6/10/2015

Df $F : \mathbb{C} \longrightarrow \mathbb{D}$ è un'equivalenza se $\exists\, G : \mathbb{D} \longrightarrow \mathbb{C}$, $\exists\, \eta : Id_{\mathbb{C}} \xrightarrow{\cdot} GF$ iso naturale, $\exists\, \xi : Id_{\mathbb{D}} \longrightarrow FG$

G si chiama QUASI-INVERSO di F ed è anch'esso un'equivalenza $(\mathbb{D} \to \mathbb{C})$

- se F iso $\Rightarrow F$ equivalenza

Df $\mathbb{C}$ e $\mathbb{D}$ sono equivalenti (come categorie) se $\exists\, F : \mathbb{C} \to \mathbb{D}$ equivalenza

---

$\mathbb{C} = Set_f$ insiemi finiti

$\mathbb{D} \hookrightarrow \mathbb{C}$ sottocategoria piena

$$ob(\mathbb{D}) = \begin{cases} \phi = 0 \\ \{\phi\} = 1 = \{0\} \\ \{0, 1\} = 2 \\ \{0, 1, \ldots, n-1\} = n \end{cases}$$

$\mathbb{D}$ è equivalente a $\mathbb{C}$ (EX)

M monoide $\longrightarrow$ $\underline{M}$ cat 1 object

$$\underline{\mathrm{Mon}} \xrightarrow{F} \mathbb{1}\text{-Cat}$$ = categorie con 1 sol oggetto

$$\begin{array}{ccc}
M & \longmapsto & F(M) = \underline{M} \\
\Big\downarrow f & & \Big\downarrow F(f) \\
N & \longmapsto & F(N) = \underline{N}
\end{array}$$

$\underline{M}$: ob $*$

$\mathrm{Hom}(*,*) = M$ Fisso $*$

$$F(f) : \underline{M} \longrightarrow \underline{N}$$

$$\begin{array}{ccc}
* & \longmapsto & * \\
\Big\downarrow m & & \Big\downarrow f(m) \\
* & & *
\end{array}$$

$$\mathbb{1}\text{-Cat} \xrightarrow{G} \underline{\mathrm{Rng}}$$

$\mathbb{C} \ni c$

$\mathrm{Hom}(c,c)$

$$\begin{array}{cccc}
c & \longmapsto & \mathrm{Hom}(c,c) & \ni f \\
\Big\downarrow H \text{ funtore} & & \Big\downarrow H & \Big\downarrow \cancel{H(f)} \\
\mathbb{D} & \longmapsto & \mathrm{Hom}(D,D) & H(f) \\
\Big\downarrow H' & & \Big\downarrow H' & \\
\mathbb{E} & \longmapsto & \mathrm{Hom}(E,E) &
\end{array}$$

$$\underline{Mon} \xrightarrow{F} \mathbb{1}\text{-Cat} \xrightarrow{G} \underline{Mon}$$

$$M \longrightarrow \underline{M} \longrightarrow Hom(*,*) = M$$

$$\underline{M} = (*, M)$$

$$G \circ F = 1_{\underline{Mon}}$$

$$\mathbb{1}\text{-Cat} \xrightarrow{G} \underline{Mon} \xrightarrow{F} \mathbb{1}\text{-Cat}$$

$$(C, Hom(c,c)) \to Hom(c,c) \longrightarrow (*, Hom(c,c))$$

$$FG \xleftarrow[\eta]{\cdot} 1_{\mathbb{1}\text{-cat}}$$

$\forall\ \mathbb{C} \in \mathbb{1}\text{-cat}$ $\quad \eta_{\mathbb{C}} : \mathbb{C} \longrightarrow FG(\mathbb{C})$ funtore

η <u>è naturale</u>! $\quad \eta_{\mathbb{C}}(c) = *$

(EX) $\quad \eta_{\mathbb{C}}(f) = f$!

