Assioma della scelta e Lemma di Zorn

L.V.

In questo capitolo ci limitiamo ad enunciare Assioma della scelta e Lemma di Zorn, aggiungendo poi un eccellente commento in inglese di C. D. Aliprantis e K. C. Border.

Assioma della scelta. Su ogni insieme non vuoto X esiste un selettore, cioè una funzione che ad ogni sottoinsieme non vuoto di X associa un suo elemento.

Un insieme X si dice parzialmente ordinato da una relazione binaria " $\leq$ " se questa relazione è

1. riflessiva ($x \leq x$ per ogni $x \in X$),
2. transitiva (se $x \leq y$ e $y \leq z$ allora $x \leq y$),
3. antisimmetrica (se $x \leq y$ e $y \leq x$ allora $x = y$).

Una catena in un insieme parzialmente ordinato $(X,\leq)$ è un sottoinsieme C di X sul quale la relazione " $\leq$ " definisce un ordine lineare, cioè ogni due elementi di C sono confrontabili.

Lemma di Zorn. Sia X un insieme parzialmente ordinato. Se ogni catena in X ammette un elemento maggiorante in X, allora X ammette un elemento massimale (cioè, un elemento m tale che $x \geq m$ implichi $x=m$).

Il testo inglese che segue
è stato tratto dal libro
Infinite Dimensional Analysis:
A Hitchhiker's Guide
di C. D. Aliprantis e K. C. Border.

The axiom of choice and axiomatic set theory

A good reference for "naïve set theory" is Halmos [4], and indeed you may wonder if there is any need to go beyond that. "Axiomatic set theory" is viewed by many as an arcane subject of little practical relevance. Indeed you may never have been exposed to the most popular axioms of set theory, the **Zermelo-Frankel (ZF) set theory**. For your edification we mention that ZF set theory proper has eight axioms. There is also a ninth axiom, the **Axiom of Choice**, and ZF set theory together with this axiom is often reffered to as ZFC set theory. We shall not list them here, but suffice it to say that the first eight axioms are designed so that the collection of objects that we call sets is closed under certain set theoretic operations, such as unions and power sets. They are also designed to avoi the **Russell's Paradox**[1], which illustrates that it is nonsense to call the collection

[1] In case you have not heard it before, Russell's Paradox goes like this. Let *S* be the set of all sets, and let

of all sets itself a set. Another important axiom of ZF set theory is the Axiom of Infinity, which asserts the existence of an infinite set. For an excellent exposition of axiomatic set theory, we recommend K. J. Devlin [2] or T. Jech [8].

The ninth axiom, the Axiom of Choice, is a seemingly innocuous set theoretic axiom with much hidden power.

Axiom of Choice. If $\{ A_i : i \in I \}$ is a nonempty set of nonempty sets, then there is a function $f : I \to \bigcup_{i \in I} A_i$ satisfying $f(i) \in A_i$ for each $i \in I$. In other words, the Cartesian product of a nonempty set of nonempty sets is itself a nonempty set.

The function f, whose existence is asserted, chooses a member of A_i for each i. Hence the term "Axiom of Choice". This axiom is both consistent with and independent of ZF set theory proper. That is, if the Axiom of Choice is dropped as an axiom of set theory, it cannot be proven by using the remaining eight axioms that the product of nonempty sets is a nonempty set. Furthermore, adding the Axiom of Choice does not make the axioms of ZF set theory inconsistent. (A collection of axioms is inconsistent if it is possible to deduce both a statement P and its negation $\neg P$ from the axioms.)

There has been some debate over the desirability of assuming the Axiom of Choice. (G. Moore [12] presents an excellent history of the Axiom of Choice and the controversy surrounding it.) Since there may be no way of describing the choice function, why should we assume it exists? Further, The Axiom of Choice has some unpleasant consequences. The Axiom of Choice makes it possible, for instance, to prove the existence of non-Lebesgue measurable sets of real numbers. R. Solovay [14] has shown that by dropping the Axiom of Choice, it is possible to construct models of set theory in which all subsets of the real line are Lebesgue measurable. Since measurability is a major headache in integration and probability theory, it would seem that dropping the Axiom of Choice would be desirable. Along the same lines is the **Banach-Tarski Paradox** due to S. Banach and A. Tarski [1]. They prove, using the Axiom of Choice, that the unit ball U in $\mathbf{R}^3$ can be partitioned into two disjoint sets X and Y with the property that X can be partitioned into five disjoint sets, which can be reassembled (after translation and rotation) into U, and the same is true of Y. That is, the ball can be cut up into pieces and reassembled to make two balls of the same size! (These pieces are obviously not Lebesgue measurable. Worse yet, this paradox shows that it is impossible to define a finitely additive volume in any reasonable manner on $\mathbf{R}^3$.) For a proof of this remarkable result, see, e.g., [7, Theorem 1.2, pp. 3-6].

