

5. The Mori cone theorem in the setting of surfaces

$S = \text{surface}$; $\text{NE}(S) = \text{closed convex cone in } N(S)$

generated by the class of the irred. curves
def. Its closure in $N(S)$, $\overline{\text{NE}(S)}$ is the
Kleiman-Mori cone.

Theorem (Kleiman ampleness crit.) Let $D \in \text{Div}(S)$
and look at it as a functional $N(S) \rightarrow \mathbb{R}$

$$D \text{ is ample} \iff D \cdot x > 0 \quad \forall x \in \overline{\text{NE}(S)} \setminus \{0\}$$

(for ampleness it is not enough $D \cdot C > 0$ for
 $C \subset S$ a curve!)

Digression on the n -dimensional context

\exists a natural intersection pairing between
curves and hypersurfaces of a proj. manifold
 X , with $\dim X = n$.

$$(Y \cap C) \stackrel{\text{R 1-cycles mod } \equiv}{=} N_1(X) \times N^1(X) \xrightarrow{\text{extension/IR}} \mathbb{R}$$

ext. of the natural pairing deriving from int. between curves C and hypersurf. Y ($C \not\sim Y$)

It is a duality $\Rightarrow \dim N_1(X) = \dim N^1(X) = p$
(for $n=2$ $N_1 = N^1$)

$$\begin{array}{ccc} N_1(X) & N^1(X) & \left| \begin{array}{l} \text{The Kleiman} \\ \text{ampleness criterion} \\ \text{holds in dim } n \end{array} \right. \\ \cup & \cup & \\ \overline{\text{NE}(X)} & \overline{C(X)} & \end{array}$$

More:

def A ray in $N(X)$ is $R = R_+[Z] = \{\lambda Z, \lambda \geq 0\}$
where Z is a 1-cycle (i.e. a lin. comb. of curves)
Here $R_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

def R is said to be an extremal ray if $K_X \cdot Z < 0$
and $\forall Z_1, Z_2 \in \overline{\text{NE}(X)}$ s.t. $Z_1 + Z_2 \in R$, we have
that $Z_1, Z_2 \in R$ themselves.

Theorem (Mori cone thm) $X = \text{smooth project. variety.}$ \exists a countable set of curves C_i , with $C_i \cdot K_X < 0$ such that

$$\overline{\text{NE}(X)} = \text{NE}(X) + \sum_{i \in I} R_+[C_i], \text{ i.e.,}$$

$\overline{\text{NE}(X)}$ is the smallest closed convex cone containing $\overline{\text{NE}_+(X)} = \{Z \in \text{NE}(X) \mid Z \cdot K_X \geq 0\}$ (the non-negative part of the cone w.r.t. K_X) and the rays generated by the C_i 's.

Moreover,
i) the rays $R_+[C_i]$ are the extremal rays of X ,

- ii) the set of curves C_i 's as above is minimal (i.e. no smaller subset is sufficient);
- iii) for any R_+ -invariant neighborhood U of $\text{NE}_+(X)$ only finitely many rays $R_+[C_i]$ do not belong to U .

def. The C_i 's in the Thm are called extremal rational curves ERC.

They are irred. rational curves, possibly singular, and satisfy $1 \leq -K_X \cdot C_i \leq n+1$.

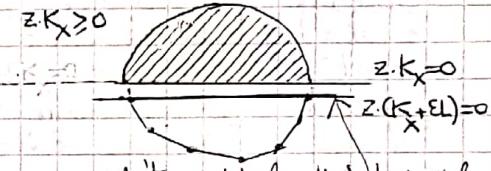
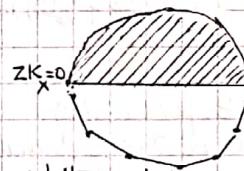
Note: Condition iii) in the Thm can be replaced by:
 $\forall \varepsilon > 0$ and $\forall L \in \text{Pic}(X)$ ample, any cycle

Z s.t. $(K_X + \varepsilon L) \cdot Z \leq 0$ can be written as

$$Z = \sum_{i \in I} \lambda_i C_i + \gamma, \quad \lambda_i \geq 0 \quad \forall i,$$

where γ (depending on ε) is an element of $\text{NE}(X)$ s.t.

$$(K_X + \varepsilon L) \cdot \gamma = 0$$



ERC in case $n=2$ are easily described.

