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6. Reider's theorem & some applications

Theorem (Reider, 1987)  $S$  surface,  $L \in \text{Pic}(S)$  nef.

I) let  $L^2 \geq 5$ . If  $x$  is a base point of  $|K_S + L|$  then  $\exists$  effective divisor  $D$  on  $S$ , containing  $x$ , s.t. either

- a)  $D \cdot L = 0$  &  $D^2 = -1$ , or
- b)  $D \cdot L = 1$  &  $D^2 = 0$ .

II) let  $L^2 \geq 9$ . If  $x, y$  (possibly infinitely near) are such that  $|K_S + L|$  fails to separate them, then  $\exists$  effective divisor  $D$  on  $S$ , containing  $x$  and  $y$  s.t. either

- a)  $D \cdot L = 0$  &  $D^2 = -2$  or  $-1$ ,
- b)  $D \cdot L = 1$  &  $D^2 = -1$ , or  $0$ ,
- c)  $D \cdot L = 2$  &  $D^2 = 0$ , or
- d)  $L \equiv 3D$  &  $D^2 = 1$

The proof requires two basic results due to Griffiths & Harris (GH) and to Bogomolov (B), respectively.

Theorem (GH.)  $S$  surface,  $L \in \text{Pic}(S)$ ,  $Z \subset S$  a 0-dimensional subscheme.

There exist: a rank-2 vector bundle  $E$  on  $S$  with  $\det E (= \wedge^2 E) = L$ , and a section  $s \in \Gamma(E)$  with  $Z = \{s=0\}$  iff

(1) every section of  $\Gamma(K_S + L)$  vanishing at all but one pts of  $Z$ , also vanishes at the remaining point.

(Cayley-Bocherach property of  $K_S + L$  w.r. to  $Z$ )

Note: for  $Z = \{x\}$  condition (1) means that  $x \in Bs |K_S + L|$

Theorem (B) If  $\exists$  a rank-2 vector bundle

$E$  on  $S$  with  $c_1(E)^2 - 4c_2(E) > 0$  (i.e.  $E$  is Bogomolov unstable), then  $\exists$  line bundles  $A, B \in \text{Pic}(S)$  and a 0-dimensional finite subscheme  $Z$  of  $S$  giving rise to an exact sequence

$$(2) \quad 0 \rightarrow A \rightarrow E \rightarrow B \otimes \mathcal{I}_Z \rightarrow 0$$

( $\mathcal{I}_Z =$  ideal sheaf of  $Z$ ). Moreover,

- i)  $(A-B)^2 > 0$
- ii)  $(A-B) \cdot H > 0$  for any ample  $H \in \text{Pic}(S)$ .

Comment on (2). Here we need few propert. of Chern classes (for the theory ref. to Hartshorne book pp 429-431).

If  $L$  and  $M$  are line bundles such that

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0,$$

then

$$c_1(E) = L + M \quad \text{and} \quad c_2(E) = L \cdot M.$$

If  $M = \mathcal{O}_S \otimes Z$  where  $Z$  is a line bundle, then  $c_2(E) = L \cdot Z + \deg Z$ . ( $c_1(E) = L + M$  even (if  $Z = x_1 + x_2 + \dots + x_r$ , then  $\deg Z = r$ ) in this case)

Pf of Reider's thm, confined to I)  
By the GH theorem (see the note),  $\exists$  a rank-2 vect. bundle  $E$  on  $S$  and a section  $s \in \Gamma(E)$  s.t.  $\{s=0\} = \{x\}$  with  $\det E = L$ .

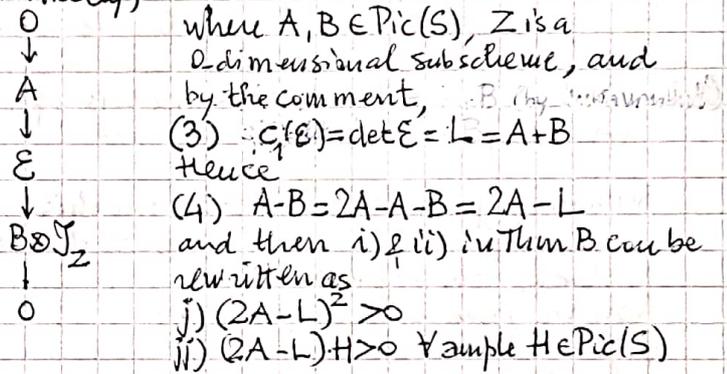
The multiplication by  $s$  defines an exact sequence

$$0 \rightarrow \mathcal{O}_S \xrightarrow{-s} E \xrightarrow{\wedge^2} L \otimes \mathcal{I}_x \rightarrow 0$$

