*) Premise. L very ample $\Rightarrow K_S+L$ spanned unless $(S,L) = \{ \mathbb{P}^2, \mathcal{O}(e)),\ e=1,2$ or scroll/smooth curve (including $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}}(1))$)

Pf. If $L^2 \geq 5$ use Reider's theorem, first part. If $L^2 \leq 4$ use the classification of surfaces of degree ≤ 4 (Weil & Swinnerton-Dyer) [BS, Prop. 8.10.1 p. 243]

Let $\phi = \phi_{K_S+L}$ be a morphism; then $\dim \phi(S) \leq 1$ for (S,L) C.D.U.B. as shown discussing adjunction for polar surfaces [LP, Thm 2.1].

The Sommese - Van de Ven theorem on the adjunction mapping

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ABSTRACT. A proof of the theorem in the title based on Reider's theorem is given with some variations with respect to the original one.

Premise

Let L be a very ample line bundle on a smooth complex projective surface S. Assume that the adjoint bundle $K_S + L$ is spanned and that the adjunction mapping ϕ has a 2-dimensional image. Let $\phi = s \circ r$ be the Stein factorization and let (S', L') be the reduction of (S, L). Then the reduction morphism $r : S \to S'$ contracts all the (-1)-lines $E_1, \dots, E_k$ of (S, L) to distinct points $x_1, \dots, x_k \in S'$. We say that (S', L') is the simple reduction of (S, L) to mean that r contracts a single (-1)-line. The theorem of Sommese and Van de Ven [SV] says the following.

Theorem. *Let things be as above; then $K_{S'} + L'$ is very ample unless (S, L) is one of the following pairs:*

a) *S is a Del Pezzo surface with $K_S^2 = 2$ and $L = -2K_S$;* $L^2=8$

b) *(S, L) has the pair in a) as a simple reduction;* $L^2=7$

c) *S is a Del Pezzo surface with $K_S^2 = 1$ and $L = -3K_S$;* $L^2=9$

d) *S is the $\mathbb{P}^1$-bundle of invariant -1 over an elliptic curve and $L \equiv 3C_0$ (C_0 being a section of minimal self-intersection).* $L^2=9$

The adjunction mapping of (S, L) is a double cover $\phi : S \to \mathbb{P}^2$ branched along a smooth plane quartic in case a), a double cover of the quadric cone $\phi : S \to \mathbb{Q}_0$, branched at the vertex and along the transverse intersection with a cubic surface in case c), and a triple cover $\phi : S \to \mathbb{P}^2$ in case d).

Proof. The proof consists of two parts according to whether $L'^2 \geq 9$ or $L'^2 \leq 8$.

A) If $L'^2 \geq 9$ then (S, L) is as in c) or d). We prove this in two steps.

by N-M

1) Since L' is ample, Reider's theorem [R] shows that if $K_{S'} + L'$ is not very ample, then there exists an effective divisor D' on S' such that one of the following cases occurs:

(i) $L'D' = 1$ and $D'^2 = 0$ or -1,

(ii) $L'D' = 2$ and $D'^2 = 0$, or

(iii) $L' \equiv 3D'$ and $D'^2 = 1$.

Cases (i) and (ii) will be ruled out by looking at the proper transform $D = r^{-1}(D')$ to get information on D' which contrast the ampleness of $K_{S'} + L'$. Case (iii)

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will lead to pairs in c), d). We have $r^*D' = D + \sum_{i=1}^{k} \nu_i E_i$, where $\nu_i = DE_i = \mathrm{mult}_{x_i}(D') \geq 0$. So

$$(*) \qquad 1 \leq LD = (r^*L' - \sum_{i=1}^{k} E_i)(r^*D' - \sum_{i=1}^{k} \nu_i E_i) \leq L'D' - \sum_{i=1}^{k} \nu_i \leq L'D'.$$

In case (i), from $(*)$ we get $1 \leq LD \leq L'D' = 1$, hence D is a line of (S, L) and D' is a smooth $\mathbb{P}^1$ not containing any x_i. From the genus formula we thus get $D'K_{S'} = -2 - D'^2 \leq -1$ and then $(K_{S'} + L')D' \leq 0$, contradicting the ampleness of $K_{S'} + L'$.

