A generalization of unitaries

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Abstract: In this talk we give a new geometric generalization of the notion of a unitary of a C*-algebra and give examples of classes of Banach spaces where such objects can be found.

Let A be a C^*-algebra with identity e and let $S = \{f \in A_1^* : f(e) = 1\}$. This is called the state space and it is well-known that $spanS = A^*$. Now let $u \in A$ be any unitary. Since $x \to ux$ is a surjective isometry of A mapping e to u,clearly, if $S_u = \{f \in A_1^* : f(u) = 1\}$, then $spanS_u = A^*$. Let $x \in A$ be any unit vector and let $S_x = \{f \in A_1^* : f(x) = 1\}$. An interesting result in C^*-algebra theory says that if $spanS_x = A^*$ then x is a unitary.

As the condition $spanS_x = A^*$ is purely a Banach space theoretic one, an abstract notion of unitary in a Banach space X, as a unit vector x such that $spanS_x = X^*$ was introduced and studied in a joint work with P. Bandyopadhyay and K. Jarosz. It turned out that these abstract unitaries share several important properties of unitaries of a C^*-algebra. In particular unitaries are preserved under the canonical embedding of X in its bidual X^{**}. One of the limitations in the general theory is that an exact analogue of the Russo-Dye theorem (the unit ball of a complex C^*-algebra is the norm closed convex hull of unitaries) is very rarely true.

1. Multismoothness

Let X be a Banach space and $x \in X$ a unit vector. It is well-known that when $S_x = \{x^*\}$, x is called a smooth point of X. Motivated by the above considerations, we call x a k-smooth point if $spanS_x$ is a vector space of dimension k and a ω-smooth point if $spanS_x$ is a closed subspace. We say that X is k-smooth if every unit vector is n-smooth for $n \leq k$.

We recall that S_x is a weak*-compact convex and extreme (face) set. Let $A(S_x)$ denote the space of affine continuous functions, equipped with the supremum norm (when the scalar field is real, we denote this space by $A_R(S_x)$). Let $\delta : S_x \to A(S_x)_1^*$ denote the evaluation map. It is easy to see that it is an affine, one-to-one and continuous map. Let Γ denote the unit circle. For any extreme point $\tau \in \partial_e A(S_x)_1^*$, since τ has an extension to an extreme point of $C(S_x)_1^*$, we have that $\tau = \delta(k)$ for some $k \in \partial_e S_x$. Therefore $A(S_x)_1^* = \overline{CO}(\Gamma\delta(S_x))$, where the closure is taken w. r. t weak*-topology. In particular in the case of real scalars, $A_R(S_x)_1^* = CO(\delta(S_x) \cup -\delta(S_x))$.

Now let $\tau \in A(S_x)_1^*$ and $\tau(1) = 1$. Since the norm-preserving extension of τ to $C(S_x)$ is a probability measure, $\tau \in A_R(S_x)_1^*$. Suppose $\tau = \lambda\delta(x_1^*) - (1-\lambda)\delta(x_2^*)$ for some $x_1^*, x_2^* \in S_x$ and $\lambda \in [0,1]$. Evaluating this equation at 1, we get $\lambda = 1$ and thus $\tau = \delta(x^*)$ for $x^* \in S_x$

on $A_R(S_x)$ and hence on $A(S_x)$. Thus $S_1 = \delta(S_x)$. Also by using the Jordan decomposition of measures, we see that $A(S_x)^* = span\delta(S_x)$.

Let $\Phi : X \to A(S_{x_0})$ be defined by $\Phi(x)(x^*) = x^*(x)$ for $x^* \in S_{x_0}$. Φ is clearly a linear contraction and $\Phi(x_0) = 1$. Therefore $\Phi^*(\delta(S_{x_0})) = S_{x_0}$ so that $\Phi^*(A(S_{x_0})) = spanS_{x_0}$.

Now our assumption $spanS_{x_0}$ is closed implies by the closed range theorem, $spanS_{x_0}$ is weak*-closed and also range of Φ is closed.

Now let M be the preannihilator of $spanS_{x_0}$. Then $(X|M)^* = M^\perp = spanS_{x_0}$. In particular $\pi(x_0)$ is a unitary of $X|M$ where $\pi : X \to X|M$ is the quotient map.

Question: Suppose for some $x_0 \in X_1$, $\pi(x_0)$ is a unitary. When can one get a multismooth or ω-smooth point $x \in X_1$ such that $\pi(x_0) = \pi(x)$?

Suppose x_0 is a multismooth point. Let $n = dim(spanS_{x_0})$. By a theorem of Carathoedary, $\partial_e S_{x_0}$ is a spanning set for $spanS_{x_0}$. As S_{x_0} is an extreme set, there are exactly n independent extreme points of X^* in S_{x_0}. This we shall call the exact independent set of extreme points.

For example in a $C(K)$ space (K is a compact set), if f is a n-smooth point, then since there are exactly n point masses in $spanS_f$, we have that f attains its norm at exactly n points of K. Since this finite subset of K is a G_δ, we see that if $C(K)$ has a n-smooth point then it has a k smooth poiny for all $k \leq n$.

Question: In general it is not clear if the existence of n smooth point implies the existence of a k smooth point for some $k < n$? This question is of particular interest in the case of non-commutative C^*-algebras.

Analogous to the duality of smoothness and strict convexity (rotundity), in this context we have the notion of k-rotundity.

A Banach space X with $dim(X) \geq k+1$ is said to be k-rotund, if for any $k+1$ independent unit vectors $\{x_i\}_{1 \leq i \leq k+1}$, $\|\frac{\sum_1^{k+1} x_i}{k+1}\| < 1$.

