About the adjunction process for polarized algebraic surfaces

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Introduction

Many powerful results are known about the adjoint system $|K_S + H|$ to a very ample divisor H on a non-singular complex projective algebraic surface S (e.g. see [5]). In this paper we study some properties of $K_S + H$ when H is simply supposed to be an ample divisor, possibly non-effective. From this standpoint, here we are mainly interested in 1) the ampleness of $K_S + H$ and 2) the dimension $h^0(K_S + H)$.

To deal with the former problem, at first we study the self-intersection index of $K_S + H$ (Th. 2. 1). This alone enables us to characterize some classes of polarized surfaces (Th. 2. 2 and Cor. 2. 3, 2. 4) as well as to recover some classical results in the wider context of ample divisors. Moreover Th. 2. 1 jointly with some arguments from Mori's theory of extremal rays is the key for obtaining our main result (Th. 2. 5) describing the polarized surfaces on which $K_S + H$ fails to be ample.

To come to the latter problem, we characterize (Th. 3. 2) the polarized surfaces (S, H) satisfying equality in the fairly obvious inequality

$$g(H) \geqq \max\{0, q(S) - p_g(S)\}, \tag{0. 1}$$

where $g(H)$ is the arithmetical virtual genus of H and $q(S)$ and $p_g(S)$ are the irregularity and the geometric genus of S respectively. Note that (0. 1) can be thought of as a generalization of the sharper inequality $g(H) \geqq q(S)$ holding for a very ample divisor H, which stems from the first Lefschetz theorem on hyperplane sections. Since $h^0(K_S + H) = p_g(S) + g(H) - q(S)$, this exactly answers problem 2 as far as the vanishing of the dimension $h^0(K_S + H)$ is concerned. Finally we give some information about the case $h^0(K_S + H) = 1$ (Prop. 3. 5). In particular we find that $|K_S + H|$ is a pencil at least, whenever $g(H) \geqq 3$.

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1. Notation and background

Here we consider only complex projective irreducible algebraic varieties. Such a variety will be called curve if it has dimension one and surface provided that it is smooth and of dimension two. Let S be a surface and D a divisor on S. In view of well known correspondences we shall not distinguish among the invertible sheaf $\mathcal{O}_S(D)$, the associated line bundle and the divisor D up to linear equivalence. We shall use the following standard notations: $h^i(D)=\dim_{\mathbb{C}} H^i(S, \mathcal{O}_S(D))$; $|D|=$ the complete linear system defined by D; $\equiv$ for the linear equivalence and $\equiv$ for the numerical equivalence; $DD'=$ the intersection index of two divisors D and D' on S; $D^2=DD$; $K_S=$ a canonical divisor on S; $p_g(S)=h^0(K_S)$ is the geometric genus of S; $q(S)=$ the dimension of the complex vector space of holomorphic 1-forms on S is the irregularity of S; $\chi(\mathcal{O}_S)=$ the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_S$ of S; $\mathbb{P}^n=$ the n-dimensional complex projective space; $\mathcal{O}_{\mathbb{P}^n}(r)=$ the r-th tensor power of the line bundle on $\mathbb{P}^n$ corresponding to the effective generator of $\mathrm{Pic}(\mathbb{P}^n)$.

For a divisor H on a surface S we shall put $g(H)=1+\frac{1}{2}(H^2+HK_S)$: the arithmetic virtual genus of H, and we shall frequently call the above "genus formula". Of course, if $|H|$ contains a smooth curve C, then $g(H)$ is the geometric genus of C.