η iso $\quad \eta_{\mathbb{C}}^{-1} : FG(\mathbb{C}) \longrightarrow \mathbb{C}$

$\qquad * \longrightarrow c$

$\Rightarrow$ F equivalenza

Df $F : \mathbb{C} \longrightarrow \mathbb{D}$ è essenzialmente suriettivo sugli oggetti se

$\forall D \in \mathbb{D}$, $\exists C_D \in \mathbb{C}$ & $\varepsilon_D : FC_D \longrightarrow D$

ε_D iso

$F : \mathbb{C} \longrightarrow \mathbb{D}$ equivalenza $\Rightarrow$

è essenz. suriettivo sugli oggetti

$\forall D$, considero $GD \in \mathbb{C}$

e $\xi_D : D \longrightarrow FGD$ iso

come $C_D = GD$ $\quad \varepsilon_D = \xi_D^{-1} : F(GD) \to D$

$F : \mathbb{C} \longrightarrow \mathbb{D}$ equiv $\Rightarrow$ F è pieno con quasi inverso $G : \mathbb{D} \to \mathbb{C}$ e fedele

Dim F è fedele

$$\begin{array}{c} C \\ f\downarrow\downarrow f' \\ C' \end{array} \qquad \begin{array}{c} F(C) \\ \downarrow \\ F(C') \end{array} \; F(f) = F(f') \Longrightarrow f = f'$$

$$\begin{tikzcd} C \arrow[r, "\eta_C", "\simeq"'] \arrow[d, "f"'] \arrow[dr, phantom, "="] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\eta_{C'}", "\simeq"'] & GF(C') \end{tikzcd}$$

$$
\begin{array}{ccc}
f & = & \eta_{C'}^{-1} \cdot GF(f) \cdot \eta_C \\
\| & & \quad \| \\
f' & = & \eta_{C'}^{-1}\, GF(f')\, \eta_C
\end{array}
$$

F è pieno: data $F(C) \xrightarrow{\ g\ } F(C')$

$$
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, shift right, "GF(f)"'] \arrow[d, shift left, "G(g)"] \\
C' \arrow[r, "\eta_{C'}"'] & GF(C')
\end{tikzcd}
$$

$$
\begin{array}{rl}
f := & \eta_{C'}^{-1}\, G(g)\, \eta_C \\
& \quad \| \\
= & \eta_{C'}^{-1}\, GF(f)\, \eta_C
\end{array}
$$

G è fedele!

$$G(F(f)) = G(g)$$
$$\Downarrow$$
$$F(f) = g$$

$\Rightarrow$ F è pieno (come pure G)

F è equiv $\Longleftrightarrow$ pieno
fedele
essenzialmente sur

Dim $\Rightarrow$ OK

$\Leftarrow$ Dobbiamo def: $G: \mathbb{D} \longrightarrow \mathbb{C}$

$\eta : Id_{\mathbb{C}} \xrightarrow{\cdot} GF \qquad \xi : id_{\mathbb{D}} \xrightarrow{\cdot} FG$

$\forall\, D \in ob(\mathbb{D}) \quad \exists\, C_D \quad \varepsilon_D : FC_D \xrightarrow[iso]{} D$

Def $G(D) = C_D$

$\xi_D : D \xrightarrow[iso]{} FG(D) = F(C_D)$

$\xi_D = \varepsilon_D^{-1}$

$$\begin{tikzcd}
D \arrow[r, "\xi_D"] \arrow[d, "h"'] \arrow[dr, phantom, "//"] & FGD \arrow[d, dashed, "FG(h)"] \\
D' \arrow[r, "\xi_{D'}"'] & FGD'
\end{tikzcd}$$

$\xi_{D'}\, h\, \xi_D^{-1} =: F(G h)$

F pieno e fedele $\exists !$ $G(h) : GD \to GD'$ t.c. $FG(h) = \xi_{D'}\, h\, \xi_D^{-1}$

G è df sulle frecce + naturalità di ξ

EX G così def è un funtore!

