

May 11, 2020

7. The adjunction mapping in the very ample setting.

S surface, $L \in \text{Pic}(S)$ very ample

Proposition. $K_S + L$ spanned, unless $(S, L) \in \mathcal{C} \cup \mathcal{B}$.

Pf. If $L^2 \geq 5$ this follows from Reider's thm I.

let $L^2 \leq 4$ and use the classification of surfaces of degree $d \leq 4$ (for varieties: Neil, Swinnerton-Dyer).

For surfaces, by hands. Let $S \subset \mathbb{P}^N$. then

$$d \geq \frac{\text{codim}}{N-2} + 1 = N-1 \quad \text{So } N \leq d+1 \leq 5.$$

Proceed by cases.

$N=5 \Rightarrow d=4$ & $S = \text{surf of minimal degree} \Rightarrow (S, L)$ rat. scroll or Veronese surface.

If $S = \mathbb{F}_e$, set $L = [C_0 + bF]$ (note: scroll $\Rightarrow C_0 \cdot F = 1$) with the usual notation. Then $4 = (C_0 + bF)^2 = -e + 2b \Rightarrow b = 2 + \frac{e}{2} > e$ by the ampleness. So $e < 4$ & even. Thus $e = 0$ or 2 and $b = 2$ or 3 correspondingly. In conclusion,

$$(S, L) = \begin{cases} (\mathbb{F}_0, [C_0 + 2F]), \text{ or} \\ (\mathbb{F}_2, [C_0 + 3F]), \text{ or} \\ (\mathbb{P}^2, [O(12)]) \end{cases} \text{ hence } (S, L) \in \mathcal{C} \cup \mathcal{B}$$

$N=4$: here $d=4$ or 3; if $d=4$ then $(S, L) = (\text{del Pezzo}, -K_S)$ is the $V_{2,2} \subset \mathbb{P}^4$ and in this case $K_S + L = 0$ is spanned (since trivial), or $S \subset \mathbb{P}^4$ is the 10th Hirzebruch projection of the Veronese surface: in this case $(S, L) \in \mathcal{A}$.

If $d=3$, S = surface of min degree \Rightarrow it is the cubic scroll of \mathbb{P}^4 , $(\mathbb{F}_1, [C_0 + 2F])$, hence $(S, L) \in \mathcal{B}$.

$N=3$: S is a hypersurface of degree $d=2, 3$ or 4.

$d=4 \Rightarrow (S, L) = \text{quartic K3}: K_S \sim 0$, so $K_S + L = L$ is very ample (in part. spanned).

$d=3 \Rightarrow (S, L) = \text{cubic surf}; \text{line } K_S + L = 0$, spanned

$d=2 \Rightarrow (S, L) = \text{quadratic surface, hence } \in \mathcal{B}$. \square

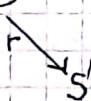
So, let $K_S + L$ be spanned, then $\phi := \phi_{K_S + L}$ is a morphism (the adjunction mapping).

If $\dim \phi(S) \leq 1$ then $(K_S + L)^2 = 0 \Rightarrow (S, L) \in \mathcal{C} \cup \mathcal{D}$ (as shown for polarized surfaces).

So we can assume that $(S, L) \notin \mathcal{C} \cup \mathcal{B} \cup \mathcal{G} \cup \mathcal{D}$ and then $\phi: S \rightarrow \mathbb{P}^N$ is a morphism with 2-dimen- sional image. Set $\Sigma := \phi(S)$.

Fact. (Sommerre (79)) Consider the Stein factorization of ϕ :

$$S \xrightarrow{\phi} \Sigma \subset \mathbb{P}^N \quad \phi = s \circ r, \text{ where}$$



Σ is a sm surface, $r: \Sigma \rightarrow S'$ is the contraction of all

(-1)-lines of (S, L) , and $s: S' \rightarrow \Sigma$ is a finite morphism.

Note: All (-1)-lines of (S, L) are disjoint each other.

