The problems I left behind

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During over forty years of studying and working on problems of metric fixed point theory, I raised some problems and asked several questions. For some I was lucky to get answer or find followers who did it for me. Some are still open and seem to be difficult. Some are of my own and some came out after fruitful discussions with my friends and colleagues. The problems are connected to the: geometry of Banach spaces, minimal invariant sets, classification of Lipschitz mappings, stability of fixed point property, minimal displacement and constructions of optimal retractions.
The aim of this talk is to present a selection.

Marseille 1989

Rotundity

Let $(X, \|\cdot\|)$ be a Banach space. The first and standard method of measuring "*the rotundity*" of the unit ball in X is via defining *the modulus of convexity* of $X, \delta_X : [0,2] \to [0,1]$,

$$\delta_X(\varepsilon) = \inf \left[1 - \left\| \frac{x+y}{2} \right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon \right]$$

and *the characteristic of convexity*

$$\varepsilon_0(X) = \sup [\varepsilon : \delta_X(\varepsilon) = 0].$$

The modulus of convexity has "two dimensional character", meaning that

$$\delta_X(\varepsilon) = \inf\left[\delta_E(\varepsilon) : E \subset X, \dim E = 2\right].$$

It is known that the Hilbert space H is the most rotund space among all Banach spaces. It is understood in the sense that

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = \delta_{E_2}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},$$

for all Banach spaces X and E_2 being the two dimensional Euclidean space.

Now, fix $a \in [0, 2)$ and consider the class $\mathcal{E}_a$ of all two dimensional spaces $(E, \|\cdot\|)$ having $\varepsilon_0(E) = a$. Which of these spaces is the most rotund? It can be formulated in the following questions.

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Question 1. For any $\varepsilon \in [a, 2)$ what is $\sup[\delta_E(\varepsilon) : E \in \mathcal{E}_a]$?

Question 2. Does there exist a space $E_a \in \mathcal{E}_a$ such that for all $E \in \mathcal{E}_a, \delta_E(\varepsilon) \leq \delta_{E_a}(\varepsilon)$?

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Question 3. If the answer to the above is yes, is such space E_a in some sense unique?

Halifax 1991

Karlovitz Lemma

It was a time soon after the new technique based on the tool known nowadays as Goebel-Karlovitz Lemma was introduced to the nonexpansive mapping theory.
The standard setting is the following. Given a convex, closed and bounded subset C of a Banach space X and a nonexpansive mapping $T : C \to C$,

$$\|Tx - Ty\| \leq \|x - y\| .$$

It may be the case that there are many convex and closed subsets $D \subset C$ invariant under T, $T(D) \subset D$. A set $D \subset C$ is said to be minimal invariant if it does not contain any proper, closed, convex and T invariant subsets. Such invariant sets always do exists if C is weakly compact. Any one point set $\{x\}$ such that $x = Tx$ is minimal invariant. However there are weakly compact sets C which do not have fixed point property. If $FixT = \emptyset$ then C contains a minimal invariant subset D with $diamD > 0$.

Sets which do not contain diametral convex subsets, other then of consisting of one point, are known as having *normal structure.* Minimal invariant sets share a lot of special properties. Any minimal invariant D is diametral, which means that all the points of D are diametral. In other words, for any $x \in D$, $\sup \{\|x - y\| : y \in D\} = diamD$. However, there are (weakly compact) diametral sets which are no minimal invariant for any nonexpansive mapping. Such is the subset of c_0 defined as

$$K = Conv \{e_i : i = 1, 2, ...\} = \left\{ x = (x_i) : x_i \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

All the points of K, and for example two points e_1 and 0, satisfy $\|e_1 - 0\| = 1 = diamK$. However for any $0 < \varepsilon < \frac{1}{2}$, the sets $K \setminus B(e_1, 1 - \varepsilon)$ and $K \setminus B(0, 1 - \varepsilon)$ have different structure. First is connected of diameter 1, and the second consists of disjoint pathways connected components of diameter smaller than ε.

