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Chapter 1

1.1 Statistical ensembles and perfect gases

1.1.1 *Microcanonical ensemble*

The aim of statistical mechanics is the derivation the macroscopic properties of matter starting from a microscopic description in terms of a collection of a large number of particles. The motion of such particles can be described by the laws of classical physics (classical statistical mechanics) or of quantum physics (quantum statistical mechanics). In certain cases when the temperature is not too low classical statistical physics provide a good description of the macroscopic properties. At low temperatures however quantum effects are very important, and one enters in the realm of quantum statistical physics. While physically the systems described by classical and quantum physics are completely different, there is a very deep connection between systems described by classical statistical mechanics in d dimensions and quantum statistical mechanics in $d-1$ dimensions (a dimension less); it will be therefore very useful to introduce classical and quantum statistical physics simultaneously.

Let we start with classical gases, which can be described as a system of N identical particles with positions $q_i \in \Lambda \subset \mathbb{R}$, with $|\Lambda| = V$ and momentum p_i, $i = 1, .., dN$ verifying the Hamilton equation, $(p, q) \in R^{dN} \times R^{dN}$

$$q_i' = \frac{\partial H}{\partial p_i} \quad p_i' = -\frac{\partial H}{\partial q_i} \tag{1.1}$$

and $H = H(\vec{p}_1 ... \vec{p}_N, \vec{q}_1 ... \vec{q}_N)$. A typical Hamiltonian has the form

$$H = \sum_{i=1}^{N} [\frac{|\vec{p}_i|^2}{2m} + U(\vec{q}_i)] + \sum_{i,j=1, i\neq j}^{N} v(|\vec{q}_i - \vec{q}_j|) \tag{1.2}$$

where the first term represent the kinetic energy, the second term the interaction with an external field and the last term the mutual interaction between articles. The interaction is typically depending on the distance between particles, and has an hard-core property forbidding that the particles are too close. As the particles are in a finite volume, one has to supplement the Hamiltonian with a condition saying what happens at the boundary. Several choices are possible, like reflecting boundary conditions (the particles are reflected elastically) or periodic boubdary conditions.

The dN-dimensional space of all the possible configurations and momenta is called *phase space*, and the complete microscopic state of an isolated classical system of N point particles is specified at any time t by a point $(p(t), q(t))$ in the phase space $\Gamma = \mathbb{R}^{dN} \times \mathbb{R}^{dN}$, evolving according to the Hamilton equations. The point $(p(t), q(t))$ lies on a surface of constant energy $p, q : H(p, q) = E$ which will be denoted by Γ_E.

For instance if we have two particles with same mass moving in a line $(0, L)$ with no force, then Γ_E is $(0, L)^R \times S^2$ where S^2 is a circle with radius $\sqrt{2mE}$.

On the other hand, the macroscopic description of the same gas is completely different, as it is given in term of a few macroscopic variables like the temperature, the pressure, the entropy and so on. In order to explain how the macroscopic properties emerge from the microscopic description, one starts from the following properties which looks reasonable for physical systems:

a) the microscopic time scale is much smaller than the macroscopic one, and a macroscopic observation correspond to a average over a long time at the microscopic level;

b) A macroscopic state can be realized in a number of different ways at a microscopic level, that is the constant energy surface in the phase space Γ_E is dived in sectors Γ_M corresponding to different macroscopic states M;

c)There exists a macrostate, M_{eq} (called equilibrium state), whose sector has a volume much bigger than all the others M (corresponding to non equilibrium) and $\Gamma_{M_{eq}} \sim |\Gamma_E|$

d)In its time evolution the system spend in a region Γ_M a time roughly proportional to its relative size.

As a consequence of the property a), the macroscopic properties are obtained by averaging over the time; for instance if $O(p, q)$ is a physical

observable, like the kinetic energy $K(p,q) = \sum_{i=1}^{dN} p_i^2/2m$, the observed value is the time average

$$< P >_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt O(p(t), q(t)) \tag{1.3}$$

Of course in most interesting physical systems one cannot compute $< P >_t$, as one has to solve $\sim 10^{23}$ nonlinear coupled equations; moreover from the above definition we see that $< P >_t$ depend from the initial position and velocity of all particles, which are of course unknown. There is therefore no hope in general to be able to compute the above time average. It is a basic *postulate* of statistical mechanics, due to Gibbs and following from (b), (c) and (d) above, that one can compute the macroscopic observables averaging over all possible configurations in the phase space. Note that the disparity in size of the Γ_M explain the approach to equilibrium: if the system is in equilibrium it remains in equilibrium for most of the time, while if it is outside equilibrium it will quickly return to equilibrium.

In statistical physics one studies averages of physical quantities over all possible configurations and postulates that they represent the macroscopic observables. Starting from this postulate and the hamiltonians describing the physical system, we can proceed in a deductive way. Of course it is also important to try to prove (not sipmly assume) such properties starting from the mechanics or the theory of dynamical systems; there are important *ergodic theorems* in the theory of dynamical systems ensuring that under certain conditions the time averages are equal averages over the phase space. There have been important progress in proving ergodicity in certain dybnamical systems as billiards (Sinai 1966) with small number of particles, but wether such theorems can be extended to provide a real fundations of statistical mechanics ($\sim 10^{23}$ is unclear.

The simplest of such averages is simply to integrate over all the points on the surface with a given energy, giving equal weight to any configuration. Such prescription is called *microcanonical ensamble* (Boltzmann (1884) and Gibbs (1902)); the word ensemble refer to the fact that we are replacing the average over time evolved systems with the average of an ensamble of systems with the same energy. experiments. Therefore, if $O(p,q)$ represents a microscopic observable property of the system (time dependent), one postulates that the observed quantity is its average over all possible states with the same energy (more exactly, in between two energies in order to avoid trivial results). *We call $T = \kappa\tau$ where τ is the temperature; κ is the Boltzmann constant.*

Definition *The entropy S of a gas of identical particles with energy $E > 0$ in a box with volume V known with accuracy $h > 0$, $0 < h/E < 1$ in the microcanonical ensemble*

$$S = \log \Delta\Gamma \tag{1.4}$$

where, in the case of indistinguishible particles

$$\Delta\Gamma = \frac{1}{N!} \int_{E-h \leq H \leq E} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} \tag{1.5}$$

while if they are dishinguible $\Delta\Gamma$ is defined without factorial. Given a $O(p, q)$ its average is defined as

$$< O >= \frac{\int_{E-h \leq H \leq E} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} O(p, q)}{\int_{E-h \leq H \leq E} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN}} \tag{1.6}$$

Finally the temperature T and the pressure p are defined as

$$\frac{1}{T} = (\frac{\partial S}{\partial E})_{E,V} \qquad \frac{p}{T} = (\frac{\partial S}{\partial V})_{E,V} \tag{1.7}$$

One postulates that the entropy and the observables defined in that way correspond to the ones experimentlly observed. According to the above definition the entropy is an increasing function of the volume of the phase space corresponding to energies between $E - h$ and E: more are the configurations possible in presence of a certain energy, the larger is the entropy. In case of identical particles it is necessary to divide by $N!$, as a permutation is an identical microscopic realization of the same state. The $N!$ is essential to ensure even in the case of the perfect gas that the entropy is an intensive quantity (that is proportional to N up to negligible corrections).

The entropy is *additive*; if we have two systems 1 and 2 and we combine together so that $\Delta\Gamma = \Delta\Gamma_1 \Delta\Gamma_2$ then $\sigma = \log(\Delta\Gamma_1 \Delta\Gamma_2) = \sigma_1 + \sigma_2$. In the example before of two particles in a line $\Gamma_E = \Gamma_{E_1} \times \Gamma_{E_2}$. Note finally that we consider the average with energies between $E - h$ and E; of course all the dependence from h should disappear in the limit.