Theorem (Mori) $S = \text{surface}$, $C \subset S$ irred. curve.

$R_+[C]$ is an extremal ray. Then either

- 1) C is a (-1) -curve, or
- 2) S is a \mathbb{P}^1 -bundle $p: X \rightarrow B$, and $C \cong \mathbb{P}^1$ is a fiber, or
- 3) $S = \mathbb{P}^2$ and C is a line.

Note. There are surfaces containing infinitely many (-1) -curves.

Example. $S = \mathbb{P}^2$ blown-up at 9 points which are the base points of a pencil of irreducible cubics (such a pencil exists!).

The proper transforms on S of the cubics are the fibers of an elliptic fibration $\varphi: S \rightarrow \mathbb{P}^1$ and each of the 9 exceptional curves E_i corresponding to the base pts of the pencil \mathcal{C}

is a section of φ . Choose e.g., E_1 as the zero section. The group structure of the general fiber of φ gives rise to a group of automorphisms of S generated by the other 8 sections, hence isomorphic to \mathbb{Z}^8 .

Any such automorphism maps an E_j to another (-1) -curve... Hence there are infinitely many such curves.

For more details see Friedman's book, p. 132.

Polarized surfaces with low sectional genus
Consider the following classes of polarized surfaces:

$$\mathcal{A} = \{(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(c)) \mid c=1,2\};$$

$$\mathcal{B} = \{(S, H) \mid S = \mathbb{P}^1\text{-bdle over } B \text{ and } H_f = \mathcal{O}_{\mathbb{P}^1}(1) \text{ for every fiber } f\}$$

pairs in \mathcal{B} are called scrolls;

$$\mathcal{C} = \{(S, H) \mid S \text{ is ruled over a surface } B \text{ and } H_f = \mathcal{O}_{\mathbb{P}^1}(2) \text{ for the general fiber } f\}$$

pairs in \mathcal{C} are called conic fibrations;

$$\mathcal{D} = \{(S, H) \mid S \text{ is del Pezzo, and } H = -K_S\}$$

pairs in \mathcal{D} are called del Pezzo pairs.

def. the sectional genus of a polar. surface (S, H) is
 $g := p_a(H)$

Rank 1. If $(S, H) \in \mathcal{B}$, then $g = q$ (recall that $q = q(\mathcal{B})$). To see this first we need to recall some properties of \mathbb{P}^1 -bundles $p: S \rightarrow B$ over a smooth proj.-curve B . (cf. Hartshorne's book, pp 369-373).

We have $S = P(E)$ where E is a rank-2 vector bundle over B , defined by S up to the twist by a line bundle. So we can suppose that E is unrelaxed in the sense that $H^0(E) \neq 0$ but $H^0(E \otimes L) = 0$ for any line bundle L with $\deg L < 0$. Any $\eta \in H^0(E)$ defines an injection $0 \rightarrow \mathcal{O}_B \rightarrow E$ and then a surjection $E \rightarrow L \rightarrow 0$ where $L := E/\mathcal{O}_B$, which is an invertible sheaf (i.e. a line bundle) on B . To this surjection it corresponds a section $\beta: B \rightarrow S$, whose image C_0 (tautological, or fundamental section) has the property that C_0^2 is the minimum of the self-interactions of all sections of S . In fact,

$$C_0^2 = -e, \text{ where } e = -\deg E = (\deg \wedge^2 E)$$

Note that C_0 (as any section) and f , a fiber, generate $\text{Num}(S)$, and $C_0 \cdot f = 1$, $f^2 = 0$.

