TC

30/11/2015

Df T monade idempotente se
μ_X è iso $\forall X$
($\Rightarrow$ $(X, \xi) \in \mathbb{C}^T$ ξ iso)

$$\mathbb{C}^T \underset{F^T}{\overset{U^T}{\rightleftarrows}} \mathbb{C} \qquad T \text{ idemp.}$$

la counità di questa aggiunzione ha come componenti $\varepsilon_{(X,\xi)} = \xi : TX = F^T U^T X \to X$ $\underline{\underline{\text{iso}}}$

$\Updownarrow$

U^T è pieno e fedele

se $F \dashv G : \mathbb{D} \to \mathbb{C}$ G pieno e fedele

$T = GF$ è una monade idempotente?

$$\mu_X = \underbrace{G(\varepsilon_{FX})}_{\text{iso}}$$

iso

$\Rightarrow$ T è idempotente

inoltre

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$$\begin{tikzcd} \mathbb{D} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] \arrow[d, "K"'] & \mathbb{C} \arrow[l, shift left, "F"] \\ \mathbb{C}^T \arrow[ur, shift left, "U^T"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

$T = GF$

$\exists\, K$ confronto con

$U^T K = G$

è pieno e fedele

pieno e fedele

$\xRightarrow{\text{EX}}$ K pieno e fedele

ma K è anche essenzialmente suriettivo: $\forall\,(X, \xi) \in \mathbb{C}^T$ $\exists\, Y$ t.c. $K(Y) \cong (X, \xi)$

$$K(Y) = (G(Y),\ G(\varepsilon_Y))$$

$(X, \xi) \in \mathbb{C}^T$

$$\begin{tikzcd} GFGFX = T^2X \arrow[r, "T(\xi)", "\text{iso}"'] \arrow[d, "G(\varepsilon_{FX}) = \mu_X"'] & TX = G(FX) \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"', "\text{iso}"] & X \end{tikzcd}$$

$K(FX)$ $\quad TX = G(FX)$

T idemp.

$$K(F(X)) = (FX,\ G(\varepsilon_{FX})) \cong (X, \xi)$$

$\Rightarrow$ G è monadico

T monade idemp. $\xleftrightarrow{(1.1)}$ $F \dashv G : \mathbb{D} \to \mathbb{C}$
G pieno e fedele (monadico)

$\Updownarrow$ a meno di equivalenze

SOTTOCATEGORIE RIFLESSIVE (di $\mathbb{C}$)

$J : \mathbb{A} \hookrightarrow \mathbb{C}$ ~~sottocategoria~~ PIENA di $\mathbb{C}$ si dice RIFLESSIVA ~~se~~

J ha aggiunto sinistro $R : \mathbb{C} \to \mathbb{A}$
$R \dashv J$ (J è pieno e fedele)

— Se parto da $F \dashv G : \mathbb{D} \to \mathbb{C}$ con G pieno e fedele, trovo che $\mathbb{D}$ è equivalente a una sottocategoria (riflessiva) di $\mathbb{C}$

$$\begin{tikzcd} \mathbb{D} \arrow[r, "G"] \arrow[dr, "G'"'] & \mathbb{C} \\ & G(\mathbb{D}) \arrow[u, hook, "J"'] \end{tikzcd}$$

G' è pieno, fedele (ess.) suriettivo $\Rightarrow$ G' equivalenza

$G(\mathcal{D})$ è una sottocategoria piena (e riflessiva)

$\underline{Ab} \hookrightarrow \underline{Grp}$ J ha aggiunto sx

$R \dashv J$ quindi J essendo pieno e fedele con aggiunto sx $\Rightarrow$ è monadico

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$$\begin{tikzcd} \underline{Ab}_{tf} \arrow[r, hook, "J"'] & \underline{Ab} \arrow[l, bend right, "R"'] \end{tikzcd}$$

J ha aggiunto sx

$R \dashv J$

$\nearrow$ liberi da torsione

J è monadico

$$\begin{tikzcd} Ab_{tf} \arrow[r, hook, "J"'] \arrow[r, phantom, "\perp"] & \underline{Ab} \arrow[l, bend right, "R"'] \arrow[r, "U"'] \arrow[r, phantom, "\perp"] & \underline{Set} \arrow[l, bend right, hook, "F"'] \end{tikzcd}$$

monadico

$F(X) = \mathbb{Z}^{(X)}$

UJ è ancora monadico?

