot L = ZL + M \cdot L \Rightarrow ZL = M \cdot L = 0$   
 $\forall \text{ nef } \forall$

$0 = M \cdot L = M \cdot mL = MZ + M^2 \Rightarrow MZ = M^2 = 0$   
 $\forall \text{ moves}$

$\Rightarrow B_S |M| = \emptyset$ , i.e.  $[M]$  is spanned, hence it defines a morphism  $\varphi: S \rightarrow \mathbb{P}^N$ , whose image is a curve, say  $\Gamma$ , possibly singular.

Take the Stein factorization of  $\varphi$ :



Then  $M = \varphi^*(\text{hyp. sect. of } \Gamma) = p^*(q^*(\text{hyp. sect. of } \Gamma))$   
 $= p^*(\text{effect. div. on } B) = \sum F_i$ ,  $F_i = \text{fiber of } p$ .

Let  $F$  be the general fiber of  $p$ ; then  $F^2 = 0$  and  $F \cdot F_i = 0 \Rightarrow M \cdot F = (\sum F_i) \cdot F = 0$   
 On the other hand  $0 = Z \cdot M \Rightarrow Z$  is contained in a union of fibers hence  $Z \cdot F = 0$ . So  $L \cdot F = \frac{1}{m} (uL) \cdot F = \frac{1}{m} (ZF + MF) = 0$   
 but  $(vH + uK_S) \cdot F = v \underbrace{H \cdot F}_0 + u \underbrace{K_S \cdot F}_0 \Rightarrow K_S \cdot F < 0$   $\left\{ \begin{array}{l} \text{genus formula} \\ \Rightarrow F \cong \mathbb{P}^1 \end{array} \right.$

Thus NE theorem  $\Rightarrow S$  is ruled.

Case (2)  $H \cdot L = 0 \Rightarrow \begin{cases} L^2 < 0, \text{ contrad. } L \text{ nef.} \\ L \equiv 0, \text{ i.e. } -K_S \equiv \frac{v}{u} H \text{ ample.} \end{cases}$

Suppose  $\rho = \text{rk}(NS(S)) \geq 2$ . Then  $A(S)$  open  $\Rightarrow \exists H' \in \text{Div}(S)$  ample s.t.  $H', H, K_S$  are not collinear in  $N(S)$

Consider  $(S, H')$ , set  $t'_0 = t_0(S, H') = \frac{u'}{KRTV'}$   
 $u', v'$  positive integers, and  $L' = v'H' + u'K_S$ ;  
 repeat the proof until we arrive at  $\text{def}$  the dichotomy (1) or (2).  
 If  $H' \cdot L' > 0 \Rightarrow S$  ruled (as in case (1)).  
 If  $H' \cdot L' = 0 \Rightarrow -K_S \equiv \frac{v'}{u'} H'$ , contradiction.

So, for  $\rho \geq 2$  the theorem is proven. Case  $\rho = 1$  is settled by the following

**Lemma.**  $S$  surface with  $\rho = 1$  &  $-K_S$  ample. Then  $S = \mathbb{P}^2$  (hence ruled)

**Pf.**  $h^i(O_S) = h^i(K_S + (-K_S)) = 0$  for  $i=1,2$   
 ample KVT

Exponential sequence  $\Rightarrow$   

$$\begin{array}{ccccccc} H^1(O_S) & \rightarrow & \text{Pic}(S) & \xrightarrow{\cong} & H^2(S, \mathbb{Z}) & \rightarrow & H^2(O_S) \\ \parallel 0 & & \cong & \cong & \parallel 0 & & \parallel 0 \\ & & NS(S) & & & & \end{array}$$

So  $1 = \rho = b_2(S)$ . Thus  $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ .

Look at  $H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \xrightarrow{\text{cup product}} \mathbb{Z}$

By the Poincaré duality,  $v$  is a unimodular symmetric bilinear form, hence  $\exists \ell \in H^2(S, \mathbb{Z})/\text{Tors}$  s.t.  $\ell \cdot \ell = 1$   
 Think of  $\ell$  as  $\ell \in \text{Pic}(S)$ , then  $\ell^2 = 1$ .  
 Up to exchanging  $\ell$  with  $-\ell$ , we can suppose that  $\ell$  is the ample generator of  $\text{Pic}(S)/\text{Tors}$ .  
 Then  $-K_S \equiv r\ell$  for some integer  $r > 0$ .