1. 1 Remark. *If H is an ample divisor, then*

$$g(H) \geqq \max\{0, q(S)-p_g(S)\}.$$

Proof. By the Kodaira vanishing and the Riemann-Roch theorem

$$h^0(K_S+H)=p_g(S)+g(H)-q(S), \tag{1. 1. 1}$$

so that $g(H) \geqq q(S)-p_g(S)$. On the other hand, by genus formula, $g(H) \geqq 0$ if $H^2>0$ and $HK_S \geqq -2$. But this fact holds trivially, by ampleness, if S has Kodaira dimension $\kappa(S) \geqq 0$, whilst, if $\kappa(S)=-\infty$, then $p_g(S)=0$ and so (1. 1. 1) gives $g(H) \geqq q(S) \geqq 0$.

Let C be a smooth curve. A $\mathbb{P}^1$-bundle over C is the projectivized of a rank two holomorphic vector bundle over C. Throughout all the paper we shall consider polarized surfaces, i.e. pairs (S, H) where S is a surface and H is an ample divisor on S. Let us introduce a special notation for some classes of pairs frequently occurring in the sequel.

$\mathcal{A}$: the class of pairs $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$, $r=1, 2$;

$\mathcal{B}$: the class of pairs (S, H) where S is a $\mathbb{P}^1$-bundle and $\mathcal{O}_S(H) \otimes \mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(1)$ for any fibre f of S;

$\mathcal{C}$: the class of pairs (S, H) which are conic bundles, i.e. S is a $\mathbb{P}^1$-bundle blown-up at $\delta \geqq 0$ points belonging to distinct fibres and $\mathcal{O}_S(H) \otimes \mathcal{O}_F = \mathcal{O}_{\mathbb{P}^1}(2)$ for the general fibre F of S;

$\mathcal{D}$: the class of pairs (S, H) where S is a Del Pezzo surface and $H \equiv -K_S$.

In the next Lemma some arguments from Mori's theory of extremal rays are needed. So far as general definitions are concerned we refer to Mori [4].

1. 2 Lemma. *Let (S, H) be a polarized surface. If there exists a curve $C \subset S$ such that*

$$(1.2.1) \qquad (K_S + H)C \leqq 0,$$

then either $(S, H) \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{D}$ or S is not a minimal model, equality holds in (1.2.1) *and $C = \sum_1^n \lambda_i l_i$ in* $\mathrm{Num}(S) \otimes \mathbb{R}$ *where each l_i is an exceptional curve such that $Hl_i = 1$.*

Proof. Every relation appearing in this proof is meant to be written in $\mathrm{Num}(S) \otimes \mathbb{R}$. Choose $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$. Mori's cone theorem ([4], Th. 1. 4 and 1. 5) gives $C = \sum_1^n \lambda_i l_i + Y$ where $\lambda_i \geqq 0$, l_i is an extremal rational curve, Y belongs to the closure of the convex cone in $\mathrm{Num}(S) \otimes \mathbb{R}$ generated by numerically effective divisors and satisfies the inequality $YK_S \geqq -\varepsilon YH$. So (1. 2. 1) reads

$$(1.2.2) \qquad 0 \geqq \sum_1^n \lambda_i (K_S + H)\, l_i + Y(K_S + H).$$

But Kleiman's ampleness criterion ([3], Ch. IV, Th. 1) gives $YH \geqq 0$, equality holding if and only if $Y = 0$. Therefore (1. 2. 2) gives

$$(1.2.3) \qquad 0 \geqq \sum_1^n \lambda_i (K_S + H)\, l_i + (1 - \varepsilon)\, YH \geqq \sum_1^n \lambda_i (K_S + H)\, l_i$$

and if $\sum \lambda_i (K_S + H)\, l_i = 0$, then $Y = 0$. Therefore there exists an extremal rational curve, say l_1, such that

$$(1.2.4) \qquad (K_S + H)\, l_1 \leqq 0.$$

By the classification theorem of extremal rational curves (cf. [4], Th. 2. 1) just one of the following cases must occur for S and $l = l_1$:

i) $S \cong \mathbb{P}^2$ and $\mathcal{O}_S(l) = \mathcal{O}_{\mathbb{P}^2}(1)$;

ii) S is a $\mathbb{P}^1$-bundle and l is a fibre;

iii) S is not a minimal model and l is an exceptional curve.

