CT

22/12/15

TH

A ab $\iff$ esatta + additiva

I PROP

A ab $\implies$ additiva

Dim

$$
\begin{tikzcd}
 & A \arrow[d, "{\langle 1,0\rangle}"] \arrow[dr, bend left, "r"] & \\
A \arrow[r, "\Delta"] & A \times A \arrow[r, "q"] & Q
\end{tikzcd}
$$

$q = \operatorname{coker}(\Delta)$

$r := q\langle 1,0\rangle$

TS r è iso $\iff$ epi + mono

- r epi

$$
\begin{tikzcd}
A \arrow[r, tail, "{\langle 1,0\rangle}"] \arrow[dr, "0"'] & A \times A \arrow[r, "q"] \arrow[d, "p_2"] & Q \\
 & A &
\end{tikzcd}
$$

p_2 è epi

EX $\langle 1,0\rangle = \ker p_2$

$$\begin{tikzcd} A \arrow[r, tail, "{\langle 0,1\rangle}"] \arrow[dr, equal, "1"'] & A\times A \arrow[d, "p_2"] \\ & A \end{tikzcd}$$

1 epi $\Rightarrow$
p_2 epi
$\Downarrow$

$$p_2 = \mathrm{coker}(\ker p_2) =$$
$$= \quad " \quad (\langle 1,0\rangle)$$

$$\begin{tikzcd} & A \arrow[d, "\Delta = \langle 1,1\rangle"] & \\ A \arrow[r, "{\langle 1,0\rangle}"] \arrow[dr, "v"'] & A\times A \arrow[r, "p_2"] \arrow[d, "q"] \arrow[dr, "zq"] & A \arrow[d, dashed, "\exists!\, t"] \\ & Q \arrow[r, "z"] & Z \end{tikzcd}$$

v è epi $\Leftrightarrow$ $\forall z$ $\quad zv = 0$
$\Rightarrow z = 0$

$\Rightarrow \quad zq\langle 1,0\rangle = 0$

poiché $p_2 = \mathrm{coker}(\langle 1,0\rangle)$ $\exists!\, t$
$t p_2 = zq$

$$t\underbrace{p_2\Delta}_{1} = z\underbrace{q\Delta}_{0} \Rightarrow t = 0$$

$$0 \cdot p_2 = 0 = z\cancel{q} \qquad q \text{ è epi}$$

$$\Rightarrow z = 0$$

$\Rightarrow r$ è epi

.. r è anche mono, cioè

se $rx = 0 \Rightarrow x = 0$

$$\begin{tikzcd}
X \arrow[r, "\exists!\, y"] \arrow[d, "x"'] & A \arrow[d, "\Delta"] & \\
A \arrow[r, tail, "{\langle 1,0\rangle}"] \arrow[dr, "r"'] & A\times A \arrow[r, "p_2"] \arrow[d, "q"] & A \\
& Q &
\end{tikzcd}$$

$q(\langle 1,0\rangle x) = 0 \qquad \Delta$ è mono

$\Delta = \ker(\underset{q}{\underset{\parallel}{\operatorname{coker}}}\,\Delta) \Rightarrow \exists!\, y$

$\Delta y = \langle 1,0\rangle x$

$$\underbrace{p_2\,\Delta}_{1}\, y = \underbrace{p_2 \langle 1, 0\rangle}_{\overset{\shortparallel}{0}}\, x$$

$$\Downarrow$$

$$y = 0 \Rightarrow \langle 1, \cancel{0}\rangle x = 0 \qquad x = 0$$

$\langle 1, 0\rangle$ mono

$\Rightarrow$ r è un iso

Allora possiamo scegliere come coker $\Delta = \sigma := r^{-1}q$

$$\begin{tikzcd} A\times A \arrow[r, "q"] \arrow[rr, bend right, "\sigma"'] & Q \arrow[r, "r^{-1}"] & A \end{tikzcd}$$

σ sta per $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ della struttura additiva che avevamo ora

a dx su A

$$f, g : X \underset{g}{\overset{f}{\rightrightarrows}} A$$

$f - g := \sigma \langle f, g \rangle$

$$X \xrightarrow{\langle f, g \rangle} A \times A \xrightarrow{\sigma} A$$

composta $f + g = f - (0 - g)$

è un'op binaria su

Hom (X, A) che si dimostra

essere gruppo ab e la

comp bilineare risp ~~alla~~ a questa

op.