On the other hand, dropping the Axiom of Choice also has some unpleasant side effects. For example, without some version of the Axiom of Choice, our previous assertion that the countable union of countable sets is countable ceases to be true. Its validity can be restored by assuming the Countable Axiom of Choice, a weaker assumption that says only that a countable product of nonempty sets is a nonempty set. Without the Countable Axiom of Choice, there exist infinite sets that have no countably infinite subset. (See, for instance, T. Jech [7, Section 2.4, pp. 20-23].)

From our point of view, the biggest problem with dropping the Axiom of Choice is that some of the most beautiful tools of analysis would be thrown out with it. J. L. Kelley [9] has shown that the Tychonoff Product Theorem[2] would be lost. Most of proofs of the Hahn-Banach Extension

---

$A = \{X \in S : X \notin X\} \in S$. If $A \in A$, then $A \notin A$. On the other hand, if $A \notin A$, then $A \in A$. This paradox is avoided by denying that the class of all sets is itself a set.

[2] **Tychonoff Product Theorem.** The product of a family of topological spaces is compact in the product topology if and only if each factor is compact.

Theorem make use of the Axiom of Choice. This is not strictly necessary though. The Hahn-Banach Theorem, which is the bread and butter of linear analysis, can be proven using the Prime Ideal Theorem of Boolean Algebra, see W. A. J. Luxembourg [11]. The Prime Ideal Theorem is equivalent to the Ultrafilter Theorem[3], which we prove using Zorn's Lemma (which is equivalent to the Axiom of Choice). J. D. Halpern [5] has shown that the Ultrafilter Theorem does not imply the Axiom of Choice. Nevertheless, M. Foreman and F. Wehrung [3] have recently shown that if the goal is to eliminate non-measurable sets, then we have to discard the Hahn-Banach Extension Theorem. That is, any superset of the ZF axioms that is strong enough to prove the Hahn-Banach Extension Theorem is strong enough to prove the existence of non-measurable sets. We can learn to live with non-measurable sets, but life would be nasty, bruttish, and short without the Hahn-Banach Extension Theorem. Therefore we might as well assume the Axiom of Choice. For additional consequences of the Axiom of Choice, we recommend the monograph by P. Howard and J. E. Rubin [6].

Zorn's Lemma

There are a number of propositions that are equivalent to the Axiom of Choice. One of the most useful of these is Zorn's Lemma, due to M.Zorn [15]. That is, Zorn's Lemma is a theorem if the Axiom of Choice is assumed, but if Zorn's Lemma is taken as an axiom, then the Axiom of Choice becomes a theorem.For a thorough discussion of Zorn's Lemma and its equivalent formulations see Rubin and Rubin [13]. In addition, Halmos [4] and Kelley [10, Chapter 0] have extended discussions of the Axiom of Choice.

Zorn's Lemma. If every chain in a partially ordered set *X* has an upper bound, then *X* has a maximal element.

…

Il libro di Aliprantis e Border contiene alcune applicazioni del Lemma di Zorn (per es. l'esistenza di una base in ogni spazio vettoriale) e il "Well Ordering Principle" (ogni insieme non vuoto può essere ben ordinato) che è equivalente all'Assioma della scelta.

Bibliografia

[1] S. Banach and A. Tarski, *Sur la d"ecomposition des ensembles de points en parties respectivement congruentes*, Fund. Math. **6** (1924), 244-277.

[2] K. J. Devlin, *The Joy of Sets: Fundamentals of Contemporary Set Theory*, 2nd ed., Springer-Verlag, New York, 1993.

[3] M. Foreman and F. Wehrung, *The Hahn-Banach theorem implies the existence of a non-Lebesgue measurable set*, Fund. Math. **138** (1991), 13-19.

[4] P. R. Halmos, *Naive Set Theory*, Springer-Verlag, New york, 1974 (reprint of the edition published by Litton Educational Printing, 1960).

[5] J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. **55** (1964), 55-64.

[6] P. Howard and J. E. Rubin, *Consequences of the axiom of choice*, American Math. Society, Providence, RI, 1998.

[7] T. Jech, *The Axiom of Choice*, North Holland, Amsterdam, 1973.

---

[3] Theorem 2.16 nel libro *Infinite Dimensional Analysis: A Hitchhiker's Guide* (L.V.)

[8] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[9] J. L. Kelley, *The Tychonoff product theorem implies the axiom of choice*, Fund. Math. **37** (1950), 75-76.

[10] J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.

[11] W. A. J. Luxemburg, *Two applications of the method of construction by ultrapowers in analysis*, Bull. Amer. Math. Soc. **68** (1962), 416-419.

[12] G. H. Moore, *Zermelo's Axiom of Choice: Its Origins, Development, and Influence*, Springer-Verlag, New York, 1982.

[13] H. Rubin and J. E. Rubin, *Equivalents of the Axiom of Choice*, North Holland, Amsterdam, 1963.

[14] R. M. Solovay, *A model of set theory in which every set is Lebesgue measurable*, Ann. Math. **92** (1970), 1-56.

[15] M. Zorn, *A remark on method in transfinite algebra*, Bull. Amer. Math. Soc. **41** (1935), 667-670.

- FINE -