Using Noether's formula, we have  $12 = 12\chi(O_S) = K_S^2 + e(S) = r^2\ell^2 + 3 = r^2 + 3$ ,  
 $1 - g + b_2$   $2 - 4g + b_2$   
 hence  $r = 3$ . Now look at  $|\ell|$ .

$h^i(\ell) = h^i(K_S + \frac{\ell - K_S}{4\ell, \text{ ample}}) \stackrel{\text{KVT}}{=} 0$  for  $i=1,2$ ; hence  
 $h^0(\ell) = \chi(\ell) = \chi(O_S) + \frac{1}{2}(\ell^2 - \ell K_S) = 1 + \frac{1}{2} \cdot 4 = 3$   
 i.e.  $\dim |\ell| = 2$  ( $|\ell|$  is a net)

Note: every element of  $|\ell|$  is irreducible. If  $A+B \in |\ell|$  with  $A, B$  effective, then  $1 = \ell^2 = \ell \cdot (A+B) = \ell A + \ell B \geq 2$ ,  $\text{contrad.}$   
 $\Rightarrow |\ell|$  has no fixed components.

Next:  $|\ell|$  is spanned. Suppose, by  $\text{contrad.}$ , that  $B \subset |\ell| \ni p - \forall x \in S, x \neq p$  consider the pencil  $\mathcal{F} \subset |\ell|$  of elements through  $x$ . Let  $\ell', \ell'' \in \mathcal{F}$ ; then  $1 = \ell^2 = \ell' \cdot \ell'' \geq 2$ , because  $\ell' \cdot \ell'' = \{x, p\}$ ,  $\text{contrad.}$   
 Therefore, the map  $\varphi = \varphi_{|\ell|}: S \dashrightarrow \mathbb{P}^2$  is a morphism.

Finally,  $\varphi$  is birational: what is  $\varphi^{-1}(p)$  for a point  $p \in \mathbb{P}^2$ ?  
 Let  $h', h''$  be two lines through  $p$  in  $\mathbb{P}^2$ , then  $\varphi^* h' \cdot \varphi^* h'' = \ell' \cdot \ell'' = 1$  pt. because  $\ell' \cdot \ell'' = 1$  with  $\ell', \ell'' \in |\ell|$

But  $\rho = 1$ , hence  $S$  is minimal. In conclusion  $S \xrightarrow{\varphi} \mathbb{P}^2$  is a birational morphism  $\Rightarrow \varphi$  iso.  $\square$

del Pezzo surfaces

S surf.

def. S is del Pezzo if  $-K_S$  is ample  
(in higher dimension: Fano manifold; in  $\dim_c=1$ : just  $\mathbb{P}^1$ )

Examples:  $S = \mathbb{P}^2, K_S = \mathcal{O}_{\mathbb{P}^2}(-3)$   
 $S = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$  as a quadric surface;  $K_S = -2H$   
 $S = S_3 \subset \mathbb{P}^3$  cubic surface;  $K_S = -H$   
 $S = \Sigma_1 \subset \mathbb{P}^4$  cubic scroll;  $K_S \sim -2e - 3f = -H$

General facts

1) S del Pezzo  $\Rightarrow$  S rational  
 $P_n(S) = h^0(nK_S) = h^2(K_S + (-n)K_S) = 0 \quad \forall n \geq 1$ .  
same ample KVT  
 $q(S) = h^1(\mathcal{O}_S) = h^1(K_S + (-K_S)) \stackrel{KVT}{=} 0$

Thus  $P_2 = q = 0 \Rightarrow$  S rational  
Castelnuovo

2) Let  $f: \tilde{S} \rightarrow S$  be a birat. morphism. If  $\tilde{S}$  is del Pezzo, then S is del Pezzo.  
 It is enough to prove it for  $f = \sigma: \tilde{S} \rightarrow S$  the blow-up of S at p.