An interesting property of the entropy defined in that way is how it changes with the volume; assume that we have a perfect gas (N non interacting particles) the difference of entropy correspondinf to particles in a volume V and $2V$ is

$$\frac{\sigma_2}{\sigma_1} = \log \frac{\Delta\Gamma_2}{\Delta\Gamma_1} = \log 2^N = N log 2 \tag{1.8}$$

Therefore, if we double the volume accessible to a gas the entropy increases. We can extend the above definition of entropy to non equilibrium situations; suppose that the system is in a macroscopic state M corresponding to a non-equilibrium situation (for instance the particle of a gas in a room occupy half of the volume) and we define the entropy corresponding to the non equilibrium macrostate M as $S(M) = \log \Delta\Gamma_M$; the probability of such situation is proportional the the normalized volume corresponding to M so that

$$P(M) \sim \frac{\Delta\Gamma_M}{\Delta\Gamma} = e^{S(M)-S} \tag{1.9}$$

so that $S(M) = \log \Delta\Gamma_M$ is the large deviation function for the macrostate M.

The definition of pressure and temperature is given requiring that the following thermodynamical relation (second law of thermodynamics) is verified

$$dE = TdS + pdV \tag{1.10}$$

1.1.2 *The perfect gas in the microcanonical ensemble: thermodynamic limit*

The Hamiltonian of the perfect gas is

$$H = \sum_{i=1}^{dN} \frac{p_i^2}{2m} \tag{1.11}$$

so that

$$Z = \frac{1}{N!} \int_{E-h \le \sum_{i=1}^{dN} \frac{p_i^2}{2m} \le E} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} =$$
$$V^N \frac{1}{N!} \int_{E \le \sum_{i=1}^{dN} \frac{p_i^2}{2m} \le E+h} dp_1 dq_1 ... dp_{dN} \tag{1.12}$$

We will define the specific volume and energy

$$v = V/N \qquad \varepsilon = E/N \tag{1.13}$$

and we call $\sigma = \frac{S}{N}$ the specific entropy for particle. In statistical physics N is typically very large, so that it is convenient to consider the *thermodynamic limit* defined as

$$N \to \infty, \quad v, e \quad \text{fixed} \tag{1.14}$$

The specific entropy in the microcanonical ensamble for the perfect gas is given in the thermodynamical limit by

$$\lim_{\substack{N\to\infty \\ V/N=v, E/N=efixed}} \sigma = \log[v(2m\varepsilon)^{d/2}\pi^{\frac{d}{2}}(\frac{d}{2e})^{-d/2}e] \tag{1.15}$$

In order to peform some computation we recall some well known formula. The volume $V_n(R)$ of a shere with radius R in n dimensions is given by

$$V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}R^n \tag{1.16}$$

where Γ is the Gamma function

$$\Gamma(z) = \int_0^\infty dtt^{z-1}e^{-t}dt \tag{1.17}$$

We recall some well known properties of the Γ function. By integrating by parts, if $Rez > 0$

$$\Gamma(z+1) = z\Gamma(z) \tag{1.18}$$

as

$$\int_0^\infty dtt^z e^{-t}dt = -t^z e^{-t}|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}dt = z\Gamma(z) \tag{1.19}$$

From the above property, if $n \in \mathbb{N}$ then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1).... = n! \tag{1.20}$$

Note also that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\Gamma(\frac{n}{2}) = \frac{(n-1)!!}{2^{(n-1)/2}\sqrt{\pi}} \tag{1.21}$$

In order to prove (1.16) we write, using spherical coordinates

$$V_n(R) = \int_{\sum_i x_i^2 \le R^2} dx_1...dx_n = \int_0^R \rho^{n-1}d\rho \int d\Omega = \Omega\frac{R^n}{n} \tag{1.22}$$

On the other hand

$$\int_{-\infty}^\infty dx_1..dx_n e^{-(x_1^2+...x_n^2)} = \pi^{\frac{n}{2}} = \int_0^\infty \rho^{n-1}e^{-\rho^2}d\rho \int d\Omega \tag{1.23}$$

and by the change of variables $t = \rho^2$

$$\int_0^\infty \rho^{n-1} e^{-\rho^2} d\rho = \frac{1}{2}\int_0^\infty dt t^{\frac{n-1}{2}} t^{-\frac{1}{2}} e^{-t} = \frac{1}{2}\Gamma(\frac{n}{2}) \tag{1.24}$$

Therefore

$$\Omega\Gamma(\frac{n}{2})\frac{1}{2} = \pi^{\frac{n}{2}} \tag{1.25}$$

and finally

$$V_n(R) = \Omega\frac{R^n}{n} = R^n\pi^{\frac{n}{2}}\frac{1}{\Gamma(\frac{n}{2})}\frac{1}{\frac{n}{2}} = R^n\pi^{\frac{n}{2}}\frac{1}{\Gamma(\frac{n}{2}+1)} \tag{1.26}$$

Another important property is that

$$V_n(R) - V_n(R-h) = \pi^{\frac{n}{2}}\frac{1}{\Gamma(\frac{n}{2}+1)}R^n(1-(1-\frac{h}{R})^n)$$

and if $0 < \frac{h}{R} < 1$ then

$$\lim_{n\to\infty}(1-\frac{h}{R})^n = 0$$

The above property say that in the computation of $\Delta\Gamma$ we can integrate over the configuration space of the systems with energies $\leq E$.

We consider now

$$\int_{\sum_{i=1}^{dN}\frac{p_i^2}{2m}\leq E} dp_1 dq_1 ... dp_{dN} = (2m)^{dN/2}\int_{\sum_{i=1}^{dN} p_i^2\leq E} dp_1 dq_1 ... dp_{dN} =$$

$$(2mE)^{dN/2}\pi^{\frac{dN}{2}}\frac{1}{\Gamma(\frac{dN}{2}+1)} \tag{1.27}$$

so that the entropy for particle is given by, in the thermodynamic limit

$$\sigma = \lim_{N\to\infty}\frac{1}{N}\log[(vN)^N(2m\varepsilon N)^{dN/2}\pi^{\frac{dN}{2}}2\frac{1}{\Gamma(\frac{dN}{2}+1)}\frac{1}{N!}] \tag{1.28}$$

We use now the Stirling formula

$$\lim_{n\to\infty}(\frac{n}{e})^n\sqrt{2\pi n}\frac{1}{n!} = 1 \tag{1.29}$$

so that

$$\sigma = \lim_{N\to\infty}\log[(vN)(2m\varepsilon N)^{d/2}\pi^{\frac{d}{2}}(\frac{dN}{2e})^{-d/2}N^{-1}e) \tag{1.30}$$

where we have used that $\frac{1}{N}\log N$ vanishes in the limit $N\to\infty$. Note that the N factors compensate so that (1.15) is found which can be written as

$$\lim_{\substack{N\to\infty \\ V/N=v, E/N=efixed}} \sigma = \log[v(2m\varepsilon)^{d/2}\pi^{\frac{d}{2}}(\frac{d}{2})^{-d/2}e] + d/2 \tag{1.31}$$

The entropy is increasing with the volume. By the above formula and according to the definition we get

$$\frac{1}{T} = (\frac{\partial \sigma}{\partial \varepsilon}) = \frac{d}{2\varepsilon} \qquad \frac{p}{T} = (\frac{\partial \sigma}{\partial v}) = \frac{1}{v} \tag{1.32}$$

which is the famous *state equation of perfect gases*; at fixed temperature, the pressure decreases as the inverse of the volume. The second relation says provides the interpretation of the temperature in terms of energy; for three dimensional gases the internal energy is related to the temperature by the relation $\varepsilon = \frac{3}{2}T$.