Cases i) and ii) lead to pairs $(S, H) \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ and to the pair $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ belonging to $\mathscr{D}$. Now assume case iii) holds. Since $l_1 K_S = -1$, (1. 2. 4) gives $Hl_1 \leqq 1$ and then $Hl_1 = 1$, by ampleness, so that equality holds in (1. 2. 4). Hence, if $n = 1$ we get $Y = 0$, while if $n > 1$ in view of (1. 2. 3) there exists another extremal rational curve, say l_2, such that

$$(1.2.5) \qquad (K_S + H)\, l_2 \leqq 0.$$

As we are dealing with a surface S containing the exceptional curve l_1, the above case i) cannot occur for $l=l_2$. If case ii) holds for $l=l_2$, then $S=B_p(\mathbb{P}^2)$ is $\mathbb{P}^2$ blown-up at a point p of course, and as is easy to see $(S, H) \in \mathcal{B} \cup \mathcal{C}$. By repeating this argument one concludes that the remaining $l_3, \ldots, l_n$ are exceptional curves and $l_i H=1$ for $i=1, 2, \ldots, n$. Hence $\sum_1^n \lambda_i (K_S+H)\, l_i=0$ and then $Y=0$.

1. 3 Remark. *If $(S, H) \notin \mathcal{A} \cup \mathcal{B}$ then K_S+H is numerically effective. Moreover if $(S, H) \notin \mathcal{B}$ then $(K_S+H)^2 \geqq 0$.*

Indeed, if K_S+H is not numerically effective then there exists a curve $C \subset S$ such that

$$(K_S+H)\,C<0. \tag{1.3.1}$$

If we look through the proof of Lemma 1. 2 with (1. 3. 1) instead of (1. 2. 1) we see that $(S, H) \in \mathcal{A} \cup \mathcal{B}$. As far as the latter statement is concerned, it is obvious when $(S, H) \in \mathcal{A}$ while it follows from Kleiman's pseudoampleness criterion ([3], Ch. III, Th. 1) when $(S, H) \notin \mathcal{A} \cup \mathcal{B}$.

1. 4 Lemma. *Let (S, H) be a polarized surface and assume $h^0(nK_S+nH) \leqq 1$ for all $n \in \mathbb{N}$. Then $(S, H) \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.*

Proof. Since H is ample, our assumption implies $h^0(nK_S)=0$ for n large enough. It turns out that $h^0(nK_S)=0$ for all $n \in \mathbb{N}$ and so S is ruled. Hence either $S \cong \mathbb{P}^2$ or

$$K_S^2 \leqq 8\,(1-q(S)). \tag{1.4.1}$$

Moreover, as $p_g(S)=0$, recalling (1. 1. 1) we obtain $1 \geqq h^0(K_S+H)=g(H)-q(S)$, so that it can only be either

$$g(H)=q(S), \tag{1.4.2}$$

or

$$g(H)=q(S)+1, \tag{1.4.3}$$

and in the latter case, as $h^0(K_S+H)=1$, our assumption gives

$$h^0(nK_S+nH)=1 \quad \text{for each} \quad n \in \mathbb{N}. \tag{1.4.4}$$

Assume (1. 4. 2) holds. If $S \cong \mathbb{P}^2$, then $g(H)=0$ by (1. 4. 2); hence $(S, H) \in \mathcal{A}$. Assume $S \ncong \mathbb{P}^2$ and consider the pair $(S, 2H)$. Of course $(S, 2H) \notin \mathcal{B}$, since $2H$ is even, so Remark 1. 3 gives $(K_S+2H)^2 \geqq 0$. Then recalling (1. 4. 2) we get