DETTAGLI HANDBOOK, II,

BORCEUX, 23-25 $\square$

II PROPOSIZIONE

A abeliana $\Rightarrow$ A

esatta

<u>Dim</u> esatta $\Leftrightarrow$ (a) rel. equivalenza effettive
(b) regolare

(a) R equivalenza su X

$$R \xrightarrow{\langle r_0, r_1\rangle} X \times X = X \oplus X$$

$$\Updownarrow$$

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \xrightarrow{q} X/R$$

$$q = \mathrm{coeq}(r_0, r_1) = \mathrm{coker}(r_0 - r_1)$$

$$\begin{tikzcd}
R \arrow[r, tail, "\langle r_0, r_1\rangle"] & X \oplus X \arrow[r, "\begin{pmatrix} q \\ -q \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} h \\ k = -h \end{pmatrix}"'] & X/R \\
X \arrow[u, "\delta"] & & {}
\end{tikzcd}$$

<u>TS</u> $\begin{pmatrix} q \\ -q \end{pmatrix} = \mathrm{coker}\ \langle r_0, r_1 \rangle$

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- $\begin{pmatrix} q \\ -q \end{pmatrix} \langle r_0, r_1 \rangle = q r_0 - q r_1 = 0$

- - Dato $\begin{pmatrix} h \\ k \end{pmatrix} : X \oplus X \to Z$

t.c. $\begin{pmatrix} h \\ k \end{pmatrix} \langle r_0, r_1 \rangle = 0 \iff$

$$h r_0 + k r_1 = 0$$

$$(h r_0 + k r_1)\delta = 0$$

$$h \underbrace{r_0 \delta}_{1} + k \underbrace{r_1 \delta}_{1} = 0 \qquad h + k = 0$$

$$\Updownarrow$$

$$k = -h$$

cioè

$$h r_0 - h r_1 = 0$$

$$h(r_0 - r_1) = 0$$

ma

$q = \mathrm{cok}(r_0 - r_1)$ $\exists! \varphi : X/R \to Z$

$$\varphi q = h$$

$$\begin{array}{c} X \\ \downarrow q \\ X/R \\ \downarrow \varphi \\ Z \end{array}$$

$$\varphi\begin{pmatrix} q \\ -q \end{pmatrix} = \begin{pmatrix} \varphi q \\ -\varphi q \end{pmatrix} = \begin{pmatrix} h \\ -h \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$$

$$\begin{tikzcd}
X \oplus X \arrow[rr, "\begin{pmatrix} q \\ -q \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} h \\ k \end{pmatrix}"'] & \arrow[d, phantom, "u"] & X/R \arrow[dl, "\varphi"] \\
& Z &
\end{tikzcd}$$

quindi $\begin{pmatrix} q \\ -q \end{pmatrix} = \mathrm{coker}\,(\langle v_0, v_1 \rangle)$

allora $\langle v_0, v_1 \rangle$ essendo

mono $= \ker(\mathrm{coker}\langle v_0, v_1 \rangle)$

$\langle v_0, v_1 \rangle = \ker \begin{pmatrix} q \\ -q \end{pmatrix}$

$\Downarrow$ TS

$R \underset{v_1}{\overset{v_0}{\rightrightarrows}} X$ è il kernel

pair di q, cioè

$$
\begin{tikzcd}
R \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & X \arrow[r, "q"] & X/R \\
S \arrow[u, "\exists ! \psi"] \arrow[ur, shift left, "s_0"] \arrow[ur, shift right, "s_1"'] & &
\end{tikzcd}
$$

$\forall \; s_0, s_1$

$$q s_0 = q s_1$$

$$q(s_0 - s_1) = 0$$

$$\|$$

$$q s_0 - q s_1$$

$$\|$$

$$\begin{pmatrix} q \\ -q \end{pmatrix} \langle s_0, s_1 \rangle$$

$$
\begin{tikzcd}
R \arrow[r, "\langle v_0, v_1 \rangle"] & X \oplus X \arrow[r, "\begin{pmatrix} q \\ -q \end{pmatrix}"] & X/R \\
S \arrow[u, "\psi"] \arrow[ur, "\langle s_0, s_1 \rangle"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

$\langle v_0, v_1 \rangle = \ker \begin{pmatrix} q \\ -q \end{pmatrix}$

$\Rightarrow \exists ! \; \psi \quad \langle v_0, v_1 \rangle \psi = \langle s_0, s_1 \rangle$

so $v_0 \psi = s_0$ & $v_1 \psi = s_1$ $\square$

R è effettiva

(b) regolarità

$\Updownarrow$

① limiti finiti

② $\exists$ coeq di see di eq

③ regolari epi stabili x pullback : da dim

se A abeliana epi = epi regolare

$$\begin{tikzcd} X' \arrow[r, "e'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathrm{I}"] & Y' \arrow[r, dashed, "x"] \arrow[d, "f"] & Z \\ X \arrow[r, "e"'] & Y & \end{tikzcd}$$