Pf: let ℓ, ℓ' be two (-1)-lines and suppose, by contrad., that $\ell \cdot \ell' > 0$. Then since $r(\ell + \ell') = x \in S'$, for an ample line bundle $H \in \text{Pic}(S')$ it must be $\phi^* H \cdot (\ell + \ell') = 0$. Since $\phi^* H$ is nef and big, the $H \cdot \ell + H \cdot \ell' \geq 0 \Rightarrow (\ell + \ell')^2 \leq 0$. But $(\ell + \ell')^2 = -1 + 2 \ell \cdot \ell' - 1 \geq 0$, contradiction. \square

Thus $r: S \rightarrow \Sigma$ is the blowing-up at a finite set $B = \{x_1, \dots, x_k\}$. Then $\exists L' \in \text{Pic}(\Sigma)$ s.t. $L = r^* L' - r^{-1}(B)$ and L' is ample by NM. Moreover, $K_S + L'$ is ample, and spanned, since

$$K_S + L = r^*(K_S + L')$$

implies that $S = \phi_{K_S + L'}$ (hence $\phi_{K_S + L'}$ is a morphism).

(Σ, L') is called the reduction of (S, L) and r the reduction morphism.

Note: In general, L' is not very ample.

Theorem (Sommerre & VandeVen) Consider

(S, L) with L very ample. Suppose that $\dim \phi(S) = 2$ and let (Σ, L') be the reduction.

Then $K_S + L'$ is very ample (i.e. S is an embedding) except for the following pairs:

- a) $S = \text{del Pezzo surf of degre } K_S^2 = 2, L = -2K_S$ ($L^2 = 8$)
- b) (S, L) has the pair in a) as simple reduction (i.e. contracts a single (-1)-line) ($L^2 = 7$)
- c) $S = \text{del Pezzo surf of degre } K_S^2 = 1$ and $L = -3K_S$ ($L^2 = 9$)
- d) $S = \mathbb{P}^1$ -bundle over an elliptic curve, with $e = -1$ and $L = 3C_0$ ($L^2 = 9$)

(In case d), the normalized rank-2 vector bundle E such that $S = \mathbb{P}(E)$ is def. by the non-split exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow E \rightarrow \mathcal{O}_{B \cap Y}(Y) \rightarrow 0 \text{ for some } Y \subset B,$$

where B is the base elliptic curve.)

Remark. By what we have seen on del Pezzo surfaces, in case a) $\phi: S \rightarrow \mathbb{P}^2$ is a double cover branched along a smooth quartic (the same holds for S in case b);

in case c) $\phi: S \rightarrow \mathbb{P}^3$ is a double cover of a quartic cone branched at the vertex and along the transverse intersections with a cubic surface.

In case d), $\phi: S \rightarrow \mathbb{P}^2$ is a triple cover.

$$(K_S + L) \equiv C_0 f, h^0(K_S + L) = 3)$$

Sketch of pf. Two distinct parts according to whether $L'^2 \geq 9$ or $L'^2 \leq 8$.

A) $L'^2 \geq 9 \Rightarrow (S, L)$ is as in c) or d).

First we prove

1) $L'^2 \geq 9 \Rightarrow (\Sigma, L')$ is as in c) or d).

L' ample; if $K_{\Sigma} + L'$ is not very ample

\Rightarrow Reider II) $\exists D' > 0$ on S' s.t.

- i) $L'D=1$ & $D'^2=0$ or -1 , or
- ii) $L'D=2$ & $D'^2=0$, or
- iii) $L'=3D'$ & $D'^2=1$.

i) and ii) rule out, looking at $D=\tilde{r}^*(D')$ to get info on D' which contrast the ampleness of $K_{S'}+L'$.

In case iii) one shows that $p_a(D')=1$, hence D' is an ample divisor of genus 1 and then what we proved for polarized surfaces of sectional genus 1 implies that (S', L') is as in c), or $(S', [D'])$ is an elliptic scroll, which leads to d).

Next, one proves:

2) If (S', L') is as in c) or d), then $(S, L)=(S', L')$

B) let $L^2 \leq 8$. Then $d=L^2 \leq L^2 \leq 8$.

We need several steps.