$$0 \leqq (K_S+2H)^2=K_S^2-8\,(1-q(S)). \tag{1.4.5}$$

By (1. 4. 1) and (1. 4. 5) one gets $K_S^2=8(1-q(S))$; hence S is a $\mathbb{P}^1$-bundle. Denote by C_0 and f a fundamental section and a fibre respectively. Then $H\equiv aC_0+bf$ where the integers a and b satisfy the ampleness conditions (cf. [2], p. 382). Recalling the expression of K_S (cf. [2], p. 373) an easy computation shows that (1. 4. 2) implies $(S,H)\in\mathscr{B}$. Now assume (1. 4. 3) holds. If $S\cong\mathbb{P}^2$, then $\mathcal{O}_S(H)=\mathcal{O}_{\mathbb{P}^2}(3)$ and so $(S,H)\in\mathscr{D}$. Assume $S\ncong\mathbb{P}^2$; as $g(H)\neq q(S)$ it turns out that $(S,H)\notin\mathscr{B}$ and so

$$(K_S+H)^2\geqq 0, \tag{1. 4. 6}$$

by Remark 1. 3. Then by (1. 4. 1) and (1. 4. 3) one gets $0\leqq(K_S+H)^2\leqq 8-4q(S)-H^2$ and since H is ample this implies $q(S)\leqq 1$. If $q(S)=0$, then $g(H)=1$ and genus formula gives $H(K_S+H)=0$. As H is ample and $h^0(K_S+H)=1$, this implies $K_S+H\equiv 0$, i.e. $(S,H)\in\mathscr{D}$. Finally let $q(S)=1$ and so $\chi(\mathcal{O}_S)=0$ and $g(H)=2$. As $((1-n)K_S-nH)H<0$ for each $n\in\mathbb{N}$, by Serre duality we get $h^2(nK_S+nH)=h^0((1-n)K_S-nH)=0$. So the Riemann-Roch theorem gives

$$h^0(nK_S+nH)\geqq\chi(\mathcal{O}_S)+\frac{1}{2}(nK_S+nH)((n-1)K_S+nH)=\frac{1}{2}n(n-1)(K_S+H)^2+n,$$

but this contradicts (1. 4. 4), in view of (1. 4. 6).

2. The ampleness of K_S+H

Our first result is the following

2. 1 Theorem. *Assume* $(S,H)\notin\mathscr{B}$; *then*

$$(K_S+H)^2\geqq 0. \tag{2. 1. 1}$$

Moreover equality holds in (2. 1. 1) *if and only if* $(S,H)\in\mathscr{C}\cup\mathscr{D}$.

Proof. Remark 1. 3 gives inequality (2. 1. 1). Now assume $(K_S+H)^2=0$. Henceforth $(S,H)\notin\mathscr{A}$. Then if $h^0(nK_S+nH)\leqq 1$ for each $n\in\mathbb{N}$, $(S,H)\in\mathscr{D}$ in view of Lemma 1. 4. On the other hand if $h^0(nK_S+nH)\geqq 2$ for n large enough, then the rational map $\Phi_n: S\longrightarrow\Sigma\subset\mathbb{P}^N$ defined by $|nK_S+nH|$ is a morphism whose image Σ has $\dim\Sigma=1$. To see this call Z and M the fixed and the moving part of $|nK_S+nH|$; then

