The Carathéodory theorem and its relatives
(Il Teorema di Carathéodory ed i suoi parenti)

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Theorem (C. Carathéodory, \(\sim 1907\))

Let \(A \subset \mathbb{R}^d\) and \(x \in \text{conv} \ A\). Then there is \(A_0 \subset A\) such that

\[|A_0| \leq d + 1\] and \(x \in \text{conv} \ A_0\).

By the \textit{dimension} of a set \(A\) in a vector space we mean the dimension of its affine hull, \(\text{aff} \ A\), i.e., the “algebraic dimension” of \(A\).

Corollary

Let \(A\) be a finite-dimensional set in a Hausdorff t.v.s. Then:

(a) every point of \(\text{conv} \ A\) is a convex combination of at most \((\dim A) + 1\) points of \(A\); [This is just a reformulation.]

(b) if \(A\) is compact then also \(\text{conv} \ A\) is. [Important corollary!]
Constantin Carathéodory
a Greek mathematician
Berlin 1873 – Munich 1950

Worked in:
- Calculus of Variations
- ODE (existence theorem)
- Measure Theory (Carathéodory extension theorem)
- Complex Analysis (conformal mappings)
- Thermodynamics
- Optics
“One point of the convex combination can be chosen in advance!”

Theorem (Carathéodory with one point given)

\[ A \subset \mathbb{R}^d, \ a_0 \in A, \ x \in \text{conv} \ A. \text{ Then there exists } A_1 \subset A \text{ such that } \]
\[ |A_1| \leq d \text{ and } x \in \text{conv}(\{a_0\} \cup A_1). \]
Proof. WLOG: \( a_0 = 0 \). Denote \( A' := A \setminus \{0\} \). We can write 
\[ x = \lambda_0 \cdot 0 + \sum_{i=1}^{n} \lambda_i a_i \] (convex combination) with \( a_i \in A', \lambda_i > 0 \).
Assume \( n > d \) and it cannot be taken smaller. Then there exists 
\[ \sum_{i=1}^{n} \alpha_i a_i = 0, \text{ a nontrivial linear combination. WLOG: } \sum_{i=1}^{n} \alpha_i \geq 0. \]
Clearly \( \alpha_i > 0 \) for at least one \( i \).

For \( t > 0 \):
\[ x = \sum_{1}^{n} \lambda_i a_i = \sum_{1}^{n} (\lambda_i - t\alpha_i) a_i, \quad \sum_{1}^{n} (\lambda_i - t\alpha_i) \leq \sum_{1}^{n} \lambda_i \leq 1. \]
Choosing
\[ t := \min\{ (\lambda_i/\alpha_i) : \alpha_i > 0 \} \]
we have also \( (\lambda_i - t\alpha_i) \geq 0 \) for each \( i \), and = 0 for at least one of them, say \( k \), (one of those for which inf is attained). Then
\[ x = \sum_{1 \leq i \leq n, i \neq k} (\lambda_i - t\alpha_i) a_i \in \text{conv} \left( \{0\} \cup \{a_i\}_{i \neq k} \right). \]

Contradiction with minimality of \( n \)! \quad [q.e.d.]
Given a set $A$ in a vector space, its convex-cone hull, $ccone A$, is the smallest convex cone containing $A$. It is easy to see that

$$ccone A = \left\{ \sum_{i=1}^{n} t_i a_i : n \in \mathbb{N}, t_i \geq 0, a_i \in A \right\}.$$ 

**Theorem (Carathéodory for convex-cone hulls)**

If $A \subset \mathbb{R}^d$, $x \in ccone A$. Then there is $A_0 \subset A$ such that

$$|A_0| \leq d \quad \text{and} \quad x \in ccone A_0.$$ 

Same proof!
Theorem (Carathéodory for connected sets; Fenchel 1929, Bunt 1934)

Let $A \subset \mathbb{R}^d$ be connected (or having at most $d$ connected components) and $x \in \text{conv } A$. Then there exists $A_0 \subset A$ such that

$$|A_0| \leq d \quad \text{and} \quad x \in \text{conv } A_0.$$
Proof.