$K_{\tilde{S}} = \sigma^* K_S + E, E = \sigma^{-1}(p)$ . Thus

$K_{\tilde{S}}^2 = K_S^2 - 1 \Rightarrow (-K_{\tilde{S}})^2 = (-K_S)^2 + 1 > 0$

$\forall C \subset S \quad K_S \cdot \sigma^* C = (\sigma^* K_S + E) \cdot \sigma^* C = K_S \cdot C$   
invariant so  $-K_{\tilde{S}} \cdot C = -K_S \cdot \sigma^* C > 0$ .  
ample effective

Therefore  $\xrightarrow{NM} -K_{\tilde{S}}$  ample.

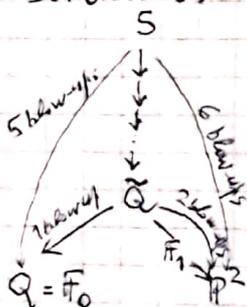
1) & 2)  $\Rightarrow$  the minimal models of del Pezzo surfaces are del Pezzo surfaces among  $\mathbb{P}^2$  and the  $F_n, n \neq 1$ .

3) S del Pezzo; if  $C \subset S$  is an irred curve with  $C^2 < 0$ , then C is a (-1)-curve.

$-2 \leq 2g(C) - 2 = C^2 + CK_S \Rightarrow C^2 = CK_S = -1$   
 $\hat{=}$   $\hat{=}$  ( $-K_S$  ample)

$F_n$  contains a curve  $\ell$  with  $\ell^2 = -n$ . Thus the minimal models of del Pezzo surfaces are  $\mathbb{P}^2$  and  $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ .

Situation:



Let  $S \neq \mathbb{P}^2$ ; then either  $S = F_0$  or  $\exists \eta: S \rightarrow \mathbb{P}^2$  a birational morphism which factors through  $K_{\mathbb{P}^2}^2 - K_S^2 = g - K_S^2$  blow-ups

def. degree of S:  $d = K_S^2 \quad (d \geq 1)$

Note:  $\sigma_i$  a blow-up of factors through. The center  $p$  of  $\sigma_i$  cannot be on a (-1)-curve introduced by previous blow-ups. Otherwise

$\begin{matrix} \downarrow \sigma_i \\ \mathbb{P}^2 \\ \downarrow \sigma_{i-1} \end{matrix} E \quad E^2 = -1 \Rightarrow (\sigma_i^{-1}(E))^2 = -2, \text{contradicting 3).$

Therefore  $\eta: S \rightarrow \mathbb{P}^2$  factors through the blow-up of  $g$ -d distinct points ( $d \leq g$ ) (i.e. can regard them as points  $p_1, \dots, p_{g-d} \in \mathbb{P}^2$ )

Theorem (classification thm of del Pezzo's)  
 S del Pezzo of degree d.

i)  $d = 9 \Rightarrow S = \mathbb{P}^2$

ii)  $d = 8 \Rightarrow S = \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathbb{F}_1 \end{cases}$

iii)  $d \leq 7 \Rightarrow S$  is  $\mathbb{P}^2$  blown up at  $g$ -d distinct points, in general position (i.e., no 3 on a line, no 6 on a conic, and no 8 on a cubic having a double point at one of them)

If of the last assertion. Suppose by contradiction:



let  $C := \eta^{-1}(\ell)$  ( $\cap \gamma, \cap \Gamma$ ). Then  $C \simeq \mathbb{P}^1$  and  $C^2 = -2$ , contradicting 3).

E.g., for  $\mathbb{P}^2$ ,

$g = \mathbb{P}^2 = (\eta^* \Gamma)^2 = (C + 2E_1 + E_2 + \dots + E_g)^2$   
 $E_i \cdot C = C^2 + 4CE_1 + 4E_1^2 + \frac{3}{2} 2E_i \cdot C + \frac{3}{2} E_i^2$   
 $= C^2 + 8 - 4 + 14 - 7 = C^2 + 11 \Rightarrow C^2 = -2 \quad \square$

Ranks S as in iii)

1)  $| -K_S | = | \eta^* \mathcal{O}_{\mathbb{P}^2}(3) - E_1 - \dots - E_{g-d} |$ ;

the anticanonical system of S corresponds to the linear system of plane cubics passing through  $p_1, \dots, p_{g-d}$ .

