The Carathéodory theorem and its relatives
(Il Teorema di Carathéodory ed i suoi parenti)

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Theorem (C. Carathéodory, $\sim$ 1907)

Let $A \subset \mathbb{R}^d$ and $x \in \operatorname{conv} A$. Then there is $A_0 \subset A$ such that

$$|A_0| \leq d+1 \quad \textit{and} \quad x \in \operatorname{conv} A_0.$$

By the *dimension* of a set A in a vector space we mean the dimension of its affine hull, $\operatorname{aff} A$, i.e., the "algebraic dimension" of A.

Corollary

Let A be a finite-dimensional set in a Hausdorff t.v.s. Then:

(a) *every point of $\operatorname{conv} A$ is a convex combination of at most $(\dim A)+1$ points of A;* [This is just a reformulation.]

(b) *if A is compact then also $\operatorname{conv} A$ is.* [Important corollary!]

Carathéodory? Who?

Constantin Carathéodory
a Greek mathematician
Berlin 1873 – Munich 1950

Worked in:

- Calculus of Variations
- ODE (existence theorem)
- Measure Theory (Carthéodory extension theorem)
- Complex Analysis (conformal mappings)
- Thermodynamics
- Optics

"One point of the convex combination can be chosen in advance!"

Theorem (Carathéodory with one point given)

$A \subset \mathbb{R}^d$, $a_0 \in A$, $x \in \operatorname{conv} A$. Then there exists $A_1 \subset A$ such that

$$|A_1| \leq d \quad \textit{and} \quad x \in \operatorname{conv}(\{a_0\} \cup A_1).$$

Proof.

WLOG: $a_0 = 0$. Denote $A' := A \setminus \{0\}$. We can write $x = \lambda_0 \cdot 0 + \sum_{i=1}^n \lambda_i a_i$ (convex combination) with $a_i \in A'$, $\lambda_i > 0$. Assume $n > d$ and it cannot be taken smaller. Then there exists $\sum_1^n \alpha_i a_i = 0$, a *nontrivial* linear combination. WLOG: $\sum_1^n \alpha_i \geq 0$. Clearly $\alpha_i > 0$ for at least one i.

For $t > 0$:

$x = \sum_1^n \lambda_i a_i = \sum_1^n (\lambda_i - t\alpha_i) a_i\,, \quad \sum_1^n (\lambda_i - t\alpha_i) \leq \sum_1^n \lambda_i \leq 1.$

Choosing

$$t := \min\big\{(\lambda_i/\alpha_i) : \alpha_i > 0\big\}$$

we have also $(\lambda_i - t\alpha_i) \geq 0$ for each i, and $= 0$ for at least one of them, say k, (one of those for which inf is attained). Then

$$x = \sum_{1 \leq i \leq n, i \neq k} (\lambda_i - t\alpha_i) a_i \in \operatorname{conv}\big(\{0\} \cup \{a_i\}_{i \neq k}\big).$$

Contradiction with minimality of n! *[q.e.d.]*

Intermezzo 1: Convex-cone hull

Given a set A in a vector space, its *convex-cone hull*, $\operatorname{ccone} A$, is the smallest convex cone containing A. It is easy to see that

$$\operatorname{ccone} A = \left\{ \sum_{i=1}^{n} t_i a_i : n \in \mathbb{N}, t_i \geq 0, a_i \in A \right\}.$$

Theorem (Carathéodory for convex-cone hulls)

$A \subset \mathbb{R}^d$, $x \in \operatorname{ccone} A$. Then there is $A_0 \subset A$ such that

$$|A_0| \leq d \quad \textit{and} \quad x \in \operatorname{ccone} A_0.$$

Same proof!

Connected (or "not too disconnected") sets

Theorem (Carathéodory for connected sets; Fenchel 1929, Bunt 1934)

Let $A \subset \mathbb{R}^d$ be connected (or having at most d connected components) and $x \in \operatorname{conv} A$. Then there exists $A_0 \subset A$ such that

$$|A_0| \leq d \quad \textit{and} \quad x \in \operatorname{conv} A_0.$$

Proof.

