

### 8. Fano manifolds

$X =$  compact complex mfd of  $\dim X = n$ .

Recall:  $X$  is projective algebraic  $\iff \exists$  ample  $H \in \text{Pic}(X)$

#### Intersection

Let  $D_1, \dots, D_n$  distinct effective divisors on  $X$  s.t.

(\*)  $I = \bigcap \text{supp}(D_i)$  is finite.

Then,  $\forall x \in I$  we define the local intersection index at  $x$  as

$$i(D_1, \dots, D_n; x) = \dim_{\mathbb{C}} (\mathcal{O}_{X,x} / (f_1, \dots, f_n))$$

where  $f_i$  is the germ defining  $D_i$  at  $x$ , and the intersection index as

$$D_1 \cdot \dots \cdot D_n = \sum_{x \in I} i(D_1, \dots, D_n; x)$$

Note:  $i(D_1, \dots, D_n; x) = 1$  iff  $D_1, \dots, D_n$  are smooth and transverse at  $x$ .

This leads to an intersection theory on  $X$  (e.g. see [Shafarevich, Basic Algebraic Geometry, Ch. IV]). In particular, due to the invariance by linear equivalence, one can define  $L_1 \cdot L_2 \cdot \dots \cdot L_n$  for  $L_i \in \text{Pic}(X)$ . Moreover,

$$L_1 \cdot L_2 \cdot \dots \cdot L_n = c_1(L_1) \cup c_1(L_2) \cup \dots \cup c_1(L_n)$$

where  $c_1(L_i)$  = Chern class of  $L_i$  and  $\cup$  = cup product. Then one can define the self-intersect.

index  $D^n = (c_1(D))^n$  for  $D \in \text{Div}(X)$ , and if  $D_1, \dots, D_n \in |D|$  and satisfy (\*), then  $D^n = D_1 \cdot D_2 \cdot \dots \cdot D_n$  as before.

For  $L \in \text{Pic}(X)$  define  $\chi(L) = \sum_{j \geq 0} (-1)^j h^j(L)$  where  $h^j(L) = \dim H^j(X, L)$

By the Riemann-Roch-Hirzebruch theorem,  $\chi(L)$  is a polynomial of degree  $n$  in  $c_1(L)$ , and the leading term is  $\frac{1}{n!} L^n$ .

def.  $X$  with  $\dim X = n$  is said to be a Fano manifold if  $-K_X$  is ample.

For  $n=1$   $X$  Fano means  $X = \mathbb{P}^1$ ,

for  $n=2$   $X$  Fano means:  $X$  is a del Pezzo surface.

#### Some properties of Fano mfd's.

1)  $h^i(\mathcal{O}_X) = 0$  for  $i=1, \dots, n$   
 $\parallel$   
 $h^i(K_X + (-K_X))$   
 ample

As a consequence,  $\chi(\mathcal{O}_X) = \sum_{j \geq 0} (-1)^j h^j(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$

2) By the exponential sequence,  $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$ , hence it is a finit. generated abelian group.

Moreover,

Proposition  $\text{Pic}(X)$  has no torsion

Pf Any  $m$ -torsion element in  $H^2(X, \mathbb{Z})$  defines an unbranched cover of order  $m$ ,  $\pi: \tilde{X} \rightarrow X$ . Then  $K_{\tilde{X}} = \pi^* K_X$ , so  $-K_{\tilde{X}}$  is ample since  $-K_X$  is so and  $\pi$  is finite. Thus  $\tilde{X}$  too is Fano, and  $\chi(\mathcal{O}_{\tilde{X}}) = 1$  by 1). But  $\chi(\mathcal{O}_{\tilde{X}}) = m \chi(\mathcal{O}_X) = m$ , hence  $m=1$ .  $\square$

3)  $\text{Pic}(X)$  fin. gener.  $\implies \exists$  finitely many divisors  $D$  and integers  $s$  s.t.  $-K_X = sD$  def. the maximum  $r$  s.t.  $-K_X = rH$  (for some  $H \in \text{Pic}(X)$ ) is uniquely determined (since it is defined only up to torsion, but there is no torsion by 2)). It is called index of  $X$ ;  $H$ , which is ample, is said the fundamental divisor of  $X$ .