$RF \dashv UJ$ è aggiunto sx

la monade associata $(UJRF)(X) = \underline{UF} = T$

FX è un gruppo libero, è libero da torsione!

$$\begin{tikzcd}
\mathrm{Ab}_{tf} \arrow[rr, "UJ"] \arrow[dr, "K"'] & & \underline{\mathrm{Set}} \\
& \underline{\mathrm{Set}}^{T'} = \mathrm{Set}^{T} \cong \underline{\mathrm{Ab}} \arrow[ur] &
\end{tikzcd}$$

K non è un'equivalenza

$\Rightarrow$ UJ non è monadico

MONADI su $\mathbb{C}$

$$(T, \mu, \eta)$$

$$T \in Ob(End(\mathbb{C}))$$

oggetti: endofuntori morfismi

$$\mu : T^2 = T \circ T \xrightarrow{t.n.} T$$

$$\eta : \underset{I}{\overset{\shortparallel}{Id_{\mathbb{C}}}} \longrightarrow T$$

Monoidi in Set

$$(M, m, e)$$

$$M \in Ob(Set)$$

$$m : M \times M \longrightarrow M$$

$$e : \underset{I}{\overset{\shortparallel}{\{*\}}} \longrightarrow M \qquad e(*) = e$$

ASSIOMI

ASSOCIATIVITÀ +
elemento neutro
a dx e a sx

$$\begin{tikzcd}
(M \times M) \times M \arrow[r, "m \times 1_M"] \arrow[d, "\cong"'] & M \times M \arrow[d, "m"] \\
M \times (M \times M) \arrow[r, "1_M \times m"] & M
\end{tikzcd}$$

$$((a,b),c) \longmapsto (a\cdot b, c) \longmapsto (a\cdot b)\cdot c = a\cdot(b\cdot c)$$

$$(a,(b,c)) \longmapsto (a, b\cdot c) \overset{m}{\longmapsto} a\cdot(b\cdot c)$$

elemento neutro m

$$M \times M \xrightarrow{m} M$$
$$(a, e) \mapsto a \cdot e$$

$$\begin{tikzcd}
M \times \{*\} \arrow[r, "1_M \times e"] \arrow[dr, "\pi_M \cong"'] & M \times M \arrow[d, "m"] \\
& M
\end{tikzcd}$$

$a = (a \cdot e)$

+ altro ax

CATEGORIE MONOIDALI

$(\mathrm{END}(\mathbb{C}), \circ, 1_{\mathbb{C}} = I)$

monoidale stretta

$(\underline{Set}, \times, \{*\} = I$ + iso canonici$)$

monoidale

$$- \circ - : \mathrm{END}(\mathbb{C}) \times \mathrm{END}(\mathbb{C}) \longrightarrow \mathrm{END}(\mathbb{C})$$

<u>BIFUNTORE</u>

$$\begin{tikzcd} G \circ F : \mathbb{C} \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \mathbb{C} \arrow[r, "G"] & \mathbb{C} \end{tikzcd}$$

<u>Df</u> $F \otimes G = G \circ F$

$$\begin{tikzcd} (F, G) \arrow[r, mapsto] \arrow[d, "\alpha \cdot\ \cdot \beta"'] & F \otimes G = GF \arrow[d, "\alpha \otimes \beta"] \\ (F', G') \arrow[r, mapsto] & F' \otimes G' = G'F' \end{tikzcd}$$

$- \otimes -$ è un bifuntore

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathbb{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathbb{C} \end{tikzcd}$$