3) If (S', L') is as in a), then r consists of a single contraction at most (so, if (S, L) is not as in a), then it can only be as in b)).

Moreover, b) is effective.

We skip the proof of the former claim and we prove the latter.

We need to show that $L = r^*L' - E$ is in fact very ample (here E is the (-1)-curve).

Writing $L = K_S + L'$, we prove it by using Reider's theorem II, again. Actually,

$$(*) L = L - \overset{r^*K_S+E}{\underset{r^*(-2K_S)-E}{\tilde{L}}} = r^*(-K_{S'}) + 2(r^*(-K_{S'}) - E)$$

$-K_{S'}$ is ample and spanned $\Rightarrow r^*(-K_{S'}) - E$ is nef.

So \tilde{L} , as a sum of 3 nef l.b. is nef.

Moreover $\tilde{L}^2 = (r^*(-3K_{S'}) - 2E)^2 = 18 - 4 > 9$.

So we can use Reider's II. Note that

\tilde{L} is ample: we have $\tilde{L} \cdot E = -2E^2 = 2$, and for any irreduc. curve $\Gamma \subset S$, not contracted by r , letting $\Gamma' = r(\Gamma)$ we have

$$\begin{aligned} \tilde{L} \cdot \Gamma &\geq r^*(-K_{S'}) \cdot \Gamma = r^*(-K_{S'}) (r^*\Gamma' - vE) = \\ &= -K_{S'} \cdot \Gamma' > 0, \end{aligned}$$

Since $-K_{S'}$ is ample. So, NM $\Rightarrow \tilde{L}$ ample.

Therefore, if $L = K_S + L'$ is not very ample, then $\exists D > 0$ on S , s.t.

$$L \cdot D = 1 \text{ f } D^2 = -1 \text{ or } 0, \text{ or}$$

$$L \cdot D = 2 \text{ f } D^2 = 0$$

In the former case, from (*) we get

$$\begin{aligned} 1 = L \cdot D &= r^*(-K_{S'}) \cdot D + 2(r^*(-K_{S'}) - E) \cdot D = \\ &= r^*(-K_{S'}) \cdot D + 2(-K_S) \cdot D. \end{aligned}$$

This implies $K_S \cdot D = 0$. As $K_S^2 > 0 \Rightarrow D^2 < 0$, hence $D^2 = -1$, but this contradicts the parity of $D^2 + DK_S$.

In the latter case, by (*) again we get

$$2 = \dots = r^*(-K_{S'}) \cdot D + 2(-K_S) \cdot D,$$

hence,

either $-K_S \cdot D = 1$, but $D^2 = 0$ gives a contradiction, or $-K_S \cdot D = 0$ and $\Rightarrow D^2 < 0$, contradiction.

Thus L is very ample.

4) Set $g=g(L)$. We can suppose $g \geq 2$ in view of the classification of polarized surfaces with $g \leq 1$ and the fact that here $\dim \phi(S)=2$.

Lemma 1: If $g \leq 3$, then either $K_S + L'$ is very ample, or (S', L') is as in a).

Pf. Can assume $d \geq 5$ in view of the known classification of surf. with $d \leq 4$. Thus

$$L \cdot K_S = 2g-2-d \leq 4-5 < 0$$

Enriques ruled. $\dim \phi(S)=2 \Rightarrow h^0(K_S + L) \geq 3$,

$$\text{hence RR+KVT} \Rightarrow$$

$$3 \leq h^0(K_S + L) = g-9 \leq 3-9.$$

Thus $g=0$, $g=3$ and $\phi(S) = \mathbb{P}^2$.