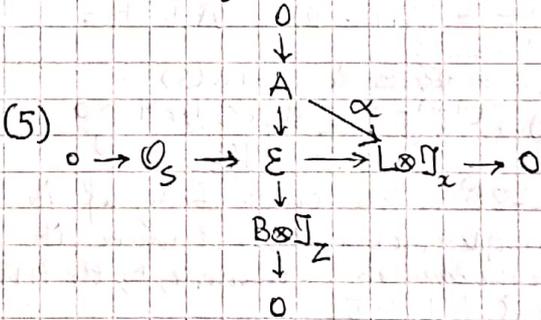
Note that  $c_2(E) = 1$  and  $c_1(E) = \det E = L$  by the comment - Thus, due to the assumption  $L^2 \geq 5$  we have

$$c_1(E)^2 - 4c_2(E) = L^2 - 4 > 0$$

and then we can apply the B theorem. Thus there is an exact sequence (we write it vertically)



Look at the diagram



and consider  $\alpha$ .

Claim.  $\alpha$  is nontrivial, and  $A = L - D$ , with  $D$  an effective divisor containing  $x$  and s.t.  $B = \mathcal{O}_S(D)$ .

Pf of the claim. Assume  $\alpha$  is trivial, i.e.  $A \in \text{Ker}[E \rightarrow L \otimes \mathcal{I}_x] = \mathcal{O}_S$  (by the exactness of the horiz. sequence in (5)).

Then  $A = \mathcal{O}_S(-C)$  where  $C$  is an effective divisor, possibly trivial.

Note that by (3),  $L = A + B = B - C$ . So

$$0 < (A-B) \cdot H = (2A-L) \cdot H = (-2C-L) \cdot H, \quad (4)$$

a contradiction, because  $C \cdot H \geq 0$ , ( $C$  being effective and  $L \cdot H \geq 0$  since  $L$  is nef).

Thus  $\alpha$  is nontrivial, hence it vanishes along a divisor on  $S$ ; let us look for it.

Set  $B = \mathcal{O}_S(D)$  for some divisor  $D$ . Then  $L = A + D$ , by (3). We show that  $D$  is effective and contains  $x$ . To do that tensor (5) by  $A^{-1}$ . We get

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{O}_S & & & & \\
 & & \downarrow & & & & \\
 0 & \rightarrow & A^{-1} & \rightarrow & E \otimes A^{-1} & \rightarrow & A^{-1} \otimes L \otimes \mathcal{I}_x \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^{-1} \otimes B \otimes \mathcal{I}_x & & \mathcal{O}_S(D) \otimes \mathcal{I}_x & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

Note that  $H^0(E \otimes A^{-1}) \neq 0$ , from the cohomology of the vertical sequence. If we knew that  $H^0(A^{-1}) = 0$ , then the cohomology of the horizontal sequence would imply that there is an injection

$$0 \rightarrow H^0(E \otimes A^{-1}) \rightarrow H^0(\mathcal{O}_S(D) \otimes \mathcal{I}_x)$$

proving the claim.

So, let us prove that  $H^0(A^{-1}) = 0$ . We know that for any ample  $H \in \text{Pic}(S)$

$$\begin{aligned}
 (A+D) \cdot H &= L \cdot H > 0 \text{ because } L \text{ is nef,} \\
 (A-D) \cdot H &= (A-B) \cdot H > 0.
 \end{aligned}$$

Summing up we get  $A \cdot H > 0$ , and this implies  $h^0(A^{-1}) = 0$ , as we need. So the claim is proved.

Now we prove some inequalities:

(6)  $(L-2D) \cdot L > 0$ .

Actually,

$$(7) \quad L-2D = A+D-2D = A-D = A+A-A-D = 2A-L$$

$\begin{matrix} \uparrow & & \uparrow \\ -L & \text{As} & -L \\ \text{shown in the claim} & & \end{matrix}$

Therefore, for any ample  $H \in \text{Pic}(S)$

$$(L-2D) \cdot H = (2A-L) \cdot H > 0$$

Hence  $(L-2D) \cdot L > 0$  since  $L$ , being nef, is limit of ample line bundles. However, it cannot be an equality; otherwise, by the HIT (recalling that  $L^2 \geq 5$ ),

$$0 \geq (L-2D)^2 = (2A-L)^2 > 0, \text{ contradiction.}$$

This proves (6). Next,

(8)  $(L \cdot D)^2 \geq L^2 \cdot D^2$ ,

this simply follows from the HIT; moreover,

(9)  $(L-D) \cdot D \leq 1$ .

Actually, by the horizontal and the vertical sequences in (5) we know that

$$\begin{aligned}
 1 = c_2(E) &= A \cdot B + \deg Z = (L-D) \cdot D + \deg Z \\
 &\quad \uparrow \qquad \qquad \qquad \uparrow \\
 &\text{comment on (2)} \qquad \text{claim}
 \end{aligned}$$

$$\geq (L-D) \cdot D$$

This proves (9). Finally,

(10)  $2D^2 < L \cdot D$ .