So $\forall D \in \text{Dir}(S)$ we have $D = aC_0 + b\beta$ for some $a, b \in \mathbb{Z}$. Moreover, since $K_S = -2C_0 + p^*(K_B + \det E)$ we have $K_S = -2C_0 + (2q-2-e)\beta$. In particular,

$$K_S^2 = -4e - 4(2q-2-e) = 8(1-q).$$

Now, since (S, H) is a scroll, we get $H \equiv C_0 + bf$ for some $b \in \mathbb{Z}$. Then

$$2g-2 = H \cdot (K_S + H) = (C_0 + bf)(-C_0 + (b+2q-2-e)\beta) \\ = -e - b + b + 2q - 2 - e = 2q - 2 \quad \square$$

Rank 2. Let $(S, H) \in \mathcal{C}$, and let $p: S \rightarrow B$ be the ruling fibration of S . Any singular fiber of p is of the form $\tilde{F}_0 = e_1 + e_2$ where e_1, e_2 are (-1) -curves and $e_1 \cdot e_2 = 1$ (immediate in view of what we proved about fibrations in rational curves).

By contracting one of the two (-1) curves on every singular fiber we get a birational morphism $\eta: S \rightarrow S_0$ where S_0 is a \mathbb{P}^1 -bundle over B . So S is S_0 blown-up at points p_1, \dots, p_s on distinct fibers of S_0 , s being the number of singular fibers of S . In particular,

$$K_S^2 = K_{S_0}^2 - s = 8(1-q) - s \leq 8(1-q),$$

equality holding iff $s=0$, i.e. iff S itself is a \mathbb{P}^1 -bundle.

Note also that $K_S + H$ restricts trivially to every fiber of S . Moreover $(K_S + H)^2 = 0$.

Actually, referring to η , there exists a divisor $D \in \text{Dir}(S_0)$ s.t. $H = \eta^*D - \sum E_i$ where $E_i = \eta^*(p_i)$ (because $H \cdot E_i = 1$). On the other hand

$$K_S = \eta^*K_{S_0} + \sum E_i, \text{ hence } (K_S + H)^2 = (K_{S_0} + D)^2.$$

Let $f = \eta(F)$ be a fiber of S_0 ; from $H \cdot F = 2$ we get $D \cdot f = 2$ and then, due to

the expression of K_{S_0} we see that

$$(K_S + H)^2 = \sum \text{fibers of } S_0, \text{ hence}$$

Finally note that $(K_S + H)^2 = 0$ obviously holds

for $(S, H) \in \mathcal{D}$.

Lemma 1 (S, H) a polarized surface. If

$K_S + H$ is not nef, then $(S, H) \in \mathcal{A} \cup \mathcal{B}$.

Pf. \exists a curve $C \subset S$ s.t. $(K_S + H) \cdot C < 0$

By Mori's cone thm., we can choose $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$ s.t. in $N(S)$

$$C = \sum_{i=1}^m \lambda_i l_i + \gamma, \text{ where } \lambda_i \geq 0, l_i \text{ are ERC and } (K_S + \varepsilon H) \cdot \gamma = 0. \text{ Thus}$$

$$0 > \sum_{i=1}^m \lambda_i (K_S + H) \cdot l_i + (K_S + H) \cdot \gamma = \sum_{i=1}^m \lambda_i (K_S + H) + (1-\varepsilon) \gamma \cdot H \\ \geq \sum_{i=1}^m \lambda_i (K_S + H) l_i. \quad \square$$

Hence \exists an ERC, say l_1 , s.t. $(K_S + H) \cdot l_1 < 0$

By the classification of ERC we see that

$$3) \Rightarrow (S, H) \in \mathcal{A}$$

$$2) \Rightarrow (S, H) \in \mathcal{B} \quad (l_1 = f \text{ & } K_f^2 = -2 \Rightarrow H \cdot f = 1)$$

In case 1), $K_S \cdot l_1 = -1$ and $H \cdot l_1 \geq 1$ due to the assumption. Thus $(K_S + H) \cdot l_1 \geq 0$, contradiction. \square

Lemma 2 If $h^0(n(K_S + H)) \leq 1 \quad \forall n \geq 1$, then

$(S, H) \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.

Pf. Since H is ample, a fortiori $h^0(nK_S) = 0$ for $n \gg 0$. Hence

$$h^0(nK_S) = 0 \quad \forall n \geq 1 \Rightarrow S$$
 ruled

enriques

So either $S = \mathbb{P}^2$ or \exists birat. morphism $S \rightarrow S_0$ where $S_0 = \mathbb{P}^1$ -bundle. Thus $p_g = 0$ and

$$(*) \quad K_S^2 \leq 8(1-q).$$

so

$$a) \quad g = q, \text{ or} \\ 1 \geq h^0(K_S + H) = Pg + g - q = g - q \Rightarrow$$

$$b) \quad g = q + 1 \& \\ h^0(n(K_S + H)) = 1 \quad \forall n \geq 1$$

Case a) if $S = \mathbb{P}^2$ then $0 = q = g \Rightarrow (S, H) \in \mathcal{A}$.