$$0=(nK_S+nH)^2=(nK_S+nH)Z+ZM+M^2.$$

Of course $ZM\geqq 0$, $M^2\geqq 0$ and $(nK_S+nH)Z\geqq 0$ by Remark 1. 3. So $ZM=0$ and as $h^0(M)=h^0(nK_S+nH)\geqq 2$ we get $Z=0$. Hence $|nK_S+nH|$ has no fixed part and as $(K_S+H)^2=0$, Φ_n is a morphism and $\dim\Sigma=1$. The Stein factorization of Φ_n gives a morphism $\pi: S\longrightarrow C$ over a smooth curve C. As a general fibre F of π is a connected component of nK_S+nH, we get $F(K_S+H)=0$ which implies $FK_S<0$. On the other hand $-2\leqq 2g(F)-2=F^2+FK_S=FK_S<0$. So $FK_S=-2$ and then $g(F)=0$ and $FH=2$. This shows that $(S,H)\in\mathscr{C}$. To see the converse, first of all note that equality holds trivially in (2. 1. 1) whenever $(S,H)\in\mathscr{D}$. Now assume $(S,H)\in\mathscr{C}$. Then S dominates a $\mathbb{P}^1$-bundle S_0 via a birational morphism $\eta: S\longrightarrow S_0$ which factors by means of $\delta\geqq 0$ blowing-ups with centers p_i $(i=1,\dots,\delta)$ lying on distinct fibres of S_0. Put $E_i=\eta^{-1}(p_i)$ and note that $HE_i=1$ by the ampleness of H. It thus easily follows the existence of a divisor D on S_0 such that

$$H\equiv\eta^*D-\sum E_i. \tag{2. 1. 2}$$

On the other hand we have $K_S \equiv \eta^* K_{S_0} + \sum E_i$, so that $(K_S + H)^2 = (K_{S_0} + D)^2$. Therefore it is enough to show that

$$(K_{S_0} + D)^2 = 0. \tag{2. 1. 3}$$

To see this call F and f the general fibre of S and of S_0 respectively and s a section of S_0. In view of (2. 1. 2) one gets $2 = HF = (\eta^* D - \sum E_i)\, \eta^* f = Df$ and this implies that $D \equiv 2s + mf\ (m \in \mathbb{Z})$. Since $K_{S_0} \equiv -2s + m'f$ (cf. [2], p. 373), one gets $K_{S_0} + D \equiv (m+m')f$ and this proves (2. 1. 3).

As a first application we can give a numerical characterization of class $\mathscr{B}$.

2. 2 Theorem. *$(S, H) \in \mathscr{B}$ if and only if $K_S^2 = 8(1 - g(H))$.*

Proof. Assume $(S, H) \in \mathscr{B}$; then $K_S^2 = 8(1 - q(S))$ and so it is enough to show that $g(H) = q(S)$. This is immediate since, S being ruled, (1. 1. 1) yields

$$h^0(K_S + H) = g(H) - q(S).$$

On the other hand, as $(S, H) \in \mathscr{B}$, $K_S f = -2$, $Hf = 1$ and so $(K_S + H) f = -1$ which implies $h^0(K_S + H) = 0$ (since $f^2 = 0$). To see the converse, assume $K_S^2 = 8(1 - g(H))$ and by contradiction let $(S, H) \notin \mathscr{B}$. In this case (2. 1. 1) holds and so

$$H^2 \leqq 4(g(H) - 1) + K_S^2 \leqq 4(1 - g(H)).$$

Since H is ample this implies $g(H) = 0$ and by the genus formula

$$H(K_S + H) = -2. \tag{2. 2. 1}$$

On the other hand, as $K_S^2 = 8(1 - g(H)) = 8$, $(S, H) \notin \mathscr{A}$ and then (2. 2. 1) contradicts Remark 1. 3.

Furthermore Theorem 2. 1 enables us to generalize some very classical results by Picard, Castelnuovo and Del Pezzo.

2. 3 Corollary. *Assume $g(H) = 0$. Then S is a rational surface and $(S, H) \in \mathscr{A} \cup \mathscr{B}$.*

Proof. First of all $H(K_S + H) = -2$, by the genus formula, thus $K_S + H$ is not numerically effective. Hence $(S, H) \in \mathscr{A} \cup \mathscr{B}$ by Remark 1. 3. In particular $p_g(S) = 0$ and formula (1. 1. 1) once again gives $h^0(K_S + H) = g(H) - q(S)$. Hence $q(S) \leqq g(H) = 0$ and so S is rational.