In case (ii), if D' is irreducible, $(*)$ shows that D is a line or a conic of (S, L) and $D' \cong D \cong \mathbb{P}^1$. So, as before we get $D'K_{S'} = -2 - D'^2 = -2$ and then $(K_{S'} + L')D' = 0$, contradiction. If D' is reducible, then due to the ampleness of L' it can only be $D' = D'_1 + D'_2$, with D'_j irreducible and $L'D'_j = 1$. Set $D_j = r^{-1}(D'_j)$. By applying $(*)$ to D'_j we get $1 \leq LD_j \leq L'D'_j = 1$, hence D_j is a line of (S, L) and D'_j is a smooth $\mathbb{P}^1$ not containing any x_i. So either $D'_1 = D'_2$ in which case $D'^2_j = 0$, or $D'_1 D'_2 = D_1 D_2 = 0$ or 1, because two lines cannot have more than one point in common. Then

$$0 = D'^2 = (D'_1 + D'_2)^2 \leq D'^2_1 + 2 + D'^2_2.$$

But then we get

$$(K_{S'} + L')D' = K_{S'}D'_1 + K_{S'}D'_2 + \overset{\overset{2}{\shortparallel}}{L'D'} = -2 - D'^2_1 - 2 - D'^2_2 + 2 = -(D'^2_1 + 2 + D'^2_2),$$

which, in view of the above inequality, gives again $(K_{S'} + L')D' \leq 0$, contradiction.

In case (iii) D' is an ample divisor and then it is an irreducible reduced curve, since $D'^2 = 1$. Assume that $D' \ni x_1$. Then from $(*)$ we get

$$1 \leq LD = L'D' - \sum_{i=1}^{k} \nu_i = 3 - \sum_{i=1}^{k} \nu_i,$$

hence, as to the positive multiplicities, it can only be

$\nu_1 = 1$,
$\nu_1 = \nu_2 = 1$, or
$\nu_1 = 2$,

and accordingly we get $LD = 2$, 1, or 1. In all cases D is a smooth $\mathbb{P}^1$, being a conic or a line of (S, L), and in the first two cases $D' \cong D$. So in the first two cases we get, by genus formula, $(K_{S'} + L')D' = -2 - D'^2 + L'D' = 0$, contradicting again the ampleness of $K_{S'} + L'$. In the third case, D would be a line of (S, L). But also E_1 is a line and $DE_1 = \nu_1 = 2$, which is clearly absurd. Therefore D' does not contain any of the points x_i. Thus $LD = L'D' = 3$ by $(*)$, hence D' is isomorphic either to a space or to a plane cubic curve, so its arithmetic genus is ≤ 1. But if D' were rational, as before we would get $D'K_{S'} = -2 - D'^2 = -3$, giving again $(K_{S'} + L')D' = 0$, a contradiction. Then the arithmetic genus of D' is 1 and by applying [LP, Corollary 2.4] we conclude that $(S', [D'])$ is either a Del Pezzo surface with $[D'] = -K_{S'}$ or a scroll over an elliptic curve B. In the former case (S', L') is as in c). In the latter case $S' = \mathbb{P}_B(\mathcal{V})$ for some rank-2 vector bundle $\mathcal{V}$ on B, $[D']$ being the tautological bundle. Consider the exact sequence

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(D') \to \mathcal{O}_{D'}(D') \to 0.$$

By taking the direct images we get the sequence on B

$$0 \to \mathcal{O}_B \to \mathcal{V} \to \mathcal{O}_B(y) \to 0$$

for some point $y \in B$. Note that $\mathcal{V}$ is ample, since so is $[D']$; therefore the above sequence cannot split. This shows that the invariant of S' is -1 and so (S', L') is as in d).

2) If (S', L') is as in c) or d), then $(S, L) = (S', L')$.

Let (S', L') be as in c). We have $h^0(-K_{S'}) = 2$ and since $-K_{S'}$ is ample with $K_{S'}^2 = 1$, the pencil $|-K_{S'}|$ has a single base point and all its elements are irreducible reduced curves. Assume that r is not an isomorphism and let $G' \in |-K_{S'}|$ be a possibly singular curve passing through x_1. Let $G = r^{-1}(G')$ and set $\mu_i = GE_i = \mathrm{mult}_{x_i}(G')$. So $\mu_i \geq 0$ and then

$$1 \leq LG = (r^*L' - \sum_{i=1}^{k} E_i)(r^*G' - \mu_1 E_1 - \sum_{i=2}^{k} \mu_i E_i) = L'G' - \mu_1 - \sum_{i=2}^{k} \mu_i \leq 3 - \mu_1.$$

Hence the positive multiplicities can only be either

$\mu_1 = 2$,
$\mu_1 = \mu_2 = 1$, or
$\mu_1 = 1$.