Recalling that

$h^i(-K_S) = h^i(K_S + (-2)K_S) = 0$  for  $i=1,2$ ,  
ample KVT  
 we have

$h^0(-K_S) = \chi(-K_S) \stackrel{RR}{=} \chi(\mathcal{O}_S) + \frac{1}{2}(K_S^2 + K_S^2) = 1 + d$ .  
 $\frac{1}{2} \frac{1}{2} \frac{1}{2}$

Thus  $| -K_S |$  defines a rat. map.  $\varphi: S \dashrightarrow \mathbb{P}^d$

2) As we will see in lect. 6,  $\varphi$  is an embedding if  $d \geq 3$ , so that  $\varphi(S) \subset \mathbb{P}^d$  is a smooth surface of degree  $d$ . Classically, all Pizzos surfaces were studied as smooth surfaces of degree  $d$  in  $\mathbb{P}^d$ .

Here we look in particular at smooth cubic surf. in  $\mathbb{P}^3$ . By what we said any such  $S$  can be seen as  $\mathbb{P}^2$  blown-up at 6 points  $p_1, \dots, p_6$ , no 3 on a line and not all on a conic, via  $\eta: S \rightarrow \mathbb{P}^2$ .

The following result goes back to Cayley and Salmon, 1848.

Theorem. On  $S$  there are exactly 27 lines, and each of these is a  $(-1)$ -curve.

Pf. Let  $l \subset S \subset \mathbb{P}^3$  be a line. So  $l \cong \mathbb{P}^1$  and  $l \cdot H = 1$  where  $H$  is a hyperplane section. Recall that  $H \sim -K_S$ .  
 $-2 = 2g(l) - 2 = l^2 + l \cdot K_S = l^2 - 1 \Rightarrow l^2 = -1$ , so  $l$  is a  $(-1)$ -curve.

The 6 curves  $E_1, \dots, E_6$ , where  $E_i = \eta^{-1}(p_i)$ , are lines, because

$$E_i \cdot H = E_i \cdot (-K_S) = 1.$$

Let  $l \subset S$  be a line,  $l \neq E_i, i=1, \dots, 6$ .

Then  $\eta(l) = C \subset \mathbb{P}^2$  is an irreducible curve, and

$$\eta^*C = l + \sum_{i=1}^6 r_i E_i \text{ with } r_i = E_i \cdot l = \begin{cases} 0, & \text{or} \\ 1 \end{cases}$$

mult $_{p_i}(C)$      2 distinct lines

$\Rightarrow r_i \leq 1 \forall i$ , hence

$C$  is a smooth rational plane curve  $\Rightarrow$

$C$  is a line or a conic i.e.  $C \in |\mathcal{O}_{\mathbb{P}^2}(b)|, b=1 \text{ or } 2$ .

We have

$$\begin{aligned} 1 &= l \cdot H = (\eta^*C - \sum_{p_i \in C} E_i) \cdot (-K_S) \\ &= (\eta^*(\mathcal{O}_{\mathbb{P}^2}(b)) - \sum_{p_i \in C} E_i) \cdot (\eta^*(\mathcal{O}_{\mathbb{P}^2}(3)) - \sum_{i=1}^6 E_i) \\ &= 3b - \#\{p_i \mid p_i \in C\}. \end{aligned}$$

Cases:  $b=1$ ; so  $1=3-\# \Rightarrow \#=2$ ,

hence  $C =$  line through  $p_i$  and  $p_j, i \neq j$ .

The number of these lines is  $\binom{6}{2} = 15$ .

$b=2$ ; so  $1=3 \cdot 2 - \# \Rightarrow \#=5$

hence  $C =$  conic through  $\hat{p}_1, \dots, \hat{p}_5, p_6$   
 ( $\hat{\phantom{x}}$  means suppression).

The number of such conics is 6.

Total:  $\begin{array}{r} 6 \text{ } E_i\text{'s} \\ 15 \text{ lines } \langle p_i, p_j \rangle \\ \hline 6 \text{ conics, each missing a } p_i \\ \hline 27 \end{array}$