4) Examples.

$\mathbb{P}^n$  is Fano,  $r = n+1$  and  $H = \mathcal{O}_{\mathbb{P}^n}(1)$ ;  
 $\mathbb{Q}^n$  (smooth hyperquadric of  $\mathbb{P}^{n+1}$ ) is Fano,  $r = n$  and  $H = \mathcal{O}_{\mathbb{Q}^n}(1)$ ;  
 $\mathbb{P}^h \times \mathbb{P}^k$  is Fano,  $r = \text{GCD}(h+1, k+1)$ ;  
 e.g., for  $h=k$ ,  $r = k+1$  and  $H = \mathcal{O}_{\mathbb{P}^h \times \mathbb{P}^h}(1, 1)$ .

5) Bound on the index:  $r \leq n+1$

Pf.  $h^i(tH) = h^i(K_X + \underbrace{tH - K_X}_{\text{ample if } t > -r}) \equiv 0 \quad \forall i > 0$

But also  $h^0(tH) = 0$  if  $t < 0$

Thus RRH  $\implies$

$\chi(t) = \chi(tH)$  vanishes at  $t = -1, -2, \dots, -(r-1)$  so there are  $r-1$  roots of  $\chi(t)$

But  $\deg \chi = n$ , hence  $r-1 \leq n$

6) Kobayashi-Ochiai theorem

Theorem (KO).  $r = n+1 \implies X = \mathbb{P}^n$   
 $r = n \implies X = \mathbb{Q}^n$

Pf (for  $\mathbb{P}^n$ ). If  $r = n+1$ ,  $-1, -2, \dots, -n$  are all the roots of  $\chi(t)$ . Thus

$\chi(t) = k(t+1)(t+2) \dots (t+n)$  for some constant  $k \neq 0$ .

By 1),  $1 = \chi(\mathcal{O}_X) = \chi(0) = k \cdot n! \implies k = \frac{1}{n!}$

Thus  $\chi(t) = \frac{1}{n!} (t+1)(t+2) \dots (t+n) = \frac{1}{n!} t^n + \dots$

where  $\dots$  = lower degree terms.

But RRH  $\implies$  for  $L = tH$  we have

$\chi(t) = \frac{1}{n!} L^n + \dots = \frac{1}{n!} H^n t^n + \dots$

hence  $H^n = 1$ , by comparison.

Now, as  $h^i(H) = 0 \forall i$ , we get  $h^0(H) = \chi(H) = \chi(1) = \frac{1}{n!} \cdot (n+1)! = n+1$

Let us continue by induction on  $n$ . For  $n=1$  the assertion is obvious, because

$2 = n+1 = h^0(H) = \deg H + 1 - g(X) = 2 - g(X)$   
hence  $g(X) = 0$ , i.e.  $X = \mathbb{P}^1$

Let  $n > 1$  and let  $Y \in |H|$ . We have  $Y \cdot H^{n-1} = H^n = 1$ , hence  $Y$  is irreducible and reduced. Moreover,  $Y^n$  implies that  $Y$  is smooth at every point. Finally, by adjunction,

$K_Y = (K_X + Y)|_Y = (-n+1)H_Y = -nH_Y$ , i.e.  $-K_Y = (dim Y + 1)H_Y$ .

Therefore  $Y$  is a Fano manifold of dimension  $n-1$  and index  $n$ .

Thus  $(Y, H_Y) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$  by induction.

Now, from the exact sequence

$0 \rightarrow -H \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$ , tensoring by  $H$ , we get

$0 \rightarrow \mathcal{O}_X \rightarrow H \rightarrow H_Y \rightarrow 0$ .