This leads to the following definition.

Definition

Let K be a convex diametral set with $diamK = d > 0$. A point $x \in K$ is said to be almost nondiametral if, there exists $\varepsilon > 0$ such that each pathways connected component of $K \setminus B\left(x, d - \varepsilon\right)$ has diameter smaller then d.

It was shown that the weakly compact minimal invariant sets for nonexpansive mappings can not contain almost diametral points. It leads to a formal generalization of classical Kirk's result.

Theorem

If any convex diametral subset K (not a singleton) of a weakly compact convex set C contains an almost nondiametral point, then C has FPP for nonexpansive mappings.

Problem

Is the above a real generalization of Kirk's theorem?

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Problem

Does there exist a Banach space X, containing nontrivial diametral sets, such that any such set contains an almost nondiametral point?

Sevilla 1995

Minimal displacement problem

Let C be a convex, bounded, and closed subset of a Banach space X and let $T : C \to C$ be continuous mapping. The *minimal displacement* for T is the number

$$d(T) = \inf \{\|x - Tx\| : x \in C\}.$$

I believe that the first examples of continuous and lipschitzian mappings with $d(T)$ were shown in 1973 by myself. It was shown that if T satisfies Lipschitz condition

$$\|Tx - Ty\| \leq k \|x - y\|,$$

with $k \geq 1$ then

$$d(T) \leq \left(1 - \frac{1}{k}\right) r(C),$$

where $r(C)$ is the Chebyshev radius of C.

In some spaces, for some sets the above is the best estimate possible. For regular spaces, whatever it means, the estimate is not sharp. The strongest qualitative result about minimal displacement came in 1985 in the work of P.K. Lin and Y. Sternfeld:

Theorem

For any convex, closed and bounded set C *and for any* $k > 1$ *there exists* $k-$*lipschitzian mapping* $T : C \to C$ *with* $d(T) > 0$.

Let $B = B_X$ denote the unit ball of the space X. To formalize the problem, let us define the *characteristic of the minimal displacement* as the function

$$\psi(k) = \psi_X(k) = \sup\{d(T) : T : B \to B, T \in \mathcal{L}(k)\}.$$

The general estimate

$$\psi_X(k) \leq 1 - \frac{1}{k},$$

is valid for all the spaces X. Equality holds for many "square" spaces like $c_0, c, C[0,1], C^n[0,1]$ and others. For all uniformly convex spaces strong inequality holds for all $k > 1$. Basic properties of the function ψ_X are presented in my books (with W.A. Kirk, and individual) and several articles.

For the Hilbert space H the old (1973) estimate

$$\psi_H(k) \le \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}},$$

has not been improved till now.
For l^1 which is "very square" the basic inequality is also not sharp.
The best known estimate is

$$\psi_{l^1}(k) \le \left[\begin{array}{ll} \frac{2+\sqrt{3}}{4}\left(1 - \frac{1}{k}\right) & \text{for } 1 \le k \le 3 + 2\sqrt{3} \\ \frac{k+1}{k+3} & \text{for } k > 3 + 2\sqrt{3} \end{array} \right. .$$

Problem

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Problem

Does there exist a space X *for which* $\psi_X(k)$ *is the smallest possible comparing with other Banach spaces, for all* $k > 1$ *or for a fixed* k*? Is this the Hilbert space?*

The presented estimates for H and l^1 do not give clear indications since no one majorizes the other.

Kazimierz Dolny 1997

Equivalents of Schauder Fixed Point Theorems

All proofs of classical Brouwer's fixed Point Theorems contain some **nonelementary elements** (whatever it means). There is a temptation to find a proof as simple as possible. One way to do so is to study various equivalents of the famous result. standard books, usually list two or three. The most common facts are

- Sphere S^{n-1} is not the retract of the ball B^n,
- Sphere S^{n-1} is not contractible to a point.

Both have topological and not metrical character. Finding "metrical" equivalents requires some tricks.