In the first case $LG = 1$, so G is a line of (S, L). But also E_1 is a line and $GE = \mu_1 = 2$, which is impossible. In the last two cases $LG \leq 2$ so that G is a smooth $\mathbb{P}^1$. But then $G' \cong G$ would be a smooth rational curve, contradicting the fact that $g(G) = g(-K_{S'}) = 1$.

Let (S', L') be as in d) and assume that r is not an isomorphism. The algebraic system of C_0 contains an elliptic curve passing through x_1. Let G be its proper transform; then $LG = 3 - 1 = 2$, contradicting the very ampleness of L.

This concludes the proof of part A).

B) Let $L'^2 \leq 8$.

We have $d := L^2 (\leq L'^2) \leq 8$ and equality implies that $(S, L) = (S', L')$. In this case the proof requires several steps.

3) If (S', L') is as in a), then r consists of a single contraction at most, so if (S, L) is not as in a), then it can only be as in b); moreover case b) is effective.

To prove the first assertion it is enough to show that if r contracts two (-1)-lines of (S, L), then L cannot be very ample. Actually this is true "a fortiori" if skip

$k > 2$. Let $\mathfrak{M}_{x_i}$ be the ideal sheaf defining x_i, $i = 1, 2$. By the Leray spectral sequence we know that

$$h^0(L) = h^0(r^*L' - E_1 - E_2) = h^0(-2K_{S'} \otimes \mathfrak{M}_{x_1} \otimes \mathfrak{M}_{x_2}).$$

On the other hand, from the exact sequence

$$0 \to -2K_{S'} \otimes \mathfrak{M}_{x_1} \otimes \mathfrak{M}_{x_2} \to -2K_{S'} \to \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2} \to 0,$$

due to the very ampleness of $-2K_{S'}$, we get

$$h^0(-2K_{S'} \otimes \mathfrak{M}_{x_1} \otimes \mathfrak{M}_{x_2}) = h^0(-2K_{S'}) - 2 = 7 - 2 = 5.$$

So $|L|$ would embed S in $\mathbf{P}^4$. On the other hand we have $d = L'^2 - 2 = 6$, $g(L) = g(-2K_{S'}) = 3$, $\chi(\mathcal{O}_S) = 1$, and putting this information into the double point formula for surfaces in $\mathbf{P}^4$

$$d(d-10) + 12\chi(\mathcal{O}_S) = 2K_S^2 + 5LK_S$$

[H, p. 434] we get a numerical contradiction.

To prove that case b) is effective we have to show that L is in fact very ample. So let (S', L') be the pair in a). Having $k = 1$, let us put for shortness $E = E_1$. Then $L = r^*(-2K_{S'}) - E$ can be rewritten as $K_S + \mathcal{L}$, where

$$(\#) \qquad \mathcal{L} = L - K_S = r^*(-3K_{S'}) - 2E = r^*(-K_{S'}) + 2(r^*(-K_{S'}) - E).$$

As $-K_{S'}$ is ample and spanned, we have that $r^*(-K_{S'}) - E$ is nef. Hence $\mathcal{L}$, which is the sum of three nef line bundles, is nef. Since $\mathcal{L}^2 = 18 - 4 = 14 > 9$, we can use Reider's theorem again to prove the very ampleness of L. Note that $\mathcal{L}E = -2E^2 = 2$; on the other hand for any irreducible curve $\Gamma \subset S$ not contracted by r, letting $\Gamma' = r(\Gamma)$ and taking into account (#), we have

$$\mathcal{L}\Gamma \geq r^*(-K_{S'})\Gamma = (-K_{S'})\Gamma' > 0$$

because $-K_{S'}$ is ample. This shows that $\mathcal{L}$ is ample. Recalling that $\mathcal{L}^2 = 14$, if L is not very ample, then by Reider's theorem [R] there exists an effective divisor D on S such that either

$\mathcal{L}D = 1$ and $D^2 = -1, 0$, or
$\mathcal{L}D = 2$ and $D^2 = 0$.

In the former case, from (#) we get

$$1 = \mathcal{L}D = r^*(-K_{S'})D + 2(r^*(-K_{S'}) - E)D = r^*(-K_{S'})D + 2(-K_S)D.$$

Then, due to the nefness of the three summands appearing on the right hand of (#) we have $K_S D = 0$. Thus, since $-K_S$ is nef and big, the Hodge index theorem implies that $D^2 < 0$. So $D^2 = -1$, but this contradicts the genus formula. In the latter case, from (#) we get either $-K_S D = 1$, which once again contradicts genus formula, or $K_S D = 0$, which by the Hodge index theorem would imply $D^2 < 0$, a contradiction.