2. 4 Corollary. *Assume $g(H) = 1$. Then either $(S, H) \in \mathscr{D}$ or $(S, H) \in \mathscr{B}$ and $q(S) = 1$.*

Proof. By the genus formula $H(K_S + H) = 0$, thus $HK_S < 0$ and so S is ruled. Once again one gets $q(S) \leqq g(H) = 1$. Assume $q(S) = 0$; then $h^0(K_S + H) = 1$ and therefore $K_S + H \equiv 0$, i.e. $(S, H) \in \mathscr{D}$. Now assume $g(H) = q(S) = 1$ and by contradiction let $(S, H) \notin \mathscr{B}$. Then by Theorem 2. 1 one gets $H^2 \leqq K_S^2 \leqq 8(1 - q(S)) = 0$, which contradicts the ampleness of H.

The main result of this section deals with the ampleness of K_S+H. It is the following

2. 5 Theorem. *Let (S, H) be a polarized surface. Then either*

i) *K_S+H is ample,*

ii) *$(S, H)\in\mathscr{A}\cup\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$, or*

iii) *S is not a minimal model; $(K_S+H)^2>0$ and $(K_S+H)C\geqq 0$ for every curve $C\subset S$, equality holding if and only if C is an exceptional curve such that $CH=1$. In this case there exists a morphism $\eta: S\to S'$ contracting all the exceptional curves $C\subset S$ such that $CH=1$ and the pair $(S', H'=\eta_* H)$ is a polarized surface where $K_{S'}+H'$ is ample.*

Proof. If K_S+H is not ample, then either $(K_S+H)^2\leqq 0$ or

(2. 5. 1) $$(K_S+H)C\leqq 0$$

for a curve $C\subset S$. In the former case $(S, H)\in\mathscr{B}\cup\mathscr{C}\cup\mathscr{D}$, by Theorem 2. 1. Now assume (2. 5. 1) holds and (S, H) is not as in ii). Then Lemma 1. 2 says that S is not a minimal model and (2. 5. 1) reads as an equality. Since $(K_S+H)^2>0$, by the algebraic index theorem we get $C^2\leqq 0$; on the other hand $CK_S=-CH<0$ by ampleness. Hence the genus formula gives $-2\leqq 2g(C)-2=C^2+CK_S<0$; so either $C^2=CK_S=-1$ and $CH=1$ or $C^2=0$, $CK_S=-2$ but in the latter case S is a ruled surface, C is a fibre of a ruling of S and since $CH=2$ we get $(S, H)\in\mathscr{C}$, absurd! The last part of the Theorem follows by adapting one of Sommese's arguments (cf. [5], p. 392). First of all, if C_1 and C_2 are two curves satisfying equality in (2. 5. 1), then $C_1C_2\leqq 1$. Indeed, as $(C_1+C_2)(K_S+H)=0$ and $(K_S+H)^2>0$, the algebraic index theorem gives $0\geqq(C_1+C_2)^2=-2+2(C_1C_2)$. Now assume that $C_iC_j=0$ for all the exceptional curves $C_i\subset S$ such that $C_iH=1$. Let $\eta: S\to S'$ be the birational morphism obtained by contracting all the exceptional curves satisfying equality in (2. 5. 1) and consider the divisor $H'=\eta_* H$. The Nakai-Moishezon criterion immediately shows that H' is an ample divisor. Moreover, as $\eta^* H'\equiv H+\sum C_i$ and $K_S\equiv\eta^* K_{S'}+\sum C_i$, one gets $K_S+H\equiv\eta^*(K_{S'}+H')$; so $(K_{S'}+H')^2=(K_S+H)^2>0$ and for any curve $\Gamma'\subset S'$,

$$(K_{S'}+H')\Gamma'=(K_S+H)\left(\eta^{-1}(\Gamma')+\sum r_iC_i\right)=(K_S+H)\eta^{-1}(\Gamma')>0,$$