We prove (10) by cases.

If  $D^2 > 0$ , then  $L \cdot D \neq 0$  by the HIT, and  $2L \cdot D < L^2$  by (6). So,

$$2L \cdot D \cdot D^2 < L^2 D^2 \leq (L \cdot D)^2$$

Note that  $L \cdot D > 0$ ; this follows from the nefness of  $L$  and the HIT, combined with the fact that  $D^2 > 0$ . Thus, cancelling  $L \cdot D$  we get (10).

If  $D^2 = 0$ , note that it cannot be  $L \cdot D = 0$  by the HIT again; actually  $D \neq 0$ , because  $D$  is effective and contains  $x$ . Hence (10) holds.

If  $D^2 < 0$ , then (10) is obvious, since  $L$  is nef &  $D$  is effective.

Now, combining (9) and (10), we get

$$L \cdot D - 1 \leq D^2 < \frac{1}{2} L \cdot D$$

This implies  $L \cdot D < 2$ , hence, either  $L \cdot D = 0$  &  $-1 \leq D^2 < 0$ , i.e.  $D^2 = -1$ , or  $L \cdot D = 1$  &  $0 \leq D^2 < \frac{1}{2}$ , i.e.  $D^2 = 0$ . □

An application

As a first application we discuss the pluricanonical maps of del Pezzo surfaces. Further applications will be given in the next lecture.

Let  $S$  be a del Pezzo surface of degree  $d (= K_S^2)$

Consider  $m(-K_S)$ ,  $m \geq 1$ .

Since  $-K_S$  is ample, so is  $m(-K_S)$ , but, is it spanned? Is it very ample?

Write  $m(-K_S) = K_S + M$  where  $M = (m-1)(-K_S)$  and use Reid's theorem. Note that

$$M^2 = \underbrace{(m-1)^2}_d \cdot d \geq 5 \text{ if } \begin{cases} \text{either } d \geq 2, \text{ or} \\ d = 1 \text{ \& } m \geq 2. \end{cases}$$

Use Reid I). If  $m(-K_S)$  is not spanned  $\Rightarrow \exists D > 0$  s.t.

$$1 \geq D \cdot M = (m-1) \cdot D \cdot (-K_S) \geq m-1 \geq 2, \text{ contradiction.}$$

So,  $m(-K_S)$  is spanned for  $d \geq 2$  and for  $d = 1$  &  $m \geq 2$ .

Clearly it is not spanned for  $d = 1 = m$ ; actually,  $h^0(-K_S) = d+1 = 2$ ; so  $| -K_S |$  is a pencil and  $d = 1 \Rightarrow$  it has a single base point.

Next, we have

$$M^2 = \underbrace{(m-1)^2}_d \cdot d > 9 \text{ if either } \begin{cases} d \geq 3, \text{ or} \\ d = 2 \text{ \& } m \geq 2, \text{ or} \\ d = 1 \text{ \& } m \geq 3. \end{cases}$$

So, in these cases, we can use Reid II)

If  $m(-K_S)$  is not very ample  $\Rightarrow \exists D > 0$  s.t.

$$2 \geq D \cdot M = (m+1) D(-K_S) \geq m+1 \geq \begin{cases} 2 & \text{if } m=1 \\ 3, \text{ contrad., if } m \geq 2 \end{cases}$$

So, let  $m=1$  and consider  $2 = D \cdot M$ .  
Then  $1 = D \cdot (-K_S)$  and  $D^2 = 0$ , by Reider II, case c).  
But this contradicts genus formula, because  $D^2 + DK_S$  must be even.

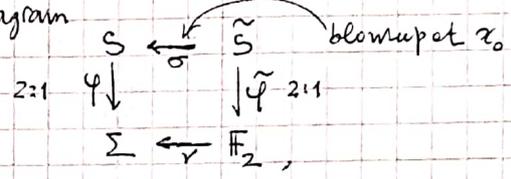
Therefore  $m(-K_S)$  is very ample if either  $\begin{cases} d \geq 3, m \geq 2 \\ d=2, m \geq 2 \\ d=1, m \geq 3 \end{cases}$

So, we have Theorem  $S = \text{del Pezzo surface of degree } d$ .