4) Let $g := g(L)$. Note that we can assume $g \geq 2$, since $\dim \phi(S) = 2$ and in view of the classif. for $g \leq 1$, done in the setting of polar surf. We have

Lemma 1. *If $g \leq 3$ then either $K_{S'} + L'$ is very ample or (S', L') is as in* a).

Proof. We can assume $d \geq 5$ in view of the known classification of surfaces of degree ≤ 4. Thus $LK_S < 0$, hence S is ruled. Since $\dim \phi(S) = 2$ we have $h^0(K_S + L) \geq 3$. Combining this with the Riemann-Roch and the Kodaira vanishing theorems we get

$$3 \leq h^0(K_S + L) = g - q \leq 3 - q,$$

hence $q = 0$, $g = 3$, and $\phi(S) = \mathbb{P}^2$. Consider a smooth curve $C \in |L|$. Note that C is not a plane curve, and $h^1(L_C) = 0$ because $\deg L_C = d \geq 5 > 2g - 2$. We thus get from Riemann-Roch

$$4 \leq h^0(L_C) = d + 1 - g.$$

So $d \geq 6$ and then the Hodge index theorem gives

$$16 = (2g-2)^2 = ((K_S + L)L)^2 \geq (K_S + L)^2 L^2 \geq 6(K_S + L)^2.$$

In view of the Stein factor. $\phi = s \circ r$, we know that $s = \phi_{K_{S'}+L'}$ is a morphism

Then $(K_{S'} + L')^2 = (K_S + L)^2 \leq 2$. On the other hand $K_{S'} + L'$ is ample and spanned and so, looking at the morphism $s : S' \to \mathbb{P}^2$ it defines, we get two possibilities: either

$(K_{S'} + L')^2 = 1$, in which case s gives an isomorphism $(S', K_{S'} + L') \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and then $K_{S'} + L'$ is very ample, or

$(K_{S'} + L')^2 = 2$, in which case s is a double cover. Let $2b$ be the degree of the branch locus. Thus

$$L' = (K_{S'} + L') - K_{S'} = s^*\mathcal{O}_{\mathbb{P}^2}(1) - s^*\mathcal{O}_{\mathbb{P}^2}(b-3) = s^*\mathcal{O}_{\mathbb{P}^2}(4-b)$$

and so genus formula gives

$$4 = 2g - 2 = (K_{S'} + L')L' = s^*\mathcal{O}_{\mathbb{P}^2}(1)\ s^*\mathcal{O}_{\mathbb{P}^2}(4-b) = 2(4-b).$$

Hence $b = 2$, which says that $s : S' \to \mathbb{P}^2$ is the Del Pezzo double plane. Having $L' = s^*\mathcal{O}_{\mathbb{P}^2}(2) = 2(K_{S'} + L')$, we conclude that (S', L') is the pair in a). □

Due to Lemma 1 we can continue the proof assuming $K_{S'} + L'$ not very ample and $g \geq 4$.

5) Let $h^0(L) = n + 1$. If $n = 3$, since $d \geq 5$ we know that $K_S + L$ itself is very ample, a contradiction. If $n = 4$ we can use the known classification of surfaces in $\mathbb{P}^4$ of degree ≤ 8 to do the same. So we can assume $n \geq 5$ and apply the Castelnuovo inequality

$$d \geq \frac{n}{2} + \sqrt{2(n-2)g + \varepsilon},$$

where $\varepsilon = 0$ or $\frac{1}{4}$ according to whether $n - 4$ is even or odd. This, combined with $g \geq 4$ (see the end of step 4), gives only the following possibilities: $d = 8$, $n = 5$, and $g = 4$ or 5, and then $(S, L) = (S', L')$, by what we said at the beginning of part B). But in case $g = 5$, for any smooth curve $C \in |L|$ we have $L_C = K_C$. So, since

skip and go to the conclusion

(S, L) is not a scroll, Weil's equivalence criterion [So, (0.9)] applies and shows that $K_S = \mathcal{O}_S$; so $K_S + L$ is very ample, contradiction. Therefore

$$d = 8, \quad n = 5, \quad \text{and} \quad g = 4.$$

In particular, S is ruled.

because $LK_S = 2g - 2 - d$ $= 8 - 6 - 8 < 0$

6) Let $g = 4$. Since $K_{S'} + L'$ is ample and spanned we have

$$3 \le h^0(K_{S'} + L') = g - q,$$

hence $q \le 1$. The Hodge theorem gives

$$36 = (2g-2)^2 = ((K_S + L)L)^2 \ge (K_S + L)^2 L^2 = 8(K_S + L)^2,$$

hence $(K_S + L)^2 \le 4$. On the other hand $(K_S + L)^2 \ge 1$ due to the ampleness.