Consider a smooth curve $C \in |L|$; C is not a plane curve ($g=3$ and $d \geq 5$) & $\deg L_C = d \geq 5 > 2g-2$

$$\text{PRfnc} \quad 4 \leq h^0(L_C) = d+1-g = d-2$$

Hence $d \geq 6$ - then HIT implies

$$16 = (2g-2)^2 = ((K_S + L) \cdot L)^2 \geq (K_S + L)^2 \cdot L^2 \geq 6(K_S + L)^2.$$

$$\text{Thus } (K_S + L)^2 \leq 16/6, \text{ i.e. } \leq 2.$$

Recall that $K_S + L'$ is ample and spanned (the Stein factorization shows that $s = \phi_{K_S + L'}$). Moreover,

$$(K_S + L')^2 = (K_S + L)^2 \leq 2$$

Thus, there are two possibilities;

either $(K_S + L')^2 = 1 \Rightarrow (S', K_S + L') \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ via s , hence $K_S + L'$ is very ample,

or $(K_S + L')^2 = 2 \Rightarrow s$ is a double cover $S \rightarrow \mathbb{P}^2$.

Let b be the degree of the branch locus. Thus

$$L' = (K_S + L') - K_S = s^*\mathcal{O}_{\mathbb{P}^2}(1) - s^*\mathcal{O}_{\mathbb{P}^2}(b-3) = s^*\mathcal{O}_{\mathbb{P}^2}(4-b)$$

and then the genus formula

$$4 = 2g-2 = (K_S + L') \cdot L' = s^*\mathcal{O}_{\mathbb{P}^2}(1) \cdot s^*\mathcal{O}_{\mathbb{P}^2}(b-4) = 2(4-b),$$

i.e. $b=2$. Therefore

$s: S \rightarrow \mathbb{P}^2$ is the del Pezzo double plane, i.e. S is the del Pezzo surf. of degree 2 and

$$-K_S = s^*\mathcal{O}_{\mathbb{P}^2}(1). \text{ But } L' = s^*\mathcal{O}_{\mathbb{P}^2}(2), \text{ hence } (S', L') \text{ is as in a).}$$

□

In view of lemma 1 we can continue assuming $K_S + L'$ not very ample, and $g \geq 4$.

By combining various facts (see Lanteri's notes for Madrid) one reduces to the case:

$$d=8, h^0(L)=6 \text{ and } g=4$$

In particular, $K_S \cdot L = 2g - 2 - d = 6 - 8 = -2 < 0$, so that S is ruled. Letting $g=4$, since $K_S + L$ is ample and spanned.

$$3 < h^0(K_S + L) = g - q = 4 - q,$$

hence $q = 0$ or 1.

5) It is $(K_S + L)^2 = 4$ and $q = 0$

By the WIT,

$$36 = (2g - 2)^2 = ((K_S + L) \cdot L)^2 \geq (K_S + L) \cdot L^2 = 8(K_S + L)^2,$$

hence $(K_S + L)^2 \leq 36/8$, i.e. ≤ 4 .

By "ad hoc" arguments one can easily exclude the values $(K_S + L)^2 = 1, 2$ and 3. Thus $(K_S + L)^2 = 4$. It remains to exclude the possibility that $q = 1$.

If $q = 1$, from

$$4 = (K_S + L)^2 = K_S^2 + 2K_S \cdot L + L^2 = K_S^2 + 2(2g - 2) - L^2 = K_S^2 + 12 - 8,$$

We see that $K_S^2 = 0$. Since S is a ruled surface over a smooth curve of genus $g = 1$, we know that $L^2 \leq 8(1-q) = 0$ hence the above equality implies that S is a \mathbb{P}^1 -bundle. With the usual notation we can write $L = aC_0 + bF$, and from S .

$$8 = L^2 = a(2b - ae)$$

$(-2 = L \cdot K_S = (aC_0 + bF)(-2C_0 - eF) = ae - 2b)$ we get $a = 4$, $b = 1+2e$. Thus the ampleness conditions $b > ae$ if $e \geq 0$ ($b > ae$ if $e = -1$) imply $e = 0$ or -1 , with $L = 4C_0 + (2e+1)F$. So, for $e = 0$ we get $L|_{C_0} = L$, while for $e = -1$ $L|_{C_0} = E$ a curve on S , $\equiv -K_S$ (a smooth curve of this type exists!). In both cases this contradicts the very ampleness of L , because both C_0 and E are elliptic curves.