From this analysis we learnt several lessons. First, when the limit is taken the dependence from h, which is not a physical observable, disappears. Second, the entropy is extensive and we get relations which are qualitatively true in real gases, like the Boyle law $pV = f(T)$, but only far from the phase transitions.

1.1.3 *The canonical ensamble*

In the canonical ensamble still the thermodynamic properties are obtained averaging over the microscopic states, but the average is extended to all points in Γ and not only in Γ_E. Of course, one has then to verify that the two ways of deducing macroscopic properties lead to the same value.

The reasonings leading to the definition of the canonical ensemble is the following. Instead of averaging over the surface in the phase space with fixed energy, we can instead average over all possible states weighted with an exponential weight which has its maximum when the energy is $\sim \kappa T$; as one expect that the fluctuation around the averaged energy are small in the thermodynamic limit, we can expect that the two ensambles are equivalent (what can indeed be proved under suitable conditions). In particular, we can imagine that the system is divided in a subsystem (s) and a reservoir (r), so that the probability that the reservoir has energy $E_t - E_s$ is proportional, up to a normalization, to

$$|\Delta\Gamma(E_t - E_s)| \sim e^{S(E_t - E_s)} \sim e^{-\partial_E S(E_t) E_s} A = A e^{-\beta E_s} \tag{1.33}$$

using the definition of temperature.

Definition *The free energy for particle of a gas of identical particles in the canonical ensemble is given by*

$$F = -\frac{1}{\beta} \log Z \tag{1.34}$$

where Z is the partition function

$$Z = \frac{1}{N!}\int dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} e^{-\beta H} \tag{1.35}$$

and $\beta = (T)^{-1}$. Moreover given a $O(p,q)$ its average is defined as

$$< O >= \frac{1}{Z}\int dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} e^{-\beta H} O(p,q) \tag{1.36}$$

Finally

$$p = -(\frac{\partial F}{\partial V}) \qquad S = -(\frac{\partial F}{\partial T}) \qquad E = -(\frac{\partial \log Z}{\partial \beta}) \tag{1.37}$$

1.2 The perfect gas in the canonical ensemble and equivalence

The computation in the canonical ensamble are easier (and this is one of the main reasons of its introduction). We recall some useful formulas of gaussian integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

This follows easily writing

$$[\int_{-\infty}^{\infty} dx e^{-x^2} dx]^2 = \int_0^{2\pi} d\vartheta \int_0^{\infty} d\rho e^{-\rho^2} = \pi \tag{1.38}$$

More by differentiating with respect to a

$$\int_{-\infty}^{\infty} dx x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

The above formulas are easily generalized; if A is a positive definite matrix

$$\int_0^{\infty} dx_1 ... dx_n e^{-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{detA}}$$

The partition function is given by

$$Z = \frac{1}{N!}\int_{-\infty}^{\infty} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} e^{-\beta \sum p_i^2/2m} = \frac{V^N}{N!}[\int dp_1 e^{-\beta p_1^2/2m}]^N =$$
$$\frac{V^N}{N!}(2\pi m T)^{dN/2} \tag{1.39}$$

Therefore the free energy for particle is given by

$$f = -\frac{1}{\beta N}\log Z = -\frac{1}{\beta}\log\frac{V}{N!^{\frac{1}{N}}}(2\pi mT)^{d/2} \tag{1.40}$$

By a similar computation we can compute the internal energy

$$< H >= dN\frac{\int_{-\infty}^{\infty} dp_1 p_1^2/2m e^{-\beta p_1^2/2m}}{\int_{-\infty}^{\infty} dp_1 e^{-\beta p_1^2/2m}} = dN\frac{1}{4m}\frac{(2mT)^{\frac{3}{2}}}{(2mT)^{\frac{1}{2}}} = \frac{1}{2}dN\beta^{-1} \tag{1.41}$$

so that the specific heat is $C_v = \frac{d}{2}N$. In a three diemensional gas of monoatomic particles $C_v = \frac{3}{2}N$.

1.2.1 *Equivalence between canonical and microcnonical in the case of perfect gas*

Note finally that from the above formulas follows the equivalence in the case of perfect gases between the microcanonical ensamble in the limit $N \to \infty, v, efixed$ and the canonical in the limit $N \to \infty$. Indeed in the canonical ensemble

$$\sigma = -\frac{\partial f}{\partial T} = \log\frac{V}{N!^{\frac{1}{N}}}(2\pi m\kappa T)^{d/2} + \frac{d}{2} \tag{1.42}$$

so that using Stirling fomula $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$ as

$$\lim_{N\to\infty}\sigma = \lim_{N\to\infty}\log ve(2\pi m\kappa T)^{d/2} + \frac{d}{2} \tag{1.43}$$

and using $\varepsilon = d\beta^{-1}/2$ we get

$$\lim_{N\to\infty}\sigma = \log vee^{\frac{d}{2}}(2\pi m 2\varepsilon/d)^{d/2} \tag{1.44}$$

as $\frac{d}{2} = \log e^{\frac{d}{2}}$, coinciding with the value found using the microcanonical ensamble. Finally $p = \frac{1}{\beta V}$.

Note that in the limit $N \to \infty$ the thermodynamic quantities found in the canonical or grand canonical ensamble coincides; this is an example of a more general property called equivalence of the ensamble, valid in the thermodynamic limit.

1.2.2 *Gases in an external potential*

Gas in a vertical cylinder

We consider in the canonical ensamble of a gas of particle in the cylinder of radius R anf height L with gravity.

The Hamiltonian is

$$H = \sum_{i=1}^{N} [\frac{p_i^2}{2m} + mgz_i]$$

Therefore the partition function is

$$Z = \frac{1}{N!}[\int_0^h e^{-\beta mgz}]R(2\pi)[2\pi mT]^{\frac{3}{2}}]^N$$

The energy is given by

$$E = -\frac{\partial \log Z}{\partial \beta} = \frac{5}{2}NT - \frac{mghN}{e^{\beta mg\beta h} - 1}$$

and the specific heat is given by $C_v = \frac{\partial T}{\partial T}$.

Gas in a rotating cylinder

We consider a gas if identical particles in a cylinder rotating with frequency ω. In order to obtain the single particle Hamiltonian we introduce lagrangian coordinates (ρ, ϑ, z) with

$$x = \rho\cos(\omega t + \vartheta) \qquad y = \rho\sin(\omega t + \phi)$$

so that the single particle Lagrangian is $L = \frac{m}{2}[\rho'^2 + \rho^2(\omega + \vartheta')^2$. Moreover the momenta are $p_\rho = m\rho'$ and $p_\vartheta = m\rho^2(\omega + \vartheta')$ so that $H = \frac{p_z^2}{2m} + \widetilde{H}$ with

$$\widetilde{H} = \rho'(p_\rho)\frac{\partial L}{\partial \rho'} + \vartheta'(\vartheta', \rho)\frac{\partial L}{\partial \vartheta'} - L = \frac{p_\rho^2}{m} + m(\frac{p_\vartheta}{m\rho^2} - \omega)\rho^2\frac{p_\vartheta}{m\rho^2} - \frac{p_\rho^2}{2m} - \frac{p_\vartheta^2}{2m\rho^2} = \frac{p_\rho^2}{2m} + \frac{p_\vartheta^2}{2m\rho^2} - m\omega p_\vartheta$$