WLOG: $x = 0$ (translation), $|A| \geq d + 1$.
Assume that 0 is not a convex combination of $d$ points of $A$.
By Carathéodory, $0 \in \text{conv } B =: C$ for some $B \subset A$, $|B| \leq d + 1$.
Necessarily, $|B| = d + 1$, $B$ is affinely independent (hence $C$ is a $d$-simplex), $0 \in \text{int } C$. Let $F_0, \ldots, F_d$ be the maximal (i.e., $(d - 1)$-dimensional) faces of $C$, and for $0 \leq i \leq d$ let
\[ \Gamma_i = \text{int}(ccone(-F_i)). \]
Notice that clearly $\bigcup_0^d \Gamma_i = \mathbb{R}^d$.

Claim 1. $A \subset \bigcup_0^d \Gamma_i$.
(If $\bar{a} \in A \cap \partial \Gamma_k$, we can write $\bar{a} = \sum_1^m t_j(-b_j)$ with $m < d$, $b_j \in B \cap F_k$, $t_j > 0$. Then $0 = \bar{a} + \sum_1^m t_j b_j$, from which easily $0 \in \text{conv}\{\bar{a}, b_1, \ldots, b_m\}$. Contradiction since $m + 1 \leq d$!)

Claim 2. $A \cap \Gamma_i \neq \emptyset$ for each $0 \leq i \leq d$.
($B \setminus F_i =: \{\bar{b}\}$. Then $\bar{b} \in B \cap \Gamma_i \subset A \cap \Gamma_i$.)
It follows that $A$ has at least $d + 1$ connected components. Contradiction!
Intermezzo 2: Carathéodory with colors

Theorem ("Colorful Carathéodory"; Bárany 1982)

\[ A_0, \ldots, A_d \subset \mathbb{R}^d \text{ (not necessarily disjoint), } x \in \bigcap_0^d \text{conv } A_i. \]

Then there exists a choice

\[ (a_0, \ldots, a_d) \in A_0 \times \cdots \times A_d \]

such that \( x \in \text{conv}\{a_0, \ldots, a_d\}. \)

Observation

The (classical) Carathéodory theorem follows by taking
\( A_0 = \cdots = A_d = A. \)
A curiosity:
“Colorful Carathéodory” with infinitely many colors!? 

Theorem (E. Behrends 2000)

Let $X$ be an Asplund Banach space (that is, if $Y \subset X$ is a separable subspace then also $Y^*$ is separable). Let

$$A_n \subset X \quad (n \in \mathbb{N})$$

be uniformly bounded sets, and let $x \in \bigcap_{n \in \mathbb{N}} \overline{\text{conv}} A_n$.

Then there is a choice

$$x_n \in A_n \quad (n \in \mathbb{N})$$

such that $x \in \overline{\text{conv}} \{x_n\}_{n \in \mathbb{N}}$. 
**Theorem (Carathéodory with interiors; E. Steinitz 1913–1916)**

Let $A \subset \mathbb{R}^d$, $x \in \text{int}(\text{conv } A)$.

Then there is $A_0 \subset A$ such that

\[ |A_0| \leq 2d \quad \text{and} \quad x \in \text{int}(\text{conv } A_0). \]

**Observation**

The number $2d$ cannot be reduced.

Indeed, consider $A = \{ \pm e_i \}_{i=1}^d$ and $x = 0$. 

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Carathéodory and relatives
Proof (Carathéodory for interiors).

WLOG: $x = 0$, $|A| > 2d$.
WLOG: $A$ is finite (by Carathéodory!). Then $C := \text{conv } A$ is cpt. Let $J$ be the set of all linear combinations of at most $(d - 1)$ points of $A$, i.e., $J = \bigcup \{\text{span } B : B \subset A, |B| = d - 1\}$. There exists a line $L \subset \mathbb{R}^d$ such that $L \cap J = \{0\}$. Write $L \cap \partial C =: \{w_1, w_2\}$, and notice that $0 \in (w_1, w_2)$.

Fix $i \in \{1, 2\}$. Let $H_i \subset \mathbb{R}^d$ be a supporting hyperplane to $C$ at $w_i$. We must have $w_i \in \text{conv}(A \cap H_i)$. By Carathéodory (applied in $H_i$), there is $A_i \subset A \cap H_i$ such that $|A_i| \leq d$ and $w_i \in \text{conv } A_i$. Since $w_i \notin J$, we must have $|A_i| = d$ and it cannot be taken smaller! Hence $\text{conv } A_i$ is a $(d - 1)$-simplex, and $w_i \in \text{int}_{H_i}(\text{conv } A_i)$.