since for $C=\eta^{-1}(\Gamma')$, formula (2. 5. 1) holds with $>$ instead of $\leqq$. Therefore it remains only to show that it cannot be $C_iC_j=1$. By contradiction let $C_1C_2=1$ and let $\sigma: S\to S''$ be the contraction of C_1. Put $F=\sigma(C_2)$ and $H''=\sigma_* H$; then (S'', H'') is a polarized surface and $(K_{S''}+H'')F=(K_S+H)\sigma^* F=(K_S+H)(C_1+C_2)=0$. On the other hand F is a rational curve on S'', $F^2=(\sigma^* F)^2=(C_1+C_2)^2=0$ and so $FH''=-FK_{S''}=2$. It turns out that S'' is a ruled surface, F is a fibre of a ruling of S'', and so $(S'', H'')\in\mathscr{C}$. Now this gives $H\sigma^* F=H(C_1+C_2)=2$ and so even (S, H) belongs to $\mathscr{C}$; absurd.

As far as $\mathbb{P}^1$-bundles are concerned, the situation can be described more precisely. Indeed if S is a $\mathbb{P}^1$-bundle and $(S, H)\notin\mathscr{B}\cup\mathscr{C}$, then K_S+H is an ample divisor by Theorem 2. 5 unless S is the rational $\mathbb{P}^1$-bundle of invariant $e=1$ (i.e. $S=B_p(\mathbb{P}^2)$ is $\mathbb{P}^2$ blown-up at a point p). Denote again by C_0 and f the fundamental section and a fibre of S respectively and write $H\equiv aC_0+bf$. Then one sees that $C(K_S+H)>0$ unless $C=C_0$, in which case one gets $C_0(K_S+H)=b-a-1$. One concludes that K_S+H is not ample only when $b=a+1$, in view of the ampleness conditions. In this case the pair (S', H') of Theorem 2. 5 corresponding to (S, H) is the polarized surface $\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a+1)\right)$ $(a>2)$.

3. On the dimension of $|K_S+H|$

Let H be an ample divisor on S. According to Remark 1. 1 we have the following inequality

$$g(H) \geqq \max\{0, q(S)-p_g(S)\}. \tag{3. 0. 1}$$

In this section we characterize equality in (3. 0. 1).

3. 1 Remark. Note that if H is a very ample divisor, the first Lefschetz theorem on hyperplane sections supplies the sharpened version of (3. 0. 1): $g(H) \geqq q(S)$. Thus the characterization we are going to give seems to be the most natural generalization of the classical theorem characterizing equality $g(H)=q(S)$ when H is a very ample divisor (e.g. cf. [5], p. 388).

3. 2 Theorem. *One has* $g(H)=\max\{0, q(S)-p_g(S)\}$ *if and only if* $(S, H) \in \mathcal{A} \cup \mathcal{B}$ *(in particular it is always* $p_g(S)=0$*).*

Proof. If $(S, H) \in \mathcal{A} \cup \mathcal{B}$, equality holds trivially in (3. 0. 1). To see the converse first of all note that cases $g(H)=0$ and $g(H)=1$ have been treaten in Corollaries 2. 3 and 2. 4. Now assume $g(H) \geqq 2$. Then $q(S) \geqq p_g(S)+2$, so that

$$\chi(\mathcal{O}_S)=1-q(S)+p_g(S) \leqq -1,$$

and so S is ruled by Castelnuovo-De Franchis theorem (cf. [1], p. 154). In particular we have $p_g(S)=0$ and

$$g(H)=q(S) \geqq 2. \tag{3. 2. 1}$$

By (3. 2. 1), $(S, H) \notin \mathcal{A}$. By contradiction let $(S, H) \notin \mathcal{B}$. Then by Theorem 2. 1 and by (3. 2. 1) we get $H^2 \leqq 4q(S)-4+K_S^2$, and S being ruled, $K_S^2 \leqq 8(1-q(S))$. Henceforth $H^2 \leqq 4(1-q(S))<0$, contradicting the ampleness of H.