Therefore the partition function is given by

$$Z = \frac{1}{N!}[\int_0^{2\pi} id\vartheta \int_0^R d\rho \int_0^h dz \int_{-\infty}^{\infty} dp_z e^{-\frac{\beta p_z^2}{2m}} \int_{-\infty}^{\infty} dp_\vartheta \int_{-\infty}^{\infty} dp_\rho e^{-\beta[\frac{p_\rho^2}{2m} + \frac{p_\vartheta^2}{2m\rho^2} - m\omega p_\vartheta]}]^N =$$

$$\frac{1}{N!}[(2\pi)h(2\pi mT)\int_0^R \rho d\rho \int_{-\infty}^{\infty} dp_\vartheta e^{-\beta[\frac{p_\vartheta^2}{2m\rho^2} - m\omega p_\vartheta]}]^N = \frac{1}{N!}[(2\pi)h(2\pi mT)^{\frac{3}{2}}\int_0^R \rho d\rho e^{\beta\frac{m}{2}\omega^2\rho^2} =$$

$$\frac{1}{N!}[(2\pi)h(2\pi mT)^{\frac{3}{2}}\frac{1}{1 - e^{-\beta\frac{m}{2}\omega^2 R^2}}\frac{1}{\beta m\omega^2}$$

so that

$$E = -\frac{\partial \log Z}{\partial \beta} = \frac{5}{2}NT - N\frac{1}{1 - e^{-\beta\frac{m}{2}\omega^2 R^2}}\frac{m\omega^2 R^2}{2} \qquad (1.45)$$

1.2.3 *Specific heat for diatomic gases; part I*

We have seen above that a gas of particles described by material points has speific heat given by $\frac{3}{2}NT$. Let us consider now a gas of diatomic particles, described by two material points with fixed distance. If $\vec{r}_1$ is the position of the atom 1 and $\vec{r}_2$ is the position of the atom 2 the constrain is $|\vec{r}_1 - \vec{r}_2| = 1$. It is convenient to use coordinates $\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Therefore the unconstrained lagrangian is given by

$$L = \frac{m_1}{2}|\vec{r}_1'|^2 + \frac{m_2}{2}|\vec{r}_2'|^2 = \frac{M}{2}|\vec{R}'|^2 + \frac{\mu}{2}|\vec{r}'|^2$$

where $M = \frac{m_1+m_2}{2}$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$. The Lagrangian parameters are (ϑ, ϕ) where $r_x = \sin\vartheta\cos\phi, r_y = \sin\vartheta\sin\phi, r_z = \rho\cos\vartheta$; therefore the constrained Lagrangian is

$$L = \frac{M}{2}|\vec{R}'|^2 + \frac{\mu}{2}(\vartheta'^2 + \sin^2\vartheta\phi'^2]$$

and the Hamiltonian is

$$H = \frac{1}{2M}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2\mu}(p_\vartheta^2 + \frac{p_\phi^2}{\sin^2\vartheta}]$$

The partition function is then given by

$$Z = \frac{1}{N!}[(\int_{-\infty}^{\infty} e^{\frac{-\beta p^2}{2M}}]^3 V(2\pi)\int_0^\pi d\vartheta\int_{-\infty}^{\infty} dp_\vartheta e^{-\frac{\beta}{\mu}p_\vartheta^2}\int_{-\infty}^{\infty} dp_\phi e^{-\frac{\beta}{\mu}\frac{p_\phi^2}{\sin^2\vartheta}} = \frac{1}{N!}[(2\pi MT)^{\frac{3}{2}}(2\pi\mu T)]^N$$

Therefore in this case $E = \frac{5}{2}NT$.

1.2.4 *Specific heat for diatomic gases; part II*

Let us assume a more realistic description for a gas of diatomic molecules in whih the distance between the two atoms is not fixed; we can assume that the force between the two atoms is elastic. We define $r_x = \rho\sin\vartheta\cos\phi, r_y = \rho\sin\vartheta\sin\phi, r_z = \rho\cos\vartheta$ and the Lagrangian is

$$L = \frac{M}{2}|\vec{R}'|^2 + \frac{\mu}{2}[\rho'^2 + \rho^2\vartheta'^2 + \rho^2\sin^2\vartheta\phi'^2] - \frac{k}{2}\rho^2 \tag{1.46}$$

The Hamiltonian is $H = H_R + H_\rho$ with

$$H = \frac{p_\rho^2}{2\mu} + \frac{1}{2\mu\rho^2}(p_\vartheta^2 + \frac{p_\phi^2}{\sin^2\vartheta}) + \frac{k}{2}\rho^2 \tag{1.47}$$

The partition function is given by $Z = \frac{1}{N!}[V(2\pi M\beta^{-1})Z_\rho^N$ and

$$Z_\rho = \int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dp_\vartheta \int_{-\infty}^{\infty} dp_\phi \int_{-\infty}^{\infty} dp_\rho e^{-\beta\frac{p_\rho^2}{2\mu}+\frac{1}{2\mu\rho^2}(p_\vartheta^2+\frac{p_\phi^2}{\sin^2\vartheta}+\frac{k}{2}\rho^2)} \tag{1.48}$$

so that

$$Z_r = (2\pi)(2\pi\mu\beta^{-1})^{\bar{2}}[\int_0^{\pi} d\vartheta \sin^2\vartheta] \int_0^{\infty} (2\pi\mu\beta^{-1}\rho)e^{-\frac{k}{2}\beta\rho^2} \tag{1.49}$$

In conclusion $E = \frac{\partial}{\partial\beta}\log Z$ and $\log Z = \log[A\beta^{-\frac{3}{2}-\frac{3}{2}-\frac{3}{2}}]^N$ so that $E = \frac{9}{2}NT$ and $C_v = \frac{9}{2}N$.

Experimentally (for instance in a gas of H_2) one see at temperature nor too large $C_v \sim \frac{3}{2}N$ for , at greater temperatures $C_v = \frac{5}{2N}$; it is like that the contribution of the rotational and vibrational degree of freedom are activated only at certain temperatures.

1.2.5 *Specific heat for solids*

A classical model for a solid is a systems of particles (ions) connected by elastic particles and a gas of free particles (conduction electrons). Let us consider a 1d solid for definiteness. The Lagrangian is

$$L = \sum_{n=1}^{N} \frac{x'^2}{2m} - \frac{\alpha}{2}\sum_{i=1}^{N}(x_n - x_{n+1})^2$$

with $x_1 = x_{N+1}$. The EL equations are

$$mx_n'' = -\alpha(x_{n+1} + x_{n-1} - 2x_n)$$

. In look for a solution usung Fourier seriers; as x_n is periodic we write it as

$$x_n = \sum_{k=1}^{N} e^{\frac{i2\pi nk}{N}} A_k(t)$$

so that by substitution

$$\sum_{k=1}^{N} e^{\frac{i2\pi nk}{N}}[mA_k''(t) + (e^{\frac{i\pi k}{N}} + e^{\frac{-i\pi k}{N}} - 2)A_k(t)] = 0 \tag{1.50}$$

By setting

$$\omega^2(k) = 2\alpha(1 - \cos\frac{2\pi k}{N})$$

the solution is

$$x_n = \sum_{k=1}^{N} e^{\frac{i2\pi nk}{N}} [e^{i\omega(k)t} C_k + e^{-i\omega_k t} D_k]$$

With a canonical transformation the Hamiltonian can be written as

$$H = \sum_{k=1}^{N} [p_k^2 + \omega(k)^2 q_k^2]$$

The partition function is given by (no $N!^{-1}$ as in the case of gases)

$$Z = \prod_{k=1}^{N} [\int dp e^{\frac{-\beta p^2}{2m}} \int dq_k e^{-\beta q_k^2 \omega^2(k)}] = \prod_{k=1}^{N} (2mT\pi)^{\frac{1}{2}} (2T\pi\omega^{-1}(k))^{\frac{1}{2}}$$

from which $E = NT$; in a three dimensional solid $E = 3NT$. A specific heat $3N$ is measured in several solid at temperatures not too low; however if we take als intom account the contribution of the electrons one would get $6N$ instead. Moreover at small T experimentally the specif heat of solids tends to vanish with temperature; again such paradoxes find an explanation in quantum mechanics.