Classical Schauder Theorem reads:

Theorem

Every convex, compact subset of a Banach space has topological fixed point property.

It means that if $K \subset X$ is convex and compact then any continuous mapping $T : K \to K$ has a fixed point.
Let us list some "metric" equivalents of this fact.

- Given $k > 1$. Any mapping $T : K \to K$ of class $\mathcal{L}(k)$ has a fixed point.
- Given integer $n \geq 1$ and $k > 1$. Any mapping of class $\mathcal{L}(k)$ has a point of period n, $T^n x = x$.

- For any continuous mapping $T : K \to K$, there exists a point $x \in K$ such that,

$$\left\|x - T^2 x\right\| \leq \|x - Tx\| .$$

- Given $n \geq 1$ and $\varepsilon > 0$. For any continuous mapping $T : K \to K$, there exists a point $x \in K$ such that,

$$\|x - T^n x\| \leq (n - \varepsilon) \|x - Tx\| .$$

- For any two continuous mappings $T, S : K \to K$, there exists a point $x \in K$ such that

$$\|x - Tx\| \leq \|x - Sx\| .$$

Haifa 2001

Optimal retractions

If X is a finite dimensional Banach space, then the unit sphere S is not the retract of the unit ball B. It means that there are no continuous mappings (*retractions*) $R : B \to S$ which keep all the points of S fixed

The problem of *optimal retraction* is closely related to the discussed problem of minimal displacement. In the first paper from 1973, written long before Benyamini-Sternfeld result was known, there is the following:

Lemma

The characteristic of minimal displacement $\psi_X(k)$ is positive if and only if there exists a lipschitzian retraction $R : B \to S$.

In 1983 Y. Benyamini and Y. Sternfeld proved:

Theorem

If $\dim X = \infty$ *then there exists a lipschitzian mapping* $R : B \to S$ *such that for all* $x \in S, x = Rx$.

The proof is very technical and it is difficult to evaluate the Lipschitz constant of R from such mapping. It raises the following *optimal retraction problem.*

For any Banach space X define the number

$$k_0(X) = \inf\{k : \text{ there exists } R : B \to S, R = I \text{ on } S \text{ and } R \in \mathcal{L}(k)\}.$$

and call it *the optimal retraction constant.*

The exact value of $k_0(X)$ is not known for any space. There are only some estimates. The progress in finding good estimates of this constant is very slow. Most of the results are obtained via constructions some tricky examples.

Basic known selected facts are:

- For any $X, k_0(X) \geq 3$ but $k_0(H) \geq 4.5$ and $k_0\left(l^1\right) \geq 4$,
- $3 \leq k_0(C[0,1]) \leq 4\left(2+\sqrt{3}\right) = 14.92..$ but $3 \leq k_0(C_0[0,1]) \leq 2\left(2+\sqrt{2}\right) = 6.83...$
- $3 \leq k_0\left(L^1(0,1)\right) \leq 8$ and $4 \leq k_0\left(l^1\right) \leq 8$,
- If $\psi_X(k) = 1 - \frac{1}{k}$, then $k_0(X) < 32$,
- $k_0(H) < 28.99$.

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The natural and more accessible challenge is:

Problem

Improve the known estimates of $k_0(X)$ for classical Banach spaces.

Valencia 2003

Back to minimal invariant sets

Working with Brailey Sims we came back to the old problem of describing peculiar properties of minimal invariant sets. Our results has been presented in Catania on the World Congress of Nonlinear Analysts.
Let us recall the standard setting.
Given a convex, closed and bounded subset C of a Banach space X and a nonexpansive mapping $T : C \to C$,

$$\|Tx - Ty\| \leq \|x - y\| .$$

It may be the case that there are many convex and closed subsets $D \subset C$ invariant under T, $T(D) \subset D$. A set $D \subset C$ is said to be minimal invariant if it does not contain any proper, closed, convex and T invariant subsets. Such invariant sets always do exists if C is weakly compact.