WLOG: $x = 0$ (translation), $|A| \geq d + 1$.
Assume that 0 is *not* a convex combination of d points of A.
By Carathéodory, $0 \in \operatorname{conv} B =: C$ for some $B \subset A$, $|B| \leq d + 1$.
Necessarily, $|B| = d + 1$, B is affinely independent (hence C is a d-simplex), $0 \in \operatorname{int} C$. Let $F_0, \ldots, F_d$ be the maximal (i.e., $(d-1)$-dimensional) faces of C, and for $0 \leq i \leq d$ let

$$\Gamma_i = \operatorname{int}(\operatorname{ccone}(-F_i)).$$

Notice that clearly $\bigcup_0^d \overline{\Gamma}_i = \mathbb{R}^d$.

Claim 1. $A \subset \bigcup_0^d \Gamma_i$.
(If $\bar{a} \in A \cap \partial\Gamma_k$, we can write $\bar{a} = \sum_1^m t_j(-b_j)$ with $m < d$, $b_j \in B \cap F_k$, $t_j > 0$. Then $0 = \bar{a} + \sum_1^m t_j b_j$, from which easily $0 \in \operatorname{conv}\{\bar{a}, b_1, \ldots, b_m\}$. Contradiction since $m + 1 \leq d$!)

Claim 2. $A \cap \Gamma_i \neq \emptyset$ for each $0 \leq i \leq d$.
($B \setminus F_i =: \{\bar{b}\}$. Then $\bar{b} \in B \cap \Gamma_i \subset A \cap \Gamma_i$.)

It follows that A has at least $d + 1$ connected components.
Contradiction!

Intermezzo 2: Carathéodory with colors

Theorem ("Colorful Carathéodory"; Bárány 1982)

$A_0, \dots, A_d \subset \mathbb{R}^d$ (not necessarily disjoint), $x \in \bigcap_0^d \operatorname{conv} A_i$. Then there exists a choice

$$(a_0, \dots, a_d) \in A_0 \times \cdots \times A_d$$

such that $x \in \operatorname{conv}\{a_0, \dots, a_d\}$.

Observation

The (classical) Carathéodory theorem follows by taking $A_0 = \cdots = A_d = A$.

A curiosity: "Colorful Carathéodory" with infinitely many colors!?

Theorem (E. Behrends 2000)

Let X be an Asplund Banach space (that is, if $Y \subset X$ is a separable subspace then also Y^* is separable). Let

$$A_n \subset X \quad (n \in \mathbb{N})$$

be <u>uniformly bounded</u> *sets, and let* $x \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}} A_n$ *. Then there is a choice*

$$x_n \in A_n \quad (n \in \mathbb{N})$$

such that $x \in \overline{\operatorname{conv}}\{x_n\}_{n\in\mathbb{N}}$.

Interior of $\operatorname{conv} A$

Theorem (Carathéodory with interiors; E. Steinitz 1913–1916)

$A \subset \mathbb{R}^d$, $x \in \operatorname{int}(\operatorname{conv} A)$.
Then there is $A_0 \subset A$ *such that*

$$|A_0| \leq 2d \quad \textit{and} \quad x \in \operatorname{int}(\operatorname{conv} A_0).$$

Observation

The number $2d$ *cannot be reduced.*
Indeed, consider $A = \{\pm e_i\}_{i=1}^d$ *and* $x = 0$.

Proof (Carathéodory for interiors).

WLOG: $x = 0$, $|A| > 2d$.
WLOG: A is finite (by Carathéodory!). Then $C := \operatorname{conv} A$ is cpt.

Let J be the set of all linear combinations of at most $(d-1)$ points of A, i.e., $J = \bigcup\{\operatorname{span} B : B \subset A, |B| = d-1\}$.

There exists a line $L \subset \mathbb{R}^d$ such that $L \cap J = \{0\}$. Write $L \cap \partial C =: \{w_1, w_2\}$, and notice that $0 \in (w_1, w_2)$.

Fix $i \in \{1, 2\}$. Let $H_i \subset \mathbb{R}^d$ be a supporting hyperplane to C at w_i. We must have $w_i \in \operatorname{conv}(A \cap H_i)$. By Carathéodory (applied in H_i), there is $A_i \subset A \cap H_i$ such that $|A_i| \leq d$ and $w_i \in \operatorname{conv} A_i$. Since $w_i \notin J$, we must have $|A_i| = d$ and it cannot be taken smaller! Hence $\operatorname{conv} A_i$ is a $(d-1)$-simplex, and $w_i \in \operatorname{int}_{H_i}(\operatorname{conv} A_i)$.