? $\eta : \mathrm{Id}_{\mathbb{C}} \longrightarrow GF$

$\forall_{C}\ \xi_{FC} : F(C) \xrightarrow[\text{iso}]{} F(GFC)$

F pieno e fedele $\Rightarrow \exists!\, \eta_C : C \longrightarrow GFC$

t.c. $F(\eta_C) = \xi_{FC}$ iso $\Rightarrow \eta_C$ iso

F riflette iso

EX η naturale $\square$

F pieno e fedele

$$\mathrm{Hom}_{\mathbb{C}}(C, C') \xleftrightarrow[(1,1)]{} \mathrm{Hom}_{\mathbb{D}}(FC, FC')$$

- $Ab \hookrightarrow \underline{Grp}$ piena e fedele non ess. suriettivo

Mat ob. $\mathbb{N} = \{1, 2, \dots n \dots\}$

k campo

$$\begin{tikzcd} m \arrow[r, "A"] & n \arrow[d, "B"] \\ & p \end{tikzcd}$$

$A \in \mathrm{Mat}([illegible], k)$

$B \circ A = B\,A$

$$\begin{tikzcd} \mathbf{Mat} \arrow[rr] & & \text{FD } k\text{-vect} \\ n \arrow[d, "A"] \arrow[rr, mapsto] & & k^n \arrow[d, dashed, "A \cdot \vec{v}"] \\ m \arrow[rr, mapsto] & & k^m \end{tikzcd}$$

ess. suriettivo, pieno e fedele

$\Rightarrow$ equivalenza

An algebraic structure $(L, \vee, \wedge)$, consisting of a set L and two binary operations $\vee$, and $\wedge$, on L is a **lattice** if the following axiomatic identities hold for all elements a, b, c of L.

$x \vee y = y \vee x$	$x \vee (y \vee z) = (x \vee y) \vee z)$	$x \wedge (x \vee y) = x,$
$x \wedge y = y \wedge x$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z)$	$x \vee (x \wedge y) = x$

A lattice $(L, \vee, \wedge)$ is **distributive** if the following additional identity holds for all x, y, and z in L:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A **complemented lattice** is a bounded lattice (with least element 0 and greatest element 1), in which every element x has a **complement**, i.e. an element y such that

$x \vee y = 1$ and $x \wedge y = 0$.

$y = \neg x$

A lattice $(L, \vee, \wedge)$ is **complete** if *all* its subsets have both a join and a meet. In particular, every complete lattice is a bounded lattice

A Boolean algebra or **Boolean lattice** is a complemented distributive lattice.

An **atom** of a Boolean algebra is an element x such that there exist exactly two elements y satisfying $y \leq x$, namely x and 0.

A Boolean algebra is said to be **atomic** when every element is a sup of some set of atoms

A **Boolean ring** R is a ring for which $x^2 = x$ for all x in R

Given a Boolean ring R, for x and y in R we can define

$$x \wedge y = xy,$$
$$x \vee y = x + y + xy,$$
$$\neg x = 1 + x.$$

These operations then satisfy all of the axioms for meets, joins, and complements in a Boolean algebra. Thus every Boolean ring becomes a Boolean algebra.

Similarly, every Boolean algebra becomes a Boolean ring thus:

$xy = x \wedge y,$
$x + y = (x \vee y) \wedge \neg(x \wedge y)$

without changing the underlying set

A map between two Boolean rings is a ring homomorphism if and only if it is a homomorphism of the corresponding Boolean algebras, that is a function $f : A \to B$ such that for all x, y in A:

$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$
$$f(0) = 0,$$
$$f(1) = 1.$$

This way we construct an isomorphism between the category of Boolean algebras and the category of Booelan rings

DUALITA' di STONE

Boole Alg equival. $(\text{Stone Sp})^{op}$

compatti Haus
totalmente
disconnessi

Compl Atomic Boolean Algebra

||

CABA equivalente

$$
\begin{tikzcd}
(\mathrm{Set})^{op} \arrow[r] & \mathrm{CABA} \text{ equivalenza} \\
X \arrow[r, mapsto] & \mathbb{P}(X) \arrow[d, "f^{-1}(\ )"] \\
Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbb{P}(Y)
\end{tikzcd}
$$