e epi $\Downarrow$ e' epi

cioè $xe' = 0 \Rightarrow x = 0$

commutatività quadrato

$\Updownarrow$

$$\begin{tikzcd} X' \arrow[r, "\langle f', e' \rangle"] \arrow[rr, bend right, "= 0"'] & X \oplus Y' \arrow[r, "\binom{e}{-f}"] & Y \end{tikzcd}$$

$$\begin{pmatrix} e \\ -f \end{pmatrix} \langle f', e' \rangle = ef - fe' = 0$$

① $\bar{e}$ un pullback $\iff$

$$\langle f', e' \rangle = \ker \begin{pmatrix} e \\ -f \end{pmatrix}$$ EX

e è epi $\Rightarrow \begin{pmatrix} e \\ -f \end{pmatrix}$ epi

$$\begin{tikzcd}
X \oplus Y' \arrow[rr, "\begin{pmatrix} e \\ -f \end{pmatrix}"] & & Y \\
X \arrow[u, "\langle 1, 0 \rangle"] \arrow[urr, "e"'] \arrow[ur, phantom, "="] & &
\end{tikzcd}$$

$$\Rightarrow \begin{pmatrix} e \\ -f \end{pmatrix} = \operatorname{coker}\left(\ker \begin{pmatrix} e \\ -f \end{pmatrix}\right) = \operatorname{coker}(\langle f', e' \rangle)$$

Data $x : Y' \longrightarrow Z \qquad xe' = 0$

costruiamo

$$\begin{tikzcd} & X \oplus Y' \arrow[r, "(e \; -f)"] \arrow[d, "\binom{0}{x}"] & Y \arrow[dl, dashed, "\exists ! \varepsilon"] \\ X' \arrow[ur, "\langle f', e' \rangle"] & Z & \end{tikzcd}$$

$$\binom{0}{x} \langle f', e' \rangle = 0 \cdot f' + x e' = 0$$

$$\exists ! \ \varepsilon : Y \to Z \qquad \varepsilon \binom{e}{-f} = \binom{0}{x}$$

$$\begin{cases} \varepsilon \cancel{e} = 0 & e \text{ epi} \\ \varepsilon(-f) = x & \Rightarrow x = 0 \end{cases}$$

$\Rightarrow$ e' epi

$\Rightarrow$ $\mathcal{A}$ è reg + rel epi eff $\Rightarrow$ $\mathcal{A}$ è esatto

$\square$

III PROPOSIZIONE

A esatta + additiva

$\Rightarrow$ A è abeliana

Dim A è ab $\Leftrightarrow$

1. A ha 0 oggetto
2. A ha prodotti e coprodotti binari

√ additività

è esatta (ha nuclei, conuclei)

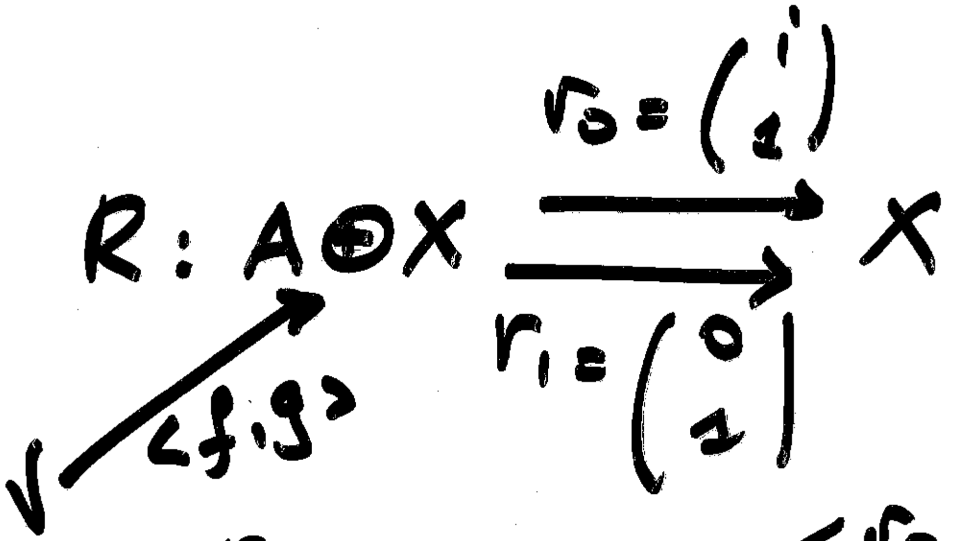
3. ogni freccia ha un kernel e un cokernel
4. ~~ogni~~ ogni mono è kernel ogni epi è cokernel

quello in rosso è da dim

cominciamo col dim

- ogni mono è kernel

Sia $i : A \rightarrowtail X$ mono



- $R = A \oplus X \xrightarrow{\langle r_0, r_1 \rangle} X \oplus X$

è mono, cioè R è una relazione su X

Sia $\langle f, g \rangle : V \longrightarrow A \oplus X$

$$\langle r_0, r_1 \rangle \langle f, g \rangle = 0 \iff \begin{cases} r_0 \langle f, g \rangle = 0 \\ r_1 \langle f, g \rangle = 0 \end{cases}$$

$$\begin{cases} \begin{pmatrix} i \\ 1 \end{pmatrix} \langle f, g \rangle = 0 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \langle f, g \rangle = 0 \end{cases} \iff \begin{cases} if + \cancel{g} = 0 \\ 0f + g = 0 \end{cases}$$