The classical characterization of equality $g(H)=q(S)$ when H is a very ample divisor can be easily extended in view of Theorem 3. 2.

3. 3 Corollary. *Let* $A \subset S$ *be a smooth curve which is an ample divisor and such that* $h^0(A) \geqq 2$ *if* $g(A) \geqq 2$. *Then* $g(A)=q(S)$ *if and only if* $(S, A) \in \mathcal{A} \cup \mathcal{B}$.

Proof. The if part is trivial. Conversely we have nothing to prove when $g(A) \leqq 1$ in view of Corollaries 2. 3 and 2. 4. Assume $g(A)=q(S) \geqq 2$ and consider the exact cohomology sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S(K_S)) \xrightarrow{j} H^0(S, \mathcal{O}_S(K_S+A)) \xrightarrow{r} H^0(A, \mathcal{O}_A(K_A))$$
$$\xrightarrow{s} H^1(S, \mathcal{O}_S(K_S)) \longrightarrow H^1(S, \mathcal{O}_S(K_S+A)) \longrightarrow \cdots.$$

By the Kodaira vanishing theorem, s is surjective, but, as $h^1(K_S)=q(S)=g(A)=h^0(K_A)$, actually s is an isomorphism. Therefore $r=0$, i.e. the linear system $|K_S+A| \cdot A$, cut by $|K_S+A|$ on A, is empty. So, since A is ample and $h^0(A) \geqq 2$, we get $|K_S+A|=\emptyset$ and the injectivity of j implies $p_g(S)=0$. Then the assumption $g(A)=q(S)$ reads $g(A)=q(S)-p_g(S)$ and so we fall into Theorem 3. 2.

3. 4 Remark. If we drop the assumption $h^0(A) \geqq 2$, then the class of pairs (S, A) such that $g(A) = q(S)$ is larger than $\mathscr{A} \cup \mathscr{B}$. To see this it is enough to consider the pair $(J(A), A)$ where A is a smooth curve of genus two embedded in its jacobian $J(A)$.

3. 5 Proposition. *Let (S, H) be a polarized surface; then $h^0(K_S + H) \geqq 1$ if and only if $(S, H) \notin \mathscr{A} \cup \mathscr{B}$. Moreover if $h^0(K_S + H) = 1$ then either $(S, H) \in \mathscr{D}$ or $g(H) = 2$ and S is not of general type.*

Proof. The first fact is a trivial consequence of Theorem 3. 2. Assume

$$h^0(K_S + H) = p_g(S) + g(H) - q(S) = 1. \tag{3. 5. 1}$$

If $g(H) < 2$ then $(S, H) \in \mathscr{D}$ by Corollaries 2. 3 and 2. 4. Now assume $g(H) \geqq 2$; then (3. 5. 1) implies $\chi(\mathscr{O}_S) \leqq 0$ and $\chi(\mathscr{O}_S) < 0$ when $g(H) \geqq 3$. Hence S cannot be of general type. Assume, by contradiction, $g(H) \geqq 3$. Then S is a ruled surface (cf. [1], p. 154) and in this case (3. 5. 1) gives

$$g(H) = q(S) + 1. \tag{3. 5. 2}$$

Now, since $(S, H) \notin \mathscr{B}$, Theorem 2. 1 shows that $H^2 \leqq 4\big(g(H) - 1\big) + K_S^2$. On the other hand, as $S \not\cong \mathbb{P}^2$ in view of (3. 5. 2), we have $K_S^2 \leqq 8\big(1 - q(S)\big)$. So, by using (3. 5. 2) we get the inequality $H^2 \leqq 8 - 4q(S)$ which contradicts the ampleness of H, as $q(S) \geqq 2$ in view of (3. 5. 2).

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