1.2.6 *Black body radiation*

Consider a cubic region V filled with electromagnetic radiation in thermal equilibrium with the surrounding walls with which it is supposed to exchange heat. The hamiltonian of this system is

$$H = \frac{1}{2} \sum_{\vec{k}} [p_{\vec{k}}^2 + c^2 |\vec{k}|^2 q_{\vec{k}}^2]$$

with $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$, $n_i = 0, .., L$. If $\nu = c|\vec{k}|$ is the frequency, the number of modes $\vec{k}$ corresponding to ν is $4\pi \frac{\nu^2}{c^3}$ (this has to be multiplied by 2 for taking into account polarization); hence the energy with frequency ν is $\frac{8\pi\nu^2}{c^3}\kappa T$ and the integarl is diverging, which again is paradoxical as the total energy is finite.

1.3 The Grand-canonical ensamble; the case of the perfect gas

In the Grand-canonical ensamble the averages are done over systems with different particle number.

Definition *The partition function for particle of a gas of identical particles in the grand canonical ensemble is given by*

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} e^{-\beta(H-\mu N)} \tag{1.51}$$

Moreover if $F = -\beta^{-1} \log Z$

$$\bar{N} = -(\frac{\partial F}{\partial \mu}) \qquad p = -(\frac{\partial F}{\partial V}) \quad S = -(\frac{\partial F}{\partial kT}) \tag{1.52}$$

where $\bar{N}$ *is the average number of particles.*

In the case of the perfect gas, using the computations done for the canonical ensamble we get

$$Z = \sum_{N=0}^{\infty} e^{\beta\mu N} \frac{V^N}{N!} (2\pi m\kappa T)^{dN/2} = e^{[e^{\beta\mu} V (2\pi m\kappa T)^{d/2}]} \tag{1.53}$$

so that

$$F = -\beta^{-1} [e^{\beta\mu} V (2\pi m\kappa T)^{d/2}] \tag{1.54}$$

Therefore

$$\bar{N} = [e^{\beta\mu} V (2\pi m\kappa T)^{d/2}] \quad p = \beta^{-1} [e^{\beta\mu} (2\pi m\kappa T)^{d/2}] \tag{1.55}$$

so that $p = \bar{N}\beta^{-1}/V$, that is again the law of perfect gases; moreover from $S = -(\frac{\partial F}{\partial kT})$ we get the entropy of perfect gases. Indeed

$$\mu = \frac{1}{\beta} \log \frac{\bar{N}}{V(2\pi\kappa T)^{\frac{d}{2}}} \tag{1.56}$$

and

$$S = -\frac{\partial F}{\partial T} = e^{\beta\mu} V (2\pi\mu\kappa T)^{\frac{d}{2}} - \beta^{-1} \frac{\mu}{(\kappa T)^2} e^{\beta\mu} V (2\pi m\kappa T)^{\frac{d}{2}} + \frac{d}{2} \beta^{-1} \beta e^{\beta\mu} V (2\pi\mu\kappa T)^{\frac{d}{2}} \tag{1.57}$$

which can be rewritten as

$$S = \bar{N} - \bar{N} \log \frac{V}{\bar{N}} (2\pi\mu\kappa T)^{\frac{d}{2}} + \frac{d}{2} \bar{N} = \bar{N} \log \frac{V}{\bar{N}} (2\pi\mu\kappa T)^{\frac{d}{2}} e e^{\frac{d}{2}} \tag{1.58}$$

and

$$\lim_{\substack{\bar{N}\to\infty \\ \frac{V}{\bar{N}}=v}} \frac{S}{\bar{N}} = \log v (2\pi\mu T)^{\frac{d}{2}} e e^{\frac{d}{2}}$$

This shows the equivalence of the three ensembles in the Thermodynamic limit in the case of the perfect gas.

1.3.1 *Spin systems and probability*

Let us consider a probability space (Ω, F, P) and a random variable $x : \Omega \to R$ *F-measurable*, that is for ant $t > 0$ the set $\{\omega : x(\omega) \leq t\}$ is in F. A gaussian random variable has density

$$f(x) = \frac{1}{(2\pi\sigma)^{\frac{1}{2}}} e^{-(x-a)^2/2\sigma}$$

Central limit Theorem *If $x_1, x_2, ..x_n$ are independent and identically distributed random variables with mean μ and variance σ^2; if $S_n = \frac{x_1+...+x_n-\mu n}{\sqrt{n}}$, then for $n \to \infty$ S_n converges in law to a gaussian random variable with zero mean and variance* 1

Proof Convergence in Law means that the characteristic function of X_n convergence to the characteristic function of x, $E(e^{i\vartheta X})$. It is easy to verify that the characteristic function of a random variable is with zero mean and variance 1 is $e^{\frac{-\vartheta^2}{2}}$. Note that

$$E(e^{i\vartheta S_n}) = E(e^{i\vartheta(\frac{x_1+...+x_n-\mu n}{\sqrt{n}})}) = [E(e^{i\vartheta(\frac{x_1-\mu}{\sqrt{n}})})]^n = e^{n \log E(e^{i\vartheta(\frac{x_1-\mu}{\sqrt{n}})})}$$

and using Taylor expansion

$$E(e^{i\vartheta(\frac{x_1-\mu}{\sqrt{n}})} = E(1 + i\vartheta(\frac{x_1-\mu}{\sqrt{n}}) - \vartheta^2(\frac{x_1-\mu}{\sqrt{n}})^2 + ... = 1 - \frac{\vartheta^2}{2n} + O(n^{\frac{-3}{2}})$$

and by $\log(1+x) \simeq x + O(x^2)$

$$\log E(e^{i\vartheta(\frac{x_1-\mu}{\sqrt{n}})}) = -\frac{\vartheta^2}{2n} + O(n^{\frac{-3}{2}})$$

we get

$$E(e^{i\vartheta S_n}) = e^{-\frac{\vartheta^2}{2} + O(n^{\frac{-1}{2}})}$$

so that for$n \to \infty$ the Theorem is proved.

Consider a set of particles, each bearing a momentum (spin) which can be parallel to antiparallel to a direction with the same probability. The σ_i are random variables identical and uniformly distributed. We define the magnetization as $\sum_{i=1}^{N} \mu\sigma_i h$ where $\sigma_i = \pm$. The possible configurations of such system is $\Gamma = (\pm 1, ..., \pm 1)$ and their number is 2^N. The entropy defined as the log of the number of possible configurations is $\sigma = \log 2^N = N \log 2$.

According to the central limit Theorem $\frac{1}{\sqrt{N}}\sum_i \sigma_i$ converges, accordin to the central limit theorem, to a gaussian variable with zero average and variance 1. Therefore the most likely configuration has zero magnetization, and the density of probability has a peak around this value with width $O(N^{-\frac{1}{2}})$.