When $(K_S + L)^2 \le 2$ we easily get contradictions. Actually, if $(K_S + L)^2 = 1$, then ϕ gives an isomorphism $(S, K_S + L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, contradicting $g = 4$. On the other hand if $(K_S + L)^2 = 2$ then either ϕ gives the isomorphism $(S, K_S + L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$, or $\phi : S \to \mathbb{P}^2$ is a double cover. In the former case we get $L = \mathcal{O}(3,3)$, contradicting the fact that $L^2 = 8$. In the latter case let $2b$ be the degree of the branch locus; then $K_S + L = \phi^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $K_S = \phi^* \mathcal{O}_{\mathbb{P}^2}(b - 3)$. So, from

$$6 = 2g - 2 = (K_S + L)L = \phi^* \mathcal{O}_{\mathbb{P}^2}(1)\ \phi^* \mathcal{O}_{\mathbb{P}^2}(4 - b) = 2(4 - b)$$

we get $b = 1$, while condition

$$8 = d = L^2 = (\phi^* \mathcal{O}_{\mathbb{P}^2}(4 - b))^2 = 2(4 - b)^2$$

gives $b = 2$.

When $(K_S + L)^2 = 3$ we also get contradictions. First of all note that

$$2g(K_S + L) - 2 = 2(K_S + L)^2 - (K_S + L)L = 6 - (2g - 2) = 0.$$

So $K_S + L$ is an ample line bundle with $g(K_S + L) = 2$. By applying again [LP, 1
Corollary 2.4] we conclude that $(S, K_S + L)$ is either a scroll over an elliptic curve, or a Del Pezzo surface with $K_S + L = -K_S$. In the former case it would be $K_S^2 = 0$, while, from $3 = (K_S + L)^2 = K_S^2 + 4$ we see that $K_S^2 = -1$, a contradiction. In the latter case, from $3 = (K_S + L)^2 = (-K_S)^2$ we would get $8 = L^2 = (-2K_S)^2 = 12$, another contradiction.

7) Let $(K_S + L)^2 = 4$ and $q = 1$. Then $K_S^2 = 0$, hence S is a $\mathbb{P}^1$-bundle over an elliptic curve. Writing $L \equiv aC_0 + bf$ as in [H, p. 382] and using $L^2 = 8$, $LK_S = -2$ and the ampleness conditions, we thus get $L \equiv 4C_0 + (2e + 1)f$, the invariant e being 0 or -1. But then $LC_0 = 1$ in case $e = 0$, while $LE = 2$, for E a smooth curve of genus 1 and $\equiv -K_S$ in case $e = -1$. In both cases this contradicts the very ampleness of L.

8) Let $(K_S + L)^2 = 4$ and $q = 0$. Look at the morphism $\phi : S \to \mathbb{P}^3$. Since $K_S + L$ is not very ample there are points $x, y \in S$ (y possibly infinitely near to x) such that $\phi(x) = \phi(y)$.

Lemma 2. *If ϕ does not separate points $x, y \in S$ (y possibly infinitely near to x), then there exists a smooth curve $C \in |L - x - y|$.*

skip

Proof. If such a curve does not exist, then (S, L) contains a line l with $x, y \in l$, by [So, (0.10.2)]. Now $L - l$ is spanned, hence nef. Think of S as embedded in $\mathbb{P}^n$ via $|L|$. If $(L - l)^2 = 0$ then by projecting S from l into a $\mathbb{P}^{n-2}$ we get a morphism $\pi : S \to \Gamma = \pi(l) = \mathbb{P}^1$ and S can be viewed as a divisor inside the projective cone of vertex l over Γ. By blowing-up $\mathbb{P}^n$ along l and taking the proper transform of this cone, we get the following situation described in [BS, Theorem 8.4.5]. Let $P := \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$; then

$$S = aH + F \quad \text{in } \mathrm{Pic}(P),$$

where H is the tautological bundle on P, F is a fibre and $a > 0$. Moreover $L = H_S$. Thus