So, let $S \neq \mathbb{P}^2$ and consider $(S, 2H)$.

Clearly $(S, 2H) \notin \mathcal{B}$ (since $2H$ is even).

$\xrightarrow{\text{Lemma 1}} K_S + 2H$ is nef $\xrightarrow{\text{Kleiman}} (K_S + 2H)^2 \geq 0$.

Thus

$$0 \leq K_S^2 + 4K_S H + 4H^2 = K_S^2 + 8(g-1) \stackrel{a)}{=} K_S^2 + 8(q-1)$$

$\Rightarrow K_S^2 \geq 8(1-q)$, which, combined with (x)

gives $K_S^2 = 8(1-q)$, hence $S = S_0$, a \mathbb{P}^1 -bundle.

Using the same notation as in Rank 1, we have

$K_S = -2G + (2q-2-e)f$; set $H = aG + bf$ with $b > ae$ if $e \geq 0$ ($b > ae$ if $e < 0$)

[complement conditions similar to those formulated for F_n . Here it can be $e < 0$ because q can also be > 0 , however, $e \geq -q$ (Nagata inequality)].

From

$$\begin{aligned} 2q-2 &= 2g-2 = (K_S + H)H = (-G + b + (2q-2-e)f) \cdot H \\ &\stackrel{a)}{=} \dots = \underbrace{(a-1)(2b-ae)}_{\geq 0} + \underbrace{a(2q-2)}_{\geq 0} \end{aligned}$$

We get $a = 1$, i.e. $(S, H) \in \mathcal{B}$, if $q \geq 2$.

For $q = 0$ or 1, a direct check shows that $a = 1$ as well.

Case b) here $g = q+1$. If $S = \mathbb{P}^2$, then $(S, H) = (\mathbb{P}^2, \cup_{i=1}^n (3)) \in \mathcal{D}$. Let $S \neq \mathbb{P}^2$.

As $g \neq q$, $(S, H) \notin \mathcal{B}$. (Rank 1)

Thus Lemma 1 $\Rightarrow (K_S + H)^2 \geq 0$. So + Kleiman

$$\begin{aligned} 0 \leq (K_S + H)^2 &= K_S^2 + 2(H \cdot K_S + H^2) - H^2 = K_S^2 + 4(g-1) - H^2 \\ &\leq 8(1-q) + 4q - H^2 < 8-4q, \end{aligned}$$

hence $q \geq 0$ or 1.

If $q = 0$ then $g = 1$ hence $H \cdot (K_S + H) = 0$; thus ample nef & $h^0(K_S + H) = 1$

$K_S + H \sim 0$, i.e. $(S, H) \in \mathcal{D}$.

If $q = 1$, then $\chi(O_S) = 0 \neq g = 2$. We have

$$h^0(n(K_S + H)) = h^0((1-n)K_S - nH) = 0$$

because some

$$\begin{aligned} ((1-n)K_S - nH) \cdot H &= (1-n)K_S \cdot H - nH^2 = \\ &= (1-n)(2g-2-H^2) - nH^2 = 2-2n-H^2 < 0 \end{aligned}$$

If $n \geq 1$. Thus RR \Rightarrow

$$\begin{aligned} h^0(n(K_S + H)) &\geq \chi(O_S) + \frac{1}{2}(nK_S + nH) \cdot \underbrace{(n-1)K_S + nH}_{(n-1)(K_S + H) + H} \\ &\geq n(n-1)(K_S + H)^2 + n, \quad \text{contrad.} \quad \square \end{aligned}$$

Theorem let $(S, H) \notin \mathcal{B}$. Then $(K_S + H)^2 \geq 0$

with = iff $(S, H) \in \mathcal{C} \cup \mathcal{D}$.