Then $0 \in (w_1, w_2) \subset \text{int}(\text{conv}(A_1 \cup A_2))$, and $A_0 := A_1 \cup A_2$ has at most $2d$ elements. We are done. Proof taken from: Danzer-Grünbaum-Klee, “Helly’s theorem and its relatives”, 1963.
A proof of Helly’s intersection theorem via the Carathéodory theorem

Theorem (Helly 1913 [publ. 1923]; first published by Radon 1921)

\[ \mathcal{F} \] a finite family of convex sets in \( \mathbb{R}^d \) such that \( \mathcal{F} \) is 
\((d + 1)\)-centered (i.e., \( \bigcap \mathcal{F}_0 \neq \emptyset \) for every \( \mathcal{F}_0 \subset \mathcal{F} \), \( |\mathcal{F}_0| \leq d + 1 \)).

Then \( \bigcap \mathcal{F} \neq \emptyset \).

Proof (via Carathéodory).

WLOG: \( |\mathcal{F}| > d + 1 \).

Claim 1. WLOG: \( \mathcal{F} \) consists only of compact (convex) sets.
Proof. For each \( \mathcal{F}_0 \subset \mathcal{F} \) with \( |\mathcal{F}_0| \leq d + 1 \), choose \( x_{\mathcal{F}_0} \in \bigcap \mathcal{F}_0 \).
Denote \( I := \{ x_{\mathcal{F}_0} \}_{\mathcal{F}_0} \). Substitute each \( C \in \mathcal{F} \) with
\( \tilde{C} := \text{conv}(I \cap C) \). Then the family \( \tilde{\mathcal{F}} := \{ \tilde{C} : C \in \mathcal{F} \} \) is
\((d + 1)\)-centered (because even \( \{ I \cap C \}_{C \in \mathcal{F}} \) is!), and \( \bigcap \tilde{\mathcal{F}} \subset \bigcap \mathcal{F} \).
OK
So let \( \mathcal{F} = \{ C_1, \ldots, C_N \} \), where each \( C_i \) is compact.
Continues. . . (proof of Helly via Carathéodory)

Proceeding by the contrary, assume that $\bigcap_{1}^{N} C_i = \emptyset$, i.e.,

$$f(x) := \max_{1 \leq i \leq N} d(x, C_i) > 0 \quad \text{for each } x \in \mathbb{R}^d.$$  

Easy (by compactness): $f$ assumes its global minimum at some $x_0 \in \mathbb{R}^d$. Denote

$$r := f(x_0) \quad \text{and} \quad K := \{1 \leq k \leq N : d(x, C_k) = r\}.$$  

For each $k \in K$ there is (a unique!) $c_k \in C_k$ s.t. $\|x_0 - c_k\| = r$.  

Claim 2. $x_0 \in \text{conv}\{c_k : k \in K\} =: D$. (See the picture!)

(Sketch. If not, separate by a hyperplane, and move a little from $x_0$ towards $D$ to some $x_1$; but this would imply $f(x_1) < f(x_0)$!)

By Carathéodory, there is $K_0 \subset K$ with $|K_0| \leq d + 1$, $x_0 \in \text{conv}\{c_k : k \in K_0\}$. Write

$$x_0 = \sum_{k \in K_0} \lambda_k c_k \quad \text{(convex combination)}.$$
Claim 3. (For $k \in K_0$ fixed.) $\forall y \in C_k, \langle c_k - x_0, y - c_k \rangle \geq 0$. 
(See the picture!)

Hence $\langle c_k - x_0, y - x_0 \rangle = \langle c_k - x_0, y - c_k \rangle + \langle c_k - x_0, c_k - x_0 \rangle \geq \|c_k - x_0\|^2 = r^2$.

Now, fix some $\bar{y} \in \bigcap_{k \in K_0} C_k$. Then

$$0 = \langle y - x_0, x_0 - x_0 \rangle = \sum_{k \in K_0} \lambda_k \langle y - x_0, c_k - x_0 \rangle \geq r^2 > 0.$$ 

Contradiction!

Thank you for your attention!