In view of 5), we have $h^0(K_S + L) = g - q = 4$ so that $s: S^1 \rightarrow \Sigma \subset \mathbb{P}^3$, and from

$$4 = (K_S + L)^2 = \deg s \cdot \deg \Sigma \Rightarrow \begin{cases} 2 \cdot 2, \\ 1 \cdot 4, \end{cases}$$

but the second case does not occur in view of Lemma 2. In the above situation suppose that $K_S + L$ is not very ample. Then $\deg s \geq 2$. (See Cautieri, notes for Madrid.)

By Lemma 2 we deduce that $s: S^1 \rightarrow \Sigma$ is a double cover of Σ , which is either a quadric cone or a smooth quadric.

By using techniques of double covers similar to those in the proof of Lemma 1, one can show that both cases lead to contradictions.

□

Surfaces with a hyperelliptic hyperplane section

By hyperelliptic curve we mean a smooth curve C of genus $g \geq 2$, which is a double cover of \mathbb{P}^1 . Recall that the canonical map φ_{K_C} of such a curve factors through this double cover and the embedding of \mathbb{P}^1 in \mathbb{P}^{g-1} as a rational normal curve.

Consider $S \subset \mathbb{P}^N$, set $L = \left(\bigcup_{\mathbb{P}^N} \text{points}\right)_S$ and assume that $|L| \ni$ a hyperelliptic curve C .

Problem. Classify the pairs (S, L) as above.

Theorems. They are the following:

1) $(S, L) \in \mathcal{B}$, with base curve $\cong C$ (scrolls over a hyperelliptic curve).

2) $(S, L) \in \mathcal{C}$, with base \mathbb{P}^1 (i.e. rational conic fibrations)

3) $S = \text{elliptic } \mathbb{P}^1 \text{ bundle with } e = -1 \& L = [2C_0 + f]$

4) $S = \text{del Pezzo surf. with } K_S^2 = 1 \& L = -3K_S$

5) $S = \text{---} \& K_S^2 = 2 \& L = -2K_S$

6) (S, L) has a pair as in 5) as simple reduction.

Pf. According to the behavior of the adjunction mapping of (S, L) . Clearly $g \geq 2$. So, if $K_S + L$ is not spanned, then $(S, L) \in \mathcal{B}$: the scroll projection $\pi: S \rightarrow B$ induces an isomorphism $C \xrightarrow{\pi|_C} B$. This gives 1).

Suppose now that $K_S + L$ is spanned and consider Φ .

If $\dim \Phi(S) = 0$, then $(S, L) \in \mathcal{D}$, but this is impossible, since $g \geq 2$. Thus $\dim \Phi(S) = 1$ or 2.

Set $\varphi_C = \varphi|_C$ and $\varphi = \varphi_{K_C}$, the canonical map of C . Recall that $(K_S + L)|_C = K_C$ and look at the exact sequence

$$\rightarrow H^0(K_S + L) \xrightarrow{\text{rest}} H^0(K_C) \rightarrow H^1(K_S).$$

Φ is given by $\text{rest} \circ \text{rest}(H^0(K_S + L)) \subseteq H^0(K_C)$; thus

$$\Phi_C = \varphi_C \circ \varphi$$

i.e. Φ_C factors through φ .

Therefore

$$\deg \Phi_C \geq \deg \varphi = 2$$

with $= 1$ if $g = h^1(K_S) = 0$. Moreover,

$$\Phi(C) = \mathbb{P}^1$$

Suppose $\dim \Phi(S) = 1$; then $(S, L) \in \mathcal{C}$, with base curve say Y . Here the Stein factorization

of Φ is given by $S \xrightarrow{\Phi} \Phi(S) = \Phi(C) = \mathbb{P}^1 \xrightarrow{r} Y \xrightarrow{s} S$ and $r: S \rightarrow Y$ is the ruling of S .

If $Y = \mathbb{P}^1$, i.e. $g = 0$, then we get 2).

If $g(Y) > 0$, then s is not an embedding and this leads to 3 (result of Sommese).