- a) If  $d \geq 3$ , then  $m(-K_S)$  is very ample  $\forall m \geq 1$ . In particular, for  $m=1$ ,  $\varphi_{-K_S} : S \hookrightarrow \mathbb{P}^d$  is an embedding (as already announced). So  $S \cong \text{sm. surface of degree } d \text{ in } \mathbb{P}^d$ .
- b) If  $d=2$ , then  $m(-K_S)$  is very ample  $\forall m \geq 2$ . For  $m=1$ , it is spanned, hence  $\varphi_{-K_S} : S \rightarrow \mathbb{P}^2$  is a morphism of degree 2; it is a double cover branched along a smooth quadric curve.
- c) If  $d=1$ , then  $m(-K_S)$  is very ample  $\forall m \geq 3$ . For  $m=1$ , it is not even spanned,  $\varphi_{-K_S} : S \dashrightarrow \mathbb{P}^1$  is a rational map, not defined at the unique base point of  $| -K_S |$ . In general this pencil contains 12 singular curves. Let  $m=2$ ;  $2(-K_S)$  is spanned and the morphism  $\varphi_{2(-K_S)} : S \rightarrow \mathbb{P}^3$  is a double cover of a quadric curve  $Q_0$ , branched at the vertex and along the transverse intersection with a cubic surface.

Thus  $\varphi : S \rightarrow \Sigma$  is not birational.  
From  $4 = (-2K_S)^2 = \text{deg } \varphi \cdot \text{deg } \Sigma$  we thus see that  $\Sigma \subset \mathbb{P}^3$  is a quadric surface, possibly singular.

In fact  $\Sigma$  is a quadric cone because  $\varphi(E)$  is a line in  $\Sigma$  and all these lines pass through  $\varphi(x_0)$  where  $x_0$  is the base point of  $| -K_S |$ . So the vertex of  $\Sigma$  is  $v := \varphi(x_0)$ .  
Desingularize, recalling that  $\Sigma$  is the image of  $\mathbb{F}_2$  via the map  $\gamma$  defined by  $|C_0 + 2f|$ , (usual notation)  $v = \gamma(C_0)$ , we get the diagram



where  $\tilde{\varphi} : \tilde{S} \rightarrow \mathbb{F}_2$  is a double cover of smooth surfaces. Call  $\Delta$  its branch locus; then  $\Delta \in |2B|$  for some  $B \in \text{Pic}(\mathbb{F}_2)$ . So, we can write  $B = [aC_0 + bf]$ .

Note that  $\varphi$  is branched at  $v$ , because  $\varphi^{-1}(v) = \{x_0\}$ . So  $C_0$  is a component of  $\Delta$  hence  $\Delta = C_0 + R$ , with  $R \in |2(a-1)C_0 + 2bf|$ . Furthermore  $\Delta$  is smooth since  $\tilde{S}$  is so, hence  $C_0 \cdot R = 0$ .  $\gamma$  maps the fibers  $f$  of  $\mathbb{F}_2$  to the generators of  $\Sigma$ , hence if  $\tilde{E} = \sigma^{-1}(E)$  is the proper transform of  $E$  via  $\sigma$ , we have that  $\tilde{\varphi}$  maps  $\tilde{E}$  2 to 1 to  $f$ , with branch divisor  $\Delta \cdot f$ . Since  $g(\tilde{E}) = g(E) = 1$  we get  $4 = \Delta \cdot f = 2(aC_0 + bf) \cdot f = 2a$ .

Here we discuss the last case (the birational map of a del Pezzo surface of degree 1) in detail.

So, let  $d=1$  and set  $\varphi = \varphi_{2(-K_S)}$ . Any element  $E \in | -K_S |$  is an irreducible curve. Consider the general one, which is a smooth curve, and the exact sequence

$$0 \rightarrow -K_S \rightarrow -2K_S \rightarrow \underbrace{(-2K_S)}_E \rightarrow 0$$

Note:  $h^i(-K_S) = h^i(K_S + (-2K_S)) = h^i(O_{\mathbb{P}^3}(x_1+x_2)) = 0$  for  $i=1,2$ . So ample KVT

$h^0(-2K_S) = h^0(-K_S) + h^0(O_{\mathbb{P}^3}(x_1+x_2)) = 2+2=4$ . Since  $2(-K_S)$  is spanned  $\varphi : S \rightarrow \mathbb{P}^3$  is a morphism. Set  $\Sigma := \varphi(S)$ .

Recall that  $\text{deg } \Sigma \geq 2$  since  $\Sigma$  is not in a hyperplane. Moreover, for any smooth  $E \in | -K_S |$ ,  $\varphi|_E$  is not an embedding, since  $g(E) = 1$ .

So  $a=2$ . Therefore, recalling that  $C_0^2 = -2$ ,  $0 = C_0 \cdot R = C_0(3C_0 + 2bf) = -6 + 2b$ , i.e.  $b=3$ . In conclusion,  $R \in |3C_0 + 6f|$ . Since  $[C_0 + 2f] = \gamma^*(O_{\Sigma}(1))$  as we reminded at the beginning, this shows that  $R \in \gamma^*(O_{\Sigma}(3))$  i.e. it is the pull back on  $\mathbb{F}_2$  of the intersection of  $\Sigma$  with a cubic surface, in accordance with the last assertion in point c) of the theorem.