We observed that there may be a situation in which C has a number of weakly compact minimal invariant subsets.
Stan Prus produced a very nice example showing that there may be a convex closed and bounded set C and a fixed point free nonexpansive mapping $T : C \to C$ such that for any $\varepsilon > 0$ there exists a minimal invariant subset $D \subset C$ with $diam(D) < \varepsilon$.
However, the set C is not weakly compact. It raises the first

Problem

Does there exist a weakly compact convex set and a nonexpansive mapping $T : C \to C$ having no fixed points but minimal invariant subsets of arbitrary small diameter?

Some geometric properties prevent such existence. Two basic facts are:

- If the space X is strictly convex then all minimal invariant sets are isometric and each one is a translation of any other.
- If the space X has Kadec-Klee property than all the minimal invariant sets are of the same diameter.

Advanced form of the above problem can be also formulated as follows:

Problem

Suppose C is a weakly compact, convex set. Assume that for any $\varepsilon > 0$ any nonexpansive mapping $T : C \to C$ has a minimal invariant subset D with $diam D < \varepsilon$. Does C have the fixed point property?

Guanajuato 2005

Rotative mappings

The notion of rotative mappings has the origin in my old result about involutions.

Theorem

Let $T : C \to C$ be the involution, $T^2 = I$. If T satisfies Lipschitz condition with constant $k < 2$, then T has a fixed point.

There are examples of continuous involutions without fixed points. It raises the natural question whether the estimate $k < 2$ is the best possible. Even more, the general problem reads:

Problem

Does there exist a set C which admits an uniformly continuous fixed point free involution?

The same question can be raised for periodic mappings, $T^n = I$, $n > 2$. The more general approach to the problem is based on the definition of *rotative mappings*

Let $T : C \to C$ be a nonexpansive mapping. It is easy to see that for any $n = 2, 3, ...$ and any $x \in C$,

$$\|x - T^n x\| \leq n \|x - Tx\| .$$

If for certain $0 \leq a < n$ the sharper inequality

$$\|x - T^n x\| \leq a \|x - Tx\|$$

holds, we call T to be (n, a) *−rotative.* Any n−periodic T is $(n, 0)$ −rotative. The basic result, obtained with M. Koter reads

Theorem

If C is convex and closed (not necessarily bounded) and $T : C \to C$ is nonexpansive and rotative, then $FixT \neq \varnothing$.

The condition of rotativeness is independent on regularity. It can be considered for any mapping, not necessarily nonexpansive. Even so, it is not as natural the following is known:

Theorem

For any $n \geq 2$ and for any $0 \leq a < n$ there exists a maximal constant $\gamma_n(a) > 1$ such that if $T : C \to C$ is $k-$lipschitzian with $0 \leq k < \gamma_n(a)$, then $FixT \neq \varnothing$.

So, rotative lipschitzian mappings have fixed points even if their Lipschitz constants exceed 1, but not too much. Present and known estimates for the function $\gamma_n(a)$ are rough and unsatisfactory.

Problem

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For $n = 2$ and $a \in (1, 2)$ there are examples of $(n, a)-$rotative mappings without fixed points. It is known that for $a \in (1, 2)$, $\gamma_2(a) \leq \frac{1}{a-1}$ but nothing is known for $a \in [0, 1)$.

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Problem

Is $\gamma_2(a) < \infty$ *for* $a \in [0, 1)$?