Finally, let $\dim \Phi(S) = 2$. The reduction morphism $r: S \rightarrow S^1$ maps C isomorphically

to a $C \in |L'|$. Consider the diagram

$$C \xrightarrow{\phi_C} \phi(C)$$

$$\tau_C \downarrow C' \xrightarrow{s} C'$$

Since $\deg \phi_C \geq 2$ we argue that s cannot be an embedding, i.e. $K_S + L'$ is not very ample. Then we can apply the S-VaV theorem.

case c) leads to 4);

case a) \longrightarrow 5);

case b) \longrightarrow 6).

It remains to show that in case d) there are no hyperelliptic curves in $|L|$. (refers to [BS book, p. 290]).

Corollary. If (S, L) , with L very ample, has sectional genus 2, then (S, L) is either a scroll over a curve of genus 2, or a rational conic fibration.

Pf. Simply because $g \geq 3$ in cases 3)-6) of the theorem.

Rmk. Set

$$\mathcal{H} = \{C \in |L| \mid C \text{ hyperelliptic}\}.$$

\mathcal{H} is called the hyperelliptic locus of $|L|$.

The problem we addressed has two distinct formula-gens.

Classical point of view: when is $\mathcal{H} = |L|$?

Modern point of view: when is $\mathcal{H} \neq \emptyset$? (This is the version answered by the Theorem.)

Note: $\mathcal{H} = |L|$ in cases 1) and 2) (obvious).

These were the only cases recognized by Castelnuovo & Enriques. But their result is not complete. Actually

Proposition (Serrano): $\mathcal{H} = |L|$ also in case 3) of the Thm (even though the g_2^1 are smooth $C \in |L|$ is not a priori evident).

Idea: $C \in |L|$ via a suitable birational morphism (composition of elementary transformations)

$$S \xrightarrow[\text{birat.}]{} S_1 = B \times \mathbb{P}^1 \xrightarrow[\text{proj.}]{} \mathbb{P}^1$$

$$C \xrightarrow[\text{isom.}]{} C_1 \in |2B + 2f| \quad \text{and}$$

$$pr_2|_{C_1}: C_1 \rightarrow \mathbb{P}^1.$$

What about \mathcal{H} in the remaining cases of the Theorem? First we mention.

Fact (Sommese): If $\mathcal{H} \neq \emptyset$, then $\dim \mathcal{H} \geq 1$.

Now look at cases 4)-6) of the Theorem.

Case 4). Here $\phi: S \rightarrow \mathbb{P}^2$ is a double cover branched along a quartic. So $L = -2K = \phi^*\mathcal{O}_{\mathbb{P}^2}(2)$ then $\mathcal{H} = \phi^*|\mathcal{O}_{\mathbb{P}^2}(2)| = \mathbb{P}^5 \subset |L| = \mathbb{P}^6$.

Case 5). Here \mathcal{H} is the subspace of the hyperelliptic locus of case 4) defined by the condition of passing through the point $x_1 \in S$ blown-up by r . Then $\mathcal{H} = \mathbb{P}^4 \subset |L| = \mathbb{P}^5$.

Case 6) Recall that S is \mathbb{P}^2 blown-up at 8 points p_1, \dots, p_8 in general position. These points define a pencil P of cubics which has a further base point say p'_8 . We know that

$$|-K_S| = |\eta^*\mathcal{O}_{\mathbb{P}^2}(3) - E_1 - \dots - E_8| \text{ where } E_i = \eta'(p_i) \quad (i=1, \dots, 8).$$

Let \mathcal{J} be the linear subsystem of $|-3K_S| = |\eta^*\mathcal{O}_{\mathbb{P}^2}(9) - 3E_1 - \dots - 3E_8|$ defined by the further condition of passing through $\eta(p'_8)$. Then for every $\Gamma \in \mathcal{J}$, its proper transform $\tilde{\Gamma}$ cuts away smooth $C \in \mathcal{J}$ at

$$\# = 27 - 24 - 1 = 2$$

3.9 3.8

points. Thus $\mathcal{H} = \mathcal{J}$ and $\dim \mathcal{J} = \dim |L| - 1 = 6 - 1 = 5$.