$$K_S = (K_P + [S])_S = ((a-3)H + (n-3)F)_S$$

by the canonical bundle formula. For $a = 1, 2$ (S, L) would be a scroll or a conic bundle respectively; hence $a \geq 3$, since $\dim \phi(S) = 2$. But then

$$K_S + L = ((a-2)H + (n-3)F)_S$$

would be the restriction to S of a very ample line bundle on P [BS, Lemma 3.2.4], a contradiction. It thus follows that $L - l$ is nef and big and so we get

$$H^1(K_S + L - l) = 0$$

by the Kawamata vanishing theorem. Note that $(K_S + L)_l = \mathcal{O}_{\mathbb{P}^1}(k)$ with $k > 0$, due to the ampleness of $K_S + L$. So $(K_S + L)_l$ is in fact very ample on l and then from the exact cohomology sequence of

$$0 \to K_S + L - l \to K_S + L \to (K_S + L)_l \to 0$$

we conclude that ϕ separates x and y, a contradiction. $\square$

So by Lemma 2 there exists a smooth curve $C \in |L|$ through x and y such that $\phi(x) = \phi(y)$. As S is ruled and $q = 0$, the exact cohomology sequence of

$$0 \to K_S \to K_S + L \to K_C \to 0$$

shows that $\phi_{|C} = \varphi$, the canonical map of C. Then $\varphi(x) = \varphi(y)$, so that C is hyperelliptic. But then a result of Sommese claims that there exists at least a 1-dimensional family $\mathcal{S} \subset |L|$ of hyperelliptic curves [BS, Theorem 8.4.2]. Thus ϕ cannot be generically one-to-one and so $\deg \phi \geq 2$. $\dim \mathcal{S} \geq N-2 \geq 1$

9) Due to 8), from

$$4 = (K_S + L)^2 = \deg \phi \cdot \deg \phi(S)$$

we conclude that ϕ is a finite morphism of degree 2 onto $\phi(S)$, which is either $\mathbb{Q} := \mathbb{P}^1 \times \mathbb{P}^1$ or the quadric cone $\mathbb{Q}_0$. Moreover from the equality $4 = (K_S + L)^2 = K_S^2 + 4$ we know that $K_S^2 = 0$.

Let $\phi(S) = \mathbb{Q}_0$; then since S is smooth, ϕ is branched at the vertex v of $\mathbb{Q}_0$. Looking at the double cover $\pi : \bar{S} \to \mathbb{F}_2$ induced by the desingularization $\mathbb{F}_2 \to \mathbb{Q}_0$ we thus see that $\bar{S}$ is S blown-up at the point $\phi^{-1}(v)$; so $K_{\bar{S}}^2 = K_S^2 - 1 = -1$. On the other hand $K_{\bar{S}} = \pi^*(K_{\mathbb{F}_2} + B)$, for some line bundle B on $\mathbb{F}_2$. This shows that $K_{\bar{S}}^2$ has to be even, a contradiction.

Finally let $\phi(S) = \mathbb{Q}$ and let $\mathcal{O}_{\mathbb{Q}}(2a, 2b)$ be the class of the branch divisor of ϕ. Computing K_S, we get $0 = K_S^2 = 2(a-2)(b-2)$, so that, up to exchanging the rulings of $\mathbb{Q}$, we can assume $a = 2$. Thus $L = (K_S + L) - K_S = \phi^*\mathcal{O}_{\mathbb{Q}}(1, 3-b)$ and from

$$6 = 2g - 2 = (K_S + L)L = \phi^*\mathcal{O}_{\mathbb{Q}}(1,1)\ \phi^*\mathcal{O}_{\mathbb{Q}}(1, 3-b) = 2(4-b)$$

we get $b = 1$. So $L = \phi^*\mathcal{O}_{\mathbb{Q}}(1,2)$. Now, recalling that

$$\phi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{Q}} \oplus \mathcal{O}_{\mathbb{Q}}(-a,-b) = \mathcal{O}_{\mathbb{Q}} \oplus \mathcal{O}_{\mathbb{Q}}(-2,-1),$$

we get

$$h^0(L) = h^0(\phi_*L) = h^0(\phi_*\mathcal{O}_S \otimes \mathcal{O}_{\mathbb{Q}}(1,2)) = h^0(\mathcal{O}_{\mathbb{Q}}(1,2)).$$

This says that $|L| = \phi^*|\mathcal{O}_{\mathbb{Q}}(1,2)|$. So the embedding given by $|L|$ would factor through ϕ, which has degree 2, a contradiction.

This concludes the proof of part B) and of the theorem.

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