One can verify explicitly such properties. The number of configurations realizing the magnetization n, *called*$W(n)$ is equal to the number of configurations so that $\frac{1}{2}(N+n)$ spins point in a way and $\frac{1}{2}(N-n)$ point in the other so that

$$W(n) = \frac{1}{2^N}\frac{N!}{[\frac{1}{2}(N+n)!][\frac{1}{2}(N-n)!]} \tag{1.59}$$

and the probability is $w(n) = \frac{1}{2^N}W(n)$. Indeed suppose we order the spin; the probability that the first spin has a certain value is $\frac{1}{2}$, and so is the second etc. Moreover we have to multiply for the number of possible configurations so that $\frac{1}{2}(N+n)$ spins point in a way and $\frac{1}{2}(N-n)$ point in the other way. One can explitile verify central limit theorme in this case by using Stirling formula $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$; we get

$$p(n) \sim (\frac{2}{\pi N})^{\frac{1}{2}} e^{-\frac{n^2}{2N}} \tag{1.60}$$

Note that the maxwell distribution of canonical ensamble appears naturally as consequence of central limit in this case.

The total number of configurations for $n = 0$ is $W(0) = \frac{N!}{[\frac{N}{2}]^2}$ and again by Stirling formula, $W(0) \sim (\frac{N}{e})^N \frac{1}{(\frac{N}{2e})^N} \sim 2^N$ so that $\log W(0) = N\log 2 + O(\log N)$. The entropy has the same value if computed on the equilibrium state or considering all possible states. The magnetization is the "macroscopic" variable; $n = 0$ is the "equilibrium" value. In the limit $N \to \infty$ deviation from the equilibrium value are very rare, and the observables, as the entropy per particle, computed averaging over all the states or over the states corresponding to equilibrium give the same result.

1.4 Exercises

1.4.1 *Exercise*

The free energy and the energy of N one dimensional harmonic oscillators with hamiltonian $H = \sum_i \frac{p_i^2}{2m} + \frac{k}{2}x_i^2$ using the canonical ensamble

The partition function is given by

$$Z = [\int_{-\infty}^{\infty} dx e^{\frac{-\beta k x^2}{2}}]^N [\int_{-\infty}^{\infty} dp e^{\frac{-\beta p^2}{2m}}]^N = (2\pi T k)^{\frac{N}{2}} (2\pi T m)^{\frac{N}{2}} \qquad (1.61)$$

and the free energy is $F = -T \log(2\pi T k)^{\frac{N}{2}} (2\pi T m)^{\frac{N}{2}}$. The energy is then given by

$$E = -\frac{\partial \log Z}{\partial \beta} = -\partial_\beta \log \beta^{-N} = NT$$

1.4.2 *Exercise*

Show the equivalence of the microcanonical (with dishinguible particles) and grand canonical ensamble for N one dimensional harmonic oscillators ($m = k = 1$) with hamiltonian $H = \sum_i \frac{p_i^2}{2} + \frac{1}{2} x_i^2$

From the formulas of the previous exercise $E = NT$ and $F = -T \log(2\pi T)^N$.

Using the microcanonical ensemble, one has to compute

$$\int_{\frac{1}{2}\sum p_i^2 + x_i^2 \leq E} dx_1 .. dx_n dp_1 ... dp_n = (2E)^N \frac{\pi^N}{\Gamma(N+1)}$$

so that, setting $E = \varepsilon N$

$$\sigma = \lim_{N \to \infty} \frac{1}{N} \log(2E)^N \frac{\pi^N}{\Gamma(N+1)} = \log(2E\pi e) \qquad (1.62)$$

so that $T = \partial_E \sigma = E$.

Moreover the entropy in the canonical ensamble is given by

$$\sigma = \lim_{N \to} \frac{\partial}{\partial T} [T \log \frac{1}{N!} (2\pi T)^{\frac{N}{2}} (2\pi T)^{\frac{N}{2}}] = \log(2\pi T) + 1$$

which coincides with the previous expression.

1.4.3 *Exercise*

Computation of the thermodynamical quantities in the canonical ensamble of a gas of particle in the cylinder of radius R anf height L with Hamiltonan $H = \sum_i [\frac{p_i^2}{2m} - \frac{m\Omega^2}{2}(x_i^2 + y_i^2)]$.

We pass to polar coordinates and the single particle Lagrangian is

$$L = \frac{m}{2} [\rho'^2 + \rho^2 \vartheta'^2 + z'^2] - \frac{m\Omega^2}{2} \rho^2 \qquad (1.63)$$

The single particle Hamiltoian is

$$H = \frac{1}{2m}[p_\rho^2 + \frac{p_\vartheta^2}{\rho^2} + p_z^2] + \frac{m\Omega^2}{2}\rho^2 \tag{1.64}$$

The partition function is

$$Z = \frac{1}{N!}Z_1^N \tag{1.65}$$

with

$$Z_1 = \int_0^{2\pi} d\vartheta \int_0^h dz \int_0^R d\rho \int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dp_\vartheta \int_{-\infty}^{\infty} dp_\rho e^{-\frac{\beta}{2m}[p_\rho^2 + \frac{p_\vartheta^2}{\rho^2} + p_z^2] + \frac{m\Omega^2}{2}\rho^2} \tag{1.66}$$

and finally

$$Z = \frac{1}{N!}([2\pi \int_0^h dz \int_0^R \rho d\rho e^{-\frac{\beta}{2}m\Omega^2\rho^2}][2\pi mT]^{\frac{3}{2}})^N \tag{1.67}$$

and

$$\int_0^R \rho d\rho e^{-\frac{\beta}{2}m\Omega^2\rho^2} = \frac{1}{m\Omega^2\beta}(e^{\beta\frac{m}{2}\Omega^2R^2} - 1) \tag{1.68}$$

so that

$$Z = \frac{1}{N!}[2\pi\frac{h}{m\Omega^2\beta}(e^{\beta\frac{m}{2}\Omega^2R^2} - 1)[2\pi mT]^{\frac{3}{2}}]^N \tag{1.69}$$

The energy is given by

$$E = -\frac{\partial \log Z}{\partial\beta} = \frac{5}{2}NT - N\frac{1}{1 - e^{-\beta\frac{m}{2}\Omega^2R^2}}\frac{m\Omega^2R^2}{2} \tag{1.70}$$

1.4.4 *Exercise*

Computation of the thermodynamical quantities in the canonical ensamble of a gas of particle constrained on the surface of a sphere $R = 1$ and in presence of gravity

We use spherical coordinates $x = \sin\vartheta\sin\phi$, $y = \sin\vartheta\cos\phi$, $z = \cos\vartheta$, $0 < \vartheta \leq \vartheta$, $0 \leq \phi < 2\pi$; therefore the Lagrangian is $\mathcal{L} = \frac{m}{2}R^2[\vartheta'^2 + \sin^2\vartheta\phi'^2) - mgR\cos\vartheta$. Moreover $p_\vartheta = mR^2\vartheta'$, $p_\phi = mR^2\sin^2\vartheta\phi'$ so that the single particle Hamiltonian is

$$H = \frac{1}{2mR^2}[p_\vartheta^2 + \frac{p_\phi^2}{\sin^2\vartheta}] + mgR\cos\vartheta$$

and the partiton function is givn by

$$Z = \frac{1}{N!}[\int_0^\pi d\vartheta \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_\phi \int_{-\infty}^{\infty} dp_\vartheta e^{-\frac{\beta}{2mR^2}[p_\vartheta^2 + \frac{p_\phi^2}{\sin^2\vartheta}] - \beta mgR\cos\vartheta}]^N = \tag{1.71}$$

$$= \frac{1}{N!}[(2\pi mTR^2)(2\pi)\int_0^\pi \sin\vartheta e^{-\beta mgR\cos\vartheta}]^N = \tag{1.72}$$

$$\frac{1}{N!}[(2\pi mTR^2)(2\pi)(T(mgR)^{-1}(e^{\beta mgR} - e^{-\beta mgR})]^N \tag{1.73}$$

from which

$$E = -\frac{\partial \log Z}{\beta} = -N(-2T + mgRcotgh(\beta mgR)) \tag{1.74}$$