Since  $h^1(\mathcal{O}_X) = 0$  we thus get a surjection

$H^0(H) \twoheadrightarrow H^0(H_Y) = H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ ,

hence  $\forall x \in Y, \exists$  a section  $s \in H^0(H_Y)$  with  $s(x) \neq 0$ . But  $\forall x \in X$  we can find a smooth  $Y \in |H|$  as above, passing through  $x$ , to which all the above applies.

So  $\forall x \in X$  we can lift the section  $s \in H^0(H_Y)$  with  $s(x) \neq 0$  to a section  $\tilde{s} \in H^0(H)$  s.t.  $\tilde{s}(x) \neq 0$ . This implies that  $H$  is spanned.

Thus  $H$  (ample and spanned) defines a finite morphism  $\psi_H: X \rightarrow \mathbb{P}^n$ .

But  $\deg \psi_H = H^n = 1$ , hence it is an isomorphism.  $\square$

Remark. The inductive procedure used in the proof is what Fujita calls the "Apollonius method", see [Fujita's book, pp. 22-23].

The next natural step is to ask about Fano manifolds of index  $n-1$ . "Essentially" they are the "del Pezzo manifolds".

del Pezzo manifolds

def. Let  $(X, L)$  be a polarized mfd of  $dim X = n$ . If  $-K_X = (n-1)L$ , then  $(X, L)$  is said to be a del Pezzo mfd;  $d = L^n$  is called the degree of  $(X, L)$ .

Note. If  $X$  is a Fano  $n$ -fold of index  $n-1$ , then  $(X, H)$  is a del Pezzo manifold, but the converse is not true. Actually, let  $(X, L)$

be a del Pezzo manifold, then  $-K_X = (n-1)L$ , but  $L$  could be divisible in  $Pic(X)$ , for instance,  $L = tA$ , with  $t \geq 2$  and  $A$  ample. Then  $-K_X = (n-1)tA$ , hence  $X$  is Fano, but its index is  $(n-1)t$  or even a multiple of it. So, we can only say that  $X$  is Fano, but its index is a multiple of  $(n-1)$ . For example,  $(\mathbb{P}^3, \mathcal{O}(2))$  is a del Pezzo 3-fold since  $-K_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(4) = 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2)$ , but  $\mathbb{P}^3$  is Fano of index 4.

Fact (Fujita) Let  $(X, L)$  be a del Pezzo mfd; then  $|L|$  contains a smooth element.

Let  $Y \in |L|$  be a smooth element. Then  $(Y, L_Y)$  is also a del Pezzo mfd, of  $dim n-1$ . Actually, by adjunction,

$K_Y = (K_X + L)|_Y = -(n-1+1)L_Y = -(2-n)L_Y$ . Thus  $-K_Y = (dim Y - 1)L_Y$ .

As a consequence, the "surface section"  $(S, L_S)$  of a del Pezzo manifold is a del Pezzo pair, i.e.  $S$  is del Pezzo and  $-K_S = L_S$ .

It thus follows from the classification of del Pezzo surfaces that if  $(X, L)$  is a del Pezzo manifold, then  $d \leq 9$ . But there are some constructions.

Example. Let  $d = 9$ . There is no del Pezzo threefold  $(X, L)$  such that  $(S, L_S) = (\mathbb{P}^2, \mathcal{O}(3))$  let  $d = 8$ , and suppose that  $(S, L_S) = (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(2))$ ; then  $n = 3$  and  $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ . In fact,  $\mathbb{Q}^2$  can be contained as an ample divisor only in  $\mathbb{P}^3$  and in  $\mathbb{Q}^3$ , but in the latter case  $S = \mathbb{Q}^2 \in |\mathcal{O}_{\mathbb{Q}^3}(1)|$ , which is not compatible with  $L_S = \mathcal{O}_{\mathbb{Q}^2}(2)$ .

The fact that there is no lift to higher dimensions is due to the following

Proposition. Let  $n \geq 3$  and let  $Y = \mathbb{P}^{n-1} \subset X$  as an ample divisor. Then  $X = \mathbb{P}^n$  and  $Y \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ .

Idea of pf.  $Y$  ample  $\xrightarrow{\text{Lefschetz}}$   $Pic(X) \xrightarrow{\text{restr}} Pic(Y)$  is: an isomorphism if  $n \geq 4$ , and injective with torsionfree cokernel if  $n = 3$ . Thus, since  $Y = \mathbb{P}^{n-1}$ ,

$Pic(X) \xrightarrow{\text{restr}} Pic(Y) \cong \mathbb{Z}$  for  $n \geq 3$ .

Let  $H$  be the ample generator of  $Pic(X)$ ; then  $Y = kH$  for some  $k \geq 1$ .

Consider the exact sequence  $0 \rightarrow -Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$  and  $\otimes H$ . Then

$0 \rightarrow (1-k)H \rightarrow H \rightarrow H_Y = \mathcal{O}_{\mathbb{P}^{n-1}}(1) \rightarrow 0$

If  $k \geq 2$  we thus get  $h^0(H) \leq h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = n$ ,  
 hence  $\varphi_H: X \dashrightarrow \mathbb{P}^{n-1}$ , and so  $\dim \varphi_H(X) < n$ ,  
 contradicting the ampleness of  $H$ .

If  $k=1$ , the sequence becomes

$$0 \rightarrow \mathcal{O}_X \rightarrow H \rightarrow H_Y = \mathcal{O}_{\mathbb{P}^{n-1}}(1) \rightarrow 0,$$

hence

$$n+1 \leq h^0(H) \leq 1 + h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = n+1,$$

$H$  ample  
 so equality holds. Thus  $\varphi_H: X \dashrightarrow \mathbb{P}^n$ , but  
 $H$  is spanned (same argument as in the proof of  
 KO theorem), so  $\varphi_H: X \rightarrow \mathbb{P}^n$  is a morphism,  
 and  $H^n = H^{n-1} \cdot Y = H_Y^{n-1} = 1$ . This says that  
 $\varphi$  is an isomorphism

Corollary. If  $(X, L)$  is a del Pezzo manifold  
 of dim  $n \geq 3$ , then  $d \leq 8$ .

Theorem Classification of del Pezzo mfd's =  
 Iskovskikh for  $n=3$ , Fujita for any  $n$ .

d	n	description of $(X, L)$
1	any	double cover of a cone $W \subset \mathbb{P}^N$ over the Veronese variety $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2))$ , branched at the vertex and along the transverse intersection with a cubic hypersurface of $\mathbb{P}^N$ ;
2	any	double cover of $\mathbb{P}^n$ branched along a smooth quartic hypersurface;
3	any	a smooth cubic hypersurface of $\mathbb{P}^{n+1}$ ;
4	any	a smooth intersection of two quadric hypersurfaces of $\mathbb{P}^{n+2}$ ;
5	$n \leq 6$	a linear section of the grassmannian $G(1,4)$ (of lines of $\mathbb{P}^4$ ) Plücker embed- ded in $\mathbb{P}^9$ .
6	$\begin{cases} n=3 \\ n \leq 4 \end{cases}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Segre embedded in $\mathbb{P}^7$ ; a linear section of $\mathbb{P}^2 \times \mathbb{P}^2$ Segre embed- ded in $\mathbb{P}^8$ (for $n=3$ , $X = P(T_{\mathbb{P}^2})$ , $L = \text{tautol. line bundle}$ );
7	$n=3$	the blow-up of $\mathbb{P}^3$ at one point, here $L = \sigma^* \mathcal{O}_{\mathbb{P}^3}(2) - E$ , where $\sigma: X \rightarrow \mathbb{P}^3$ is the blow-up and $E$ the exceptional divisor;
8	$n=3$	$(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$

Remark.  $L$  is very ample for  $d \geq 3$  ample  
 and spanned for  $d=2$ , ample with  $Bs|L =$   
 a single point if  $d=1$ .