Pf. Note that $(K_S + H)^2 > 0$ if $(S, H) \in \mathcal{A}$.

Then lemma 1 & Kleiman imply $(K_S + H)^2 \geq 0$. Suppose = holds.

If $h^0(n(K_S + H)) \leq 1 \forall n \geq 1$, then lemma 2 $\Rightarrow (S, H) \in \mathcal{A}$.

If $h^0(n(K_S + H)) \geq 2$ for some $n \gg 0$, look at the rational map $\varphi: S \dashrightarrow \mathbb{P}^N$ defined by $|n(K_S + H)|$.

Arguing as in the proof of the key lemma we see that φ is a morphism and $T := \varphi(S)$ has $\dim = 1$. Taking the Stein factorization gives a surjective morphism $\pi: S \rightarrow B$ to a sm. curve B .

Let F be the general fiber of π .

$$F \cdot (K_S + H) = 0 \Rightarrow F \cdot K_S < 0$$

and then $F^2 = 0$ implies $g(F) = 0$; so $F \cdot K_S = -2$, hence $F \cdot H = 2$, i.e. $(S, H) \in \mathcal{C}$.

The converse was observed in Rank 2. \square

Corollary. (S, H) polar surface of sect genus g .

0) if $g = 0$, then S is rational & $(S, H) \in \mathcal{A} \cup \mathcal{B}$.

1) if $g = 1$, then either $(S, H) \in \mathcal{B}$ if $q = 1$ (elliptic scroll), or $(S, H) \in \mathcal{D}$.

Pf. 0) $(K_S + H) \cdot H = -2 < 0 \Rightarrow K_S + H$ not nef $\xrightarrow{\text{Lemma 1}} (S, H) \in \mathcal{A} \cup \mathcal{B}$. In particular, $p_g = 0$,

hence $0 \leq h^0(K_S + H) = g - q \Rightarrow q = 0$, hence S is rational.

1) $(K_S + H) \cdot H = 0 \Rightarrow K_S + H < 0 \Rightarrow S$ ruled; in particular, $p_g = 0$. Then

$$0 \leq h^0(K_S + H) = g - q \Rightarrow q = 0 \text{ or } 1.$$

If $q = 0$, then $h^0(K_S + H) = 1$ and $(K_S + H) \cdot H = 0$ implies $K_S + H \sim 0$ i.e. $(S, H) \in \mathcal{D}$. \square

If $q = 1$ ($= g$), suppose, by contrad. that $(S, H) \notin \mathcal{B}$. Then $(K_S + H)^2 \geq 0$ by the Thm.

Thus

$$0 \leq (K_S + H)^2 = K_S^2 + 4(g-1) - H^2 = K_S^2 + H^2. \text{ So}$$

$$0 < H^2 \leq K_S^2 \leq 8(1-q) = 0, \text{ contrad.} \quad \square$$

The picture is completed by the following

Theorem (L + Palleschi, Gruelle, 1984), Thm 2.5) (S, H) polar surface. Then either

i) $K_S + H$ is ample, a

ii) $(S, H) \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, or

iii) S is not minimal, $(K_S + H)^2 > 0$ and $(K_S + H) \cdot C \geq 0$ for every sm. curve $C \subset S$ and equality holds iff C is a (-1)-curve with

$CH = 1$ (a (-1)-line of (S, H)). All (-1)-lines

are disjoint each other. There's a birat.

morphism $\eta: S \rightarrow S_0$ contracting all (-1)-lines, and there exists an ample line bundle $H_0 \in \text{Pic}(S_0)$ such that

$$\eta^*(K_S + H_0) = K_S + H, \text{ and}$$

$K_{S_0} + H_0$ is ample.

Def. (S_0, H_0) is called the reduction of (S, H) and η , the reduction morphism.

Note: point iii) was first established by Sommese (1979) when H is very ample.

For H very ample, Sommese also proved that

$$g = q \iff (S, H) \in \mathcal{A}_0 \cup \mathcal{B},$$

For H simply ample, this is not true, even if $1+H$ contains a smooth curve.

For example, let C be a smooth curve of genus 2, let $S = J(C)$ be its jacobian and $H = [C]$, the line bundle corresponding to C embedded in it. Then H is ample and $g = q$.