Since state spaces consist of unit vectors, it is easy to see that if X^* is k-rotund then X is k-smooth.

2. Higher duals

Let X be a non-reflexive Banach space. Consider the canonical embedding $J_0 : X \to X^{**}$. Let us denote by J_2 the canonical embedding of X^{**} in its bidual $X^{(4)}$. It is easy to see that $X^{(4)} = J_2(X^{**}) \oplus J_1((X^*))^\perp$. Similarly since $X^{***} = J_1(X^*) \oplus (J_0(X))^\perp$, we also have, $X^{(4)} = J_0(X)^{\perp\perp} \oplus J_1((X^*))^\perp$. Also $J_2(X^{**})$ is canonically isometric to $(J_0(X))^{\perp\perp} = J_0^{**}(X^{**})$. Now let $x^{**} \in X^{**} \setminus J_0(X)$. Then $0 < d(x^{**}, J_0(X)) = d(J_2(x^{**}), J_0(X))^{\perp\perp}) \leq \|J_2(x^{**}) - J_0^{**}(x^{**})\|$. Thus for a non-reflexive X and $x^{**} \in X^{**} \setminus J_0(X)$, $J_2(x^{**})$ and $J_0^{**}(x^{**})$ are two distinct vectors. These are well-known observation of Dixmier.

Theorem 1. *Suppose $X^{(4)}$ is k-rotund. Then every k-smooth point of X^* attains its norm.*

Proof. By our earlier observation, X^{***} is k-smooth. Let x^* be a unit vector that is k-smooth in X^* and suppose it does not attain its norm. Let $x^{**}(x^*) = 1 = \|x^{**}\|$. By our assumption $x^{**} \in X^{**} \setminus J_0(X)$. Thus by Dixmier' observation, $J_2(x^{**})$ and $J_0^{**}(x^{**})$ are two distinct vectors. Therefore every vector in the state space of x^* generates two distinct vectors in the state space of $J_1(x^*)$. This contradicts the k-smoothness of $J_1(x^*)$.

□

We recall that a closed subspace $Y \subset X$ is said to be a U-subspace if every $y^* \in Y^*$ has a unique norm-preserving extension in X^*. In particular a Banach space X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**} under the canonical embedding (see [10]

Chapter III). It is well-known that $c_0 \subset \ell^\infty$ and for $1 < p < \infty$, $\mathcal{K}(L^p(\mu)) \subset \mathcal{L}(Ł^p(\mu))$ are examples of this phenomenon.

Remark 2. *If X is a Hahn-Banach smooth subspace then since the state space of an $x \in S(X)$ remains the same in X^{**}, it is easy to see that x is k-smooth in X^{**} if and only if it is k-smooth point in X. We do not know a general local geometric condition to ensure that the state of a unit vector in X and its bidual remain the same.*

Example 3. *Let X be a smooth, non-reflexive Banach space such that X is an L-summand in its bidual under the canonical embedding (i.e., $X^{**} = X \oplus_1 M$, for a closed subspace M, see Chapter IV of* [10]*). The Hardy space H_0^1 is one such example (see page 167 of* [10]*) . Since X is non-reflexive, it is easy to see that when $X^{**} = X \oplus_1 M$, M is infinite* *dimensional. Now every unit vector x of X is a smooth point of X but for no k, x is a k-smooth point in X^{**}.*

We next use the notion of an intersection property of balls, from [15] to establish a relation between k-smooth points in the subspace and the whole space in the case of U-subspaces. In the next two results we assume that X is a real Banach space.

Definition 4. *Let $n \geq 3$. A closed subspace $M \subset X$ is said to have the $n.X.$-intersection property ($n.X.I.P$) if when ever $\{B(a_i, r_i)\}_{1 \leq i \leq n}$ are n closed balls in M with $\cap_1^n B(a_i, r_i) \neq \emptyset$ in X (when they are considered as closed balls in X) then $M \cap \cap_1^n B(a_i, r_i + \epsilon) \neq \emptyset$ for all $\epsilon > 0$.*

We note that if X is an L^1-predual space, then for $n \geq 4$, X has the $n.Y.I.P$ in any Y that isometrically contains X. To see this, let

$\{B(a_i, r_i)\}_{1\leq i\leq n}$ be n closed balls in X with $\cap_1^n B(a_i, r_i) \neq \emptyset$ in Y. Let $\epsilon > 0$. These balls thus pair-wise intersect in X. As X is an L^1-predual space, it follows from Theorem 6 in section 21 of [14] that $X \cap \cap_1^n B(a_i, r_i + \epsilon) \neq \emptyset$.

Proposition 5. *Suppose $M \subset X$ has the k.X.I.P and M is a U-subspace. If $x \in M$ is a k-smooth point in X then it is a k-smooth point in M.*

Proof. Let $\{x_i^*\}_{1\leq i\leq k} \subset S_x$ be a linearly independent set. Let $f_i = x_i^*|M$. Note that $\|x_i^*\| = 1 = \|f_i\|$. We claim that the f_i's are linearly independent. Suppose $\sum_1^k \alpha_i f_i = 0$ for some scalars α_i. By Theorem 3.1 in [15] it follows that there exists norm preserving extensions $f_i' \in X^*$ of $\alpha_i f_i$ such that $\sum_1^k f_i' = 0$. But by the uniqueness of the extensions this implies $\sum_1^k \alpha_i x_i^* = 0$ and hence $\alpha_i = 0$ for $1 \leq i \leq k$. On the other hand if $\{g_i\}_{1\leq i\leq l}$ is any linearly independent set in the state space of x in M, the corresponding Hahn-Banach extensions are clearly linearly independent in S_x. Thus $l \leq k$. $\square$

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