Chiang Mai 2007

Mean nonexpansive mappings

Maria Japon Pineda, in connection with her talk, raised a simple question. I new the answer. However in the discussion we proposed a notion of $\alpha-$*nonexpansive mappings.*

Let $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$ be multiindex such that $\alpha_1 > 0, \alpha_n > 0, \alpha_i \geq 0$, for $i = 1, 2, ..., n$ and $\sum_{i=1}^{i=n} \alpha_i = 1$. A mapping $T : C \to C$ is said to be $\alpha-$*nonexpansive,* if for any $x, y \in C$

$$\sum_{i=1}^{n} \alpha_i \left\| T^i x - T^i y \right\| \leq \|x - y\| .$$

For the case $n = 2$ the formula reads

$$\alpha_1 \|Tx - Ty\| + \alpha_2 \left\|T^2x - T^2y\right\| \leq \|x - y\|$$

All nonexpansive mappings are $\alpha-$nonexpansive for any index α. However, there are $\alpha-$nonexpansive mappings such that none of their powers is nonexpansive. Today it is known that on any convex set C and any α the class of $\alpha-$nonexpansive mappings is properly wider than the class of nonexpansive ones.
A surprising finding concerning such mappings is:

Theorem

If C has the FPP for nonexpansive mappings, then all the $\alpha-$nonexpansive mappings $T : C \to C$ with $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$ such that $\alpha_1 \geq \frac{1}{\sqrt[n-1]{2}}$ also have fixed points.

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Problem

For given $n > 2$. How can one describe the set of all α's of length n, for which the above theorem holds?

It is known that condition $\alpha_1 \geq \frac{1}{\sqrt[n-1]{2}}$ is sufficient but not necessary. For example for $n = 3$ the conclusion of the Theorem hold also for any $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_1 \geq \alpha_2 \geq \alpha_3$ and $\alpha_1 \geq \frac{1}{2}$.

Changhua 2009

Commuting mappings

During our discussions Art Kirk often mentioned the following question originally raised (around 40 years ago) by J.B.Baillon,

Problem

Do two commuting nonexpansive mappings have a joint approximate fixed point?

It means: is it true that for any $\varepsilon > 0$ there exists a point $x_\varepsilon \in C$ such that $\|x_\varepsilon - Tx_\varepsilon\| < \varepsilon$ and $\|x_\varepsilon - Sx_\varepsilon\| < \varepsilon$?

The answer to this question is unknown!!!

There are cases for which the answer is affirmative. Observe first that if both mappings have fixed points then $S : FixT \to FixT$ and $T : FixS \to FixS$. If one of the mappings, say T, is affine, then the set $Fix(T)$ is convex and we have

$$\inf [\|x - Sx\| : x \in F_{\varepsilon}(T)] = 0.$$

If the space is strictly convex, then both sets $FixT$ and $FixS$, are convex. For the same reason as above, if at least one of them is nonempty, the answer is also affirmative. If both are nonempty we have

$$\inf [\|x - Tx\| : x \in FixS] = \inf [\|x - Sx\| : x \in FixT] = 0.$$

Example

In c_0 space define two mappings:

$$Tx = T(x_1, x_2, x_3, \ldots) = (x_1, 1 - |x_1|, x_2, x_3, \ldots)$$

$$Sx = S(x_1, x_2, x_3, \ldots) = (-x_1, x_2, x_3, \ldots).$$

Both are nonexpansive (isometric) and map the unit ball B into itself. Also they commute,

$$ST(x) = TS(x) = (-x_1, 1 - |x_1|, x_1, x_2, \ldots).$$

Both mappings have fixed points. $FixT$ consists of two points

$$FixT = \{(1,0,0,0,...), (-1,0,0,0,...)\}$$

and $FixS$ in the whole space, as well as in B is the defined by the condition $x_1 = 0$,

$$FixS = \{x : x_1 = 0\} = \{x : x = (0, x_2, x_3, x_4, ...)\}.$$

Of course $FixT \cap FixS = \varnothing$. Also, since $FixS$ is convex, we have
$\inf [\|x - Tx\| : x \in FixS] = 0$ but
$\inf [\|x - Sx\| : x \in FixT] = 2 \neq 0$.

Cluj-Napoca 2012

Stability of FPP

The notion of *stability of FPP* has its origin in our paper with Art Kirk.

We found that, if X is uniformly convex, then in fact the bounded closed convex subsets of X have the fixed point property for a broader class of mappings then nonexpansive ones.