Then $0 \in (w_1, w_2) \subset \operatorname{int}\big(\operatorname{conv}(A_1 \cup A_2)\big)$, and $A_0 := A_1 \cup A_2$ has at most $2d$ elements. We are done. *Proof taken from: Danzer-Grünbaum-Klee, "Helly's theorem and its relatives", 1963.*

A proof of Helly's intersection theorem via the Carathéodory theorem

Theorem (Helly 1913 [publ. 1923]; first published by Radon 1921)

$\mathcal{F}$ a finite family of convex sets in $\mathbb{R}^d$ such that $\mathcal{F}$ is $(d+1)$-centered (i.e., $\bigcap \mathcal{F}_0 \neq \emptyset$ for every $\mathcal{F}_0 \subset \mathcal{F}$, $|\mathcal{F}_0| \leq d+1$). Then $\bigcap \mathcal{F} \neq \emptyset$.

Proof *(via Carathéodory)***.**

WLOG: $|\mathcal{F}| > d+1$.

Claim 1. WLOG: $\mathcal{F}$ consists only of compact (convex) sets.
Proof. For each $\mathcal{F}_0 \subset \mathcal{F}$ with $|\mathcal{F}_0| \leq d+1$, choose $x_{\mathcal{F}_0} \in \bigcap \mathcal{F}_0$. Denote $I := \{x_{\mathcal{F}_0}\}_{\mathcal{F}_0}$. Substitute each $C \in \mathcal{F}$ with $\tilde{C} := \operatorname{conv}(I \cap C)$. Then the family $\tilde{\mathcal{F}} := \{\tilde{C} : C \in \mathcal{F}\}$ is $(d+1)$-centered (because even $\{I \cap C\}_{C \in \mathcal{F}}$ is!), and $\bigcap \tilde{\mathcal{F}} \subset \bigcap \mathcal{F}$. OK

So let $\mathcal{F} = \{C_1, \ldots, C_N\}$, where each C_i is compact.

Continues. . . (proof of Helly via Carathéodory)

Proceeding by the contrary, *assume* that $\bigcap_1^N C_i = \emptyset$, i.e.,

$$f(x) := \max_{1\le i\le N} \mathrm{d}(x, C_i) > 0 \quad \text{for each } x \in \mathbb{R}^d.$$

Easy (by compactness): f assumes its global minimum at some $x_0 \in \mathbb{R}^d$. Denote

$$r := f(x_0) \quad \text{and} \quad K := \big\{1 \le k \le N : \mathrm{d}(x, C_k) = r\big\}.$$

For each $k \in K$ there is (a unique!) $c_k \in C_k$ s.t. $\|x_0 - c_k\| = r$.

Claim 2. $x_0 \in \mathrm{conv}\{c_k : k \in K\} =: D$. *(See the picture!)*

(Sketch. If not, separate by a hyperplane, and move a little from x_0 towards D to some x_1; but this would imply $f(x_1) < f(x_0)$!)

By Carathéodory, there is $K_0 \subset K$ with $|K_0| \le d+1$, $x_0 \in \mathrm{conv}\{c_k : k \in K_0\}$. Write

$$x_0 = \sum_{k\in K_0} \lambda_k c_k \quad \text{(convex combination).}$$

Still continues. . . (proof of Helly via Carathéodory)

Claim 3. (For $k \in K_0$ fixed.) $\forall\, y \in C_k,\ \langle c_k - x_0, y - c_k \rangle \geq 0$.
(See the picture!)

Hence $\langle c_k - x_0, y - x_0 \rangle = \langle c_k - x_0, y - c_k \rangle + \langle c_k - x_0, c_k - x_0 \rangle \geq \|c_k - x_0\|^2 = r^2$.

Now, fix some $\bar{y} \in \bigcap_{k \in K_0} C_k$. Then

$$0 = \langle y - x_0, x_0 - x_0 \rangle = \sum_{k \in K_0} \lambda_k \langle y - x_0, c_k - x_0 \rangle \geq r^2 > 0.$$

Contradiction!

Thank you for your attention!