$g = 0$

$\cancel{i}f = 0 \implies f = 0$

• R è RIFLESSIVA

$$X \xrightarrow{\delta := \langle 0,1\rangle} R = A \oplus X \underset{\binom{0}{1}}{\overset{\binom{i}{1}}{\rightrightarrows}} X$$

$$\binom{i}{1}\langle 0,1\rangle = 1$$

$$\binom{0}{1}\langle 0,1\rangle = 1$$

A abel ⇒ A Mal'cev ⇒ R è di equivalenza, quindi effettiva, cioè (R, v_0, v_1) = kernel pair di $q = \mathrm{coeq}(v_0, v_1)$

$$R \underset{\binom{0}{1}}{\overset{\binom{i}{1}}{\rightrightarrows}} X \xrightarrow{q} X/R$$

<u>TS</u> $\quad i = \ker q$

$$\underset{\substack{\| \\ r_0}}{q\begin{pmatrix} i \\ 1 \end{pmatrix}} = \underset{\substack{\| \\ r_1}}{q\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$q\, i = 0 \quad .$$

adesso q'e $\quad x : T \longrightarrow X$ t.c.

$$\begin{tikzcd} T \arrow[r, "x"] \arrow[d, "t"'] & X \\ A \arrow[ur, tail, "i"'] & \end{tikzcd}$$

$q\,x = 0$

TS $\quad \exists!\, t \quad i\,t = x$

$$\begin{tikzcd} T \arrow[d, "\exists!"] \arrow[dr, bend left, "x"] & \\ A\oplus X = R \arrow[r, "\begin{pmatrix} i \\ 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[d, bend right=60, "0"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "q"] \\ X \arrow[r, "q"'] & X/R \end{tikzcd}$$

per pr. and pb, $\exists!\, \langle t_0, t_1\rangle : T \longrightarrow A \oplus X$

$$\begin{cases} \begin{pmatrix} i \\ 1 \end{pmatrix} \langle t_0, t_1 \rangle = x \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \langle t_0, t_1 \rangle = 0 \end{cases}$$

$$\begin{cases} i\,t_0 + \cancel{t_1} = x \\ 0\cdot t_0 + s\cdot t_1 = 0 \end{cases} \quad \Rightarrow t_1 = 0$$

$$\Rightarrow \quad x = i\,t_0$$

$\exists\ t = t_0$ che

$$\begin{tikzcd} T \arrow[r, "x"] \arrow[d, "t_0"'] & X \\ A \arrow[ur, "i"'] & \end{tikzcd}$$

(t_1 è unica perché i è mono)

$\Rightarrow \quad i = \ker q$

inoltre q cokernel di $v_0 - v_1$

$$\Rightarrow \quad q = \operatorname{coker}(\ker q) = \operatorname{coker}(i)$$

quindi ogni mono ha un cokernel

Sia ora f una freccia qualunque

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \arrow[d, "q"] \\ & \cdot \arrow[ur, tail, "i"'] & {} \end{tikzcd}$$

(e, i) fatt. (epi reg, mono) di f per regolarità di A

se $q = \mathrm{coker}(i)$

Allora $q = \mathrm{coker}(f)$ <u>ex</u>

quindi ogni freccia ha un cokernel

se prendo f epi

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, "e"'] & & \cdot \arrow[r, "q"] & \cdot \\ & \cdot \arrow[ur, "i"'] & & \end{tikzcd}$$

$q = \mathrm{coker} f$

$\Rightarrow q f = 0$

$\Rightarrow q = 0 \qquad i = \ker q = \ker 0$

$\Rightarrow i$ iso

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, "e"'] & & Y \\ & {} \arrow[ur, "\cong"'] & \end{tikzcd}$$

f è un regolare epi

$\Downarrow$

$$f = \mathrm{coeq}(R[f]) = R[f] \overset{t_0}{\underset{t_1}{\rightrightarrows}} \xrightarrow{f}$$

$= \mathrm{coker}\,(t_0 - t_1)$

$\Rightarrow$ ogni epi è un coker

$\Rightarrow A$ è abeliana $\square$