1.4.5 *Exercise*

Computation of the thermodynamical quantities in the canonical ensamble of a system of particles with single particle hamiltonian $\frac{p^2}{2m} + \varepsilon(\sigma)$ *with* $\sigma = 1, 2, 3, 4$ *and* $\varepsilon(1) = -E$, $\varepsilon(2) = \varepsilon(3) = 0$, $\varepsilon(4) = E$

The partition function is given by

$$Z = \frac{1}{N!}[V(e\pi mT)^3(e^{\beta\varepsilon} + 2 + e^{-\beta\varepsilon})]^N \tag{1.75}$$

1.5 Existence of the thermodynamic limit

In the previous sections we have discussed the thermodynamic limit assuming that the particles do not interact. Of course interactions plays a crucial role in the macroscopic properties of matter; a preliminary problem one has to worry about the existence of the thermodynamic limit exists. It will turn out that this limit exists only if certain assumptions on the two body potential are done.

Theorem *Let us consider the canonical partition function*

$$Z(V, N, T) = \frac{1}{N!}\int d\vec{p}_1 d\vec{p}_N d\vec{q}_1 ... d\vec{q}_N e^{-\beta H} \tag{1.76}$$

with

$$H = \sum_{i=1}^N \frac{|\vec{p}_i|^2}{2m} + \sum_{i<j}^N v(|\vec{q}_i - \vec{q}_j|) \tag{1.77}$$

with: $V(r) = \infty$ *for* $r \leq r_0$*;* $-\varepsilon < V(x) < 0$ *for* $r_0 < r < h$*;* $V = 0$ *for* $r \geq h$ *(Van Hove potential). Let us consider a sequence of cubes* B'_k *in* $\mathbb{R}^3$*,* $k = 1, 2, ..$ *with side* $2^k L_0 - r_0/2$ *where* L_0 *is an arbitrary length scale and contained in a box* B_k *with sides* $2^n L_0$*. Assume that the density* ρ *is dyadic, which means that is of the form* $\rho = m2^{-3s} L_0^{-3}$ *with* m, s *positive integers so that* $N_k = \rho |B_k| = m2^{3(k-s)}$*. Then calling* $f_k = \frac{1}{N_k} \log Z(|B'_k|, N_k, T)$ *then the existence of the limit '*$lim_{k\to\infty} f_k = f(\rho, \beta)$ *follows.*

Proof. Note first that

$$Z(|B'_{k+1}|, N_{k+1}, T) = \frac{1}{N_{k+1}!} \int dp_1 dq_1 ... dp_{3N_{k+1}} \times \tag{1.78}$$

$$\int_{B'_{k+1} \times B'_{k+1} .. \times B'_{k+1}} \times d\vec{q}_1 ... d\vec{q}_{N_{k+1}} e^{-\beta \sum_{i=1}^{N_{k+1}} \frac{|\vec{p}_i|^2}{2m}} e^{-\beta \sum_{i<j}^{N_{k+1}} v(|\vec{q}_i - \vec{q}_j|)} \tag{1.79}$$

$$\frac{(2\pi m \beta^{-1})^{3/2N_{k+1}}}{N_{k+1}!} \int_{B'_{k+1} \times B'_{k+1} \cdots \times B'_{k+1}} d\vec{q}_1 ... d\vec{q}_{N_{k+1}} e^{-\beta \sum_{i<j}^{N_{k+1}} v(|\vec{q}_i - \vec{q}_j|)} \tag{1.80}$$

where the N_{k+1} particles are in the box B'_{k+1} enclosed and centered in B_{k+1}; one may think to B'_{k+1} as a box with sides $2^n L_0$ with thickness $r_0/2$.

We want now compare $Z(|B'_{k+1}|, N_{k+1}, T)$ with $Z(|B'_k|, N_k, T)$. First of all, we can divide B_{k+1} in 2^3 cubes B_k with $|B_k| = 2^{3k} L_0$, and to each B_k is associated the corresponding B'_k. The integral is over all the possible configurations and the integrand is positive; therefore

$$Z(|B'_{k+1}|, N_{k+1}, T) \geq$$
$$\frac{(2\pi m \beta^{-1})^{3/2N_{k+1}}}{N_{k+1}!} \int \bar{\chi}(\underline{q}) d\vec{q}_1 ... d\vec{q}_{N_{k+1}} e^{-\beta \sum_{i<j}^{N_{k+1}} v(|\vec{q}_i - \vec{q}_j|)} \tag{1.81}$$

where $\bar{\chi}(\underline{q})$ is non vanishing only if there are $N_k = \rho B_k = m2^{3(k-s)}$ particles in each of the B'_k. (of course $2^3 N_k = N_{k+1}$). We introduce now a function $\widehat{\chi}(\underline{q})$ which is non vanishing only if there are $N_k = \rho B_k = m2^{3(k-s)}$ particles in each of the B'_k *in a definite order*, say particles $q_1 ... q_{N_k}$ in the first box and so on. It is clear that the r.h.s. is equal to the integral over a defined order times all the possible permutations, that is

$$Z(|B'_{k+1}|, N_{k+1}, T) \geq$$
$$\frac{(2\pi m \beta^{-1})^{3/2N_{k+1}}}{N_{k+1}!} \frac{N_{k+1}!}{(N_k!)^8} \int \widehat{\chi}(\underline{q}) d\vec{q}_1 ... d\vec{q}_{N_{k+1}} e^{-\beta \sum_{i<j}^{3N_{k+1}} v(|\vec{q}_i - \vec{q}_j|)} \tag{1.82}$$

Note that the distance between particles in different B'_k is greater than r_0, therefore the potential between particles in different boxes is negative;

therefore we can neglect in $e^{-\beta \sum_{i<j}^{3N_{k+1}} v(|q_i - q_j|)}$ the contribution from particles in different boxes, that is only the interaction between particles in the same boxes must be kept. Therefore the integral factorizes and using that $N_{k+1} = 8N_k$ we get

$$Z(|B'_{k+1}|, N_{k+1}, T) \geq [Z(|B'_{k+1}|, N_{k+1}, T)]^8 \tag{1.83}$$

Therefore the sequence of free energies

$$f_{k+1} = \frac{1}{N_{k+1}} \log Z(|B'_{k+1}|, N_{k+1}, T) = \frac{1}{8N_k} \log Z(|B'_{k+1}|, N_{k+1}, T) \geq$$
$$\frac{1}{8N_k} \log[Z(|B'_k|, N_k, T)]^8 = f_k \tag{1.84}$$

hence is monotonically increasing; moreover f_k is bounded as

$$\sum_{i<j}^{*} v(q_j - q_j) \geq -BN \tag{1.85}$$

where $\sum^*$ means that the sum is restricted over $|q_i - q_j| \geq r_0$; the above bound follows from the fact that fixed q_j the sum over i is over all the number of spheres of side r_0 which can be contained in a sphere of radius b. Therefore

$$f_k \leq \frac{1}{N_k} \log[\frac{1}{N_k!} V^{N_k} e^{\beta B N_k} (2\pi m \beta^{-1})^{3/2N_k}] \tag{1.86}$$

and again using Stirling formula $\log N! \geq N \log N - N$ we get

$$f_k \leq 1 + \beta B + 3/2 \log (2\pi m \beta^{-1}) - \log \rho \tag{1.87}$$

so that f_k is bounded and monotone so the limit exists.■

The above result has been proved by Ruelle (1963). There has been several development in order to consider more general interactions between particles; for instance one can relax the short range condition and assume that the interaction decay with a power law (Fisher (1964), Dyson Lenard (1967), Lebowitz Lieb (1969)). The hard core condition is essential for the proof, and in abssece of it one can find that the free eergy diverges in the limit. This kind of condition is quite natural for quantum mechanical particles, at least if the particles are fermions.