$\alpha \otimes \beta$ = comp orizzontale di α e β

$$\alpha \otimes \beta : GF \xrightarrow{\ \cdot\ } G'F'$$

$$\begin{tikzcd} (\alpha \otimes \beta)_C \quad GFC \arrow[r, "G(\alpha_C)"] & GF'C \arrow[d, "\beta_{F'C}"] \\ & G'F'C \end{tikzcd}$$

$I := \mathrm{Id}_{\mathcal{C}} \qquad F \otimes I = I \otimes F = F$

$$(F \otimes G) \otimes H = F \otimes (G \otimes H)$$

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

$(\mathrm{End}(\mathcal{C}), \otimes, I = \mathrm{Id}_{\mathcal{C}})$ è un esempio di cat monoidale stretta

<u>Df</u> $(\mathcal{V}, \otimes, I)$ è una cat monoidale stretta con

$$- \otimes - : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V} \quad \text{bifuntore}$$

$$I \in \mathrm{ob}(\mathcal{V})$$

tali che

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$

$$I \otimes X = X \otimes I = X$$

$$1_I \otimes f = f \otimes 1_I = f$$

perché $- \otimes -$ è un bifuntore

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[dd, bend right=60, "f'f"'] & Y \arrow[d, "g"'] \arrow[dd, bend left=60, "g'g"] \\ X' \arrow[d, "f'"] & Y' \arrow[d, "g'"'] \\ X'' & Y'' \end{tikzcd}$$

$$\begin{tikzcd} X \otimes Y \arrow[d, "f \otimes g"] \arrow[dd, bend left=60, "(f'f)\otimes(g'g)"] \\ X' \otimes Y' \arrow[d, "f' \otimes g'"] \\ X'' \otimes Y'' \end{tikzcd} =$$

$$(f' \otimes g')(f \otimes g) = f'f \otimes g'g$$

$$1_X \otimes 1_Y = 1_{X \otimes Y}$$

<u>ES</u>

- $(\mathrm{End}(\mathcal{C}), \otimes, I = \mathrm{id}_{\mathcal{C}})$
- M un monoide visto come categoria DISCRETA

  $m \otimes m' = mm'$ $\qquad I = e$

- $\mathbb{D}$ = oggetti di $\mathbb{D}$

  $\forall$ $\underline{n} = \{1, .., n\}$ ordinati
  $n \geq 1$

  $n = 0$ $\quad \underline{0} = \emptyset$ $\quad f : \underline{n} \longrightarrow \underline{m}$

  f monotona

$n \otimes m = n + m = \{1 \dots n, n+1 \dots n+m\}$

$I = 0$

$f : \underline{n} \longrightarrow \underline{m}$

$g : \underline{n'} \longrightarrow \underline{m'}$

$f \otimes g : n \otimes n' \longrightarrow m \otimes m'$

$(Set, \times, \{*\} = I)$ non è stretta

$A \otimes B = A \times B = \{ (a,b) \; a \in A, \; b \in B \}$

$(A \otimes B) \otimes C \neq A \otimes (B \otimes C)$

ma sono isomorfi!

$A \otimes I = A \times \{*\} \neq A$

è però una categoria monoidale (non stretta)

Df Una categoria monoidale è data da una cat V

con

- $- \otimes - : V \times V \longrightarrow V$ funtore
- $I \in Ob(V)$
- $\forall X, Y, Z$ $\quad \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ naturale in X, Y, Z

- $\forall X \quad \lambda_X : I \otimes X \xrightarrow{\cong} X$

naturale in X

- $\forall X \quad \rho_X : X \otimes I \xrightarrow{\cong} X$

naturale in X

$(\mathcal{V}, \otimes, \alpha, \lambda, \rho)$

$$
\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W\otimes X,Y,Z}"'] \arrow[dr, "\alpha_{W,X,Y}\otimes \mathrm{Id}_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y\otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "\mathrm{Id}_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes B"'] & & A \otimes (I \otimes B) \arrow[dl, "A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
$$