Let K be a bounded closed convex subset of a uniformly convex space suppose $T : K \to K$ is *uniformly lipschitzian* in the sense that

$$\|T^n x - T^n y\| \leq k \|x - y\|$$

for all $x, y \in K$ and $n = 1, 2, \cdots$. It was shown that if $k > 1$ is sufficiently near 1 (how near depends on the modulus of convexity), then T always has a fixed point. The class of all uniformly lipschitzian mappings is also fully characterized by the fact that such mappings are nonexpansive with respect to some equivalent metrics but not necessarily generated by equivalent norms. This justifies the word "stability" and prompted further the study of this property.

Let us illustrate the problem in a simplified version. For any Banach space X define the constant

$$\gamma_0(X) = \sup\left\{ k : \begin{array}{l} \text{any closed bounded convex subset } K \subset X \\ \text{has the FPP for } k\text{-uniformly lipschitzian mappings} \end{array} \right\}$$

Obviously, if X does not have FPP, then $\gamma_0(X) = 1$. The mentioned above, basic result states that if X is uniformly convex then $\gamma_0(X) > 1$. There are several results concerning estimates of $\gamma_0(X)$ for various spaces having FPP but exact value of it is not known for any. In case of Hilbert space we have

$$\sqrt{2} \le \gamma_0(H) \le \frac{\pi}{2}.$$

The natural problem which was recently solved was

Problem

Does there exists a Banach space X having FPP for nonexpansive mappings for which $\gamma_0(X) = 1$?

It came as a surprise to many when Pei-Kee Lin showed that l^1 can be given a new equivalent norm $\|\cdot\|^0$ for which the space $Z = \left(l^1, \|\cdot\|^0\right)$ has the FPP. Lin's norm is given by

$$\|x\|^0 = \max\left\{\gamma_n \sum_{k=n}^{\infty} |x_k| : n = 1, 2, ...\right\},$$

where $\gamma_n = \frac{8^n}{8^n+1}$.
Recent results of T. D. Benavides state that l^1 can not be renormed to have $\gamma_0 > 1$. There is also a nice example of K. Bolibok of a convex subset U of Lin's space Z such that for any $\varepsilon > 0$ there exists $(1+\varepsilon)-$uniformly lipschitzian, fixed point free mapping. this solves the problem, $\gamma_0(Z) = 1$.

Unsolved remain some problems concerning the minimal displacement of uniformly nonexpansive mappings. It is known that for the unit ball B in any Banach space X, for any $\varepsilon > 0$ there exists a uniformly lipschitzian mapping $T : B \to B$ with $d(T) > 1 - \varepsilon$.

To formalize the problem, let us define the *characteristic of the minimal displacement uniformly lipschitzian mappings* as the function

$$\psi_u(k) = \psi_{u,X}(k) = \sup\{d(T) : T : B \to B, T \in \mathcal{UL}(k)\}.$$

For all spaces $\psi_{u,X}(k) = 0$ on $[1, \gamma_0(X)]$ and $\lim_{k\to\infty} \psi_{u,X}(k) = 1$.

It prompts us to define next constant

$$\begin{aligned}\gamma_0(X) &\leq \gamma_1(X) = \sup\left\{k : \begin{array}{c}\text{Any } k-\text{uniformly lipschitzian}\\ \text{mapping } T : B \to B \text{ has } d(T) = 0\end{array}\right\}\\ &= \sup\{k : \psi_{u,X}(k) = 0\}.\end{aligned}$$

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Does there exist a space for which $\gamma_0(X) < \gamma_1(X)$?

Problem

Find some estimates for $\psi_{u,X}(k)$ for classical spaces.

It prompts us to define next constant

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Problem

Does there exist a space for which $\gamma_0(X) < \gamma_1(X)$?

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(Risky and naive) *Is $\gamma_0(H) = \sqrt{2}$ and $\gamma_1(H) = \frac{\pi}{2}$?*

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Does there exist a space X having FPP such that $\gamma_1(X) = 1$? Is it the space Z?