Let be ρ_1, ρ_2 two dyadic densities, and let us call $Z(|B'_k|, N_k, T) \equiv Z(|B'_k|, \rho, T)$; assume that we put $N_k^1 = \rho_1 |B_k|$ particles in 4 boxes and

$N_k^2 = \rho_2 |B_k|$ in the remaining boxes; the assumption that the densities are diadic imply that N_k^1, N_k^2 are integers. Proceeding as above

$$Z(|B'_{k+1}|, 4N_k^1 + 4N_k^2, T) \geq [Z(|B'_k|, N_k^1, T)]^4 [Z(|B'_k|, N_k^2, T)]^4 \tag{1.88}$$

or equivalently

$$Z(|B'_{k+1}|, (\rho_1 + \rho_2)/2, T) \geq [Z(|B'_k|, \rho_1, T)]^4 [Z(|B'_k|, \rho_2, T)]^4 \tag{1.89}$$

and defining $g_k(\rho) = \frac{1}{|B_k|} \log Z(|B'_k|, \rho, T)$ so that in the limit $k \to \infty$ (the limit of g_k exists as $N_k/|B_k|$ is a constant), using $N_{k+1} = 8N_k$

$$g((\rho_1 + \rho_2)/2) \geq \frac{1}{2}[g(\rho_1) + g)(\rho_2)] \tag{1.90}$$

that is $g(\rho)$ is concave function of the density. The proof is only for diadic ρ but it can be extended to generic ρ by continuity, and it easily implies the nonnegativity of the pressure, defined as the derivative with respect to $v = \rho^{-1}$.

Similar analysis could be repeated for the microcanonical and grand canonical ensemble. Among the most interesting question that statistical mechanics want to solve are phase transitions: how a system can have different states despite is composed always by the same objects. Phase transitions appear as non analyticity points: from the above analysis we have learnt that a)there are no phose transition at finite N, V b)there are no phase transitions in the perfect gas. To understand phase transition we have then to consider interacting systems in the thermodynamic limit.

1.6 Phase transitions and real gases

1.6.1 *Phase transitions*

In real gases, one measures the pressure p as functions of v, at fixed temperature; this curves are called isotherms. If the temperature is large enough one gets curves somewhat similar to the ideal gas state equations $pv = 1/\beta$; the pressure increases as v decreases. However when the temperature is lower than some critical value, then the pressure decrease up to v_1, remains constant $v_1 \leq v \leq v_2$; this corresponds to a gas-liquid *phase transition*. This kind of behavior is completely missed by the perfect gas, and one can hope that is predicted by statistical physics once that the interactions between particles are included; this however make the computation of the (infinite dimensional) integral expressing the entropy enormously more complicated.

The basic problem in equilibrium statistical physics is the computation of the free energy and its derivatives and of the other thermodynamic functions or correlations. As should be clear by the previous physical discussion, phase transitions are associated with a singular behavior in the thermodynamical properties. It is then natural the following definition.

Definition: *Any singularity point of the free energy* $\lim_{|\Lambda|\to\infty} f_\beta(h)$ *as function of* β^{-1} *is said to be a phase transition point. In particular if the free energy is continuous but its derivative is non continuous the phase transition is said to be of first kind, while if the first derivative is continuous but the second is non continuous the phase transition is of second kind.*

Of course, it is not obvious at all a priori that the phenomenon of phase transition can be understood from the above definition of the free energy, given in agreement of postulates of statistical mechanics. It is in any case clear that phase transition can emerge in presence of interaction and that there is no hope of getting such singularity, with short range regular forces, without taking the thermodynamic limit; hence the problem of finding phase transitions is a genuinely infinite dimensional problem. Even such two conditions are not sufficient and the dimensionality also plays an important role, as it will appear clear from the following section.

1.6.2 *One dimensional gases*

As a first attempt to include the interaction we consider a gas of hard rods in one dimension, with an hard core potential given by

$$\begin{aligned} v(x-y) &= \infty \quad |x-y| \le a \\ v(x-y) &= 0 \quad |x-y| > a \end{aligned} \tag{1.91}$$

and $x_i \in (0, L)$. The hard core potential implies that

The canonical partition function is then given by

$$Z = \frac{1}{N!}\int_{-\infty}^{\infty} dp_1..dp_N dx_1...dx_N e^{-\beta\sum p_i^2/2m - \beta\sum_{1\le i<j\le N} v(x_i-x_j)} \tag{1.92}$$

Note that the hard core potential has the effect that the particles have a definite ordering along the chain; moreover we can divide the integration over the coordinate as sum of $N!$ integrals so that

$$Z = (2\pi m\kappa T)^{N/2}\int_{0<x_1<x_2...<L} dx_1...dx_N \prod_{1\le i<j\le N} S(|x_i-x_j|) \tag{1.93}$$

where

$$\begin{aligned} S(|x-y|) &= 0 \quad |x-y| \le a \\ S(|x-y|) &= 1 \quad |x-y| > a \end{aligned} \tag{1.94}$$

Note that the integrand is non vanishing if $x_2 - x_1 \geq a$ therefore is nonvanishing for $0 < x_1 \leq x_2 - a$ and $x_2 \geq x_1 + a \geq a$; moreover $x_3 - x_2 \geq a$, that is $a \leq x_2 \leq x_3 - a$ and $x_3 \geq x_2 + a \geq 2a$; and generically $(j-1)a \leq x_j \leq x_{j+1} - a$ for $j < N$ while $(N-1)a \leq x_N \leq L$ so that

$$Z = (2\pi m\kappa T)^{N/2} \int_{(N-1)a}^{L} dx_N \int_{(N-2)a}^{x_N - a} dx_{N-1} ... \int_{a}^{x_3 - a} dx_2 \int_{0}^{x_2 - a} dx_1 \tag{1.95}$$

The integral can be easily performed calling $y_j = x_j - (j-1)a$ so that

$$Z = (2\pi m\kappa T)^{N/2} \int_{0}^{L-(N-1)a} dy_N \int_{(0}^{y_N} dy_{N-1} ... \int_{0}^{y_3} dy_2 \int_{0}^{y2} dy_1 =$$

$$= (2\pi m\kappa T)^{N/2} \frac{(L-(N-1)a)^N}{N!} \tag{1.96}$$

We perform now the thermodynamic limit $L/N = v$ constant as $L, N \to \infty$ and $v > a$ (the case $v < a$ is forbidden by the haed core condition) so that in the limit, using Stirling formula

$$-\beta f = -\lim_{\substack{N,L\to\infty \\ L/N=v}} \frac{1}{N} \log Z =$$

$$-\lim_{\substack{N,L\to\infty \\ L/N=v}} \frac{1}{N} \log[(2\pi m\kappa T)^{N/2} \frac{(L-(N-1)a)^N}{N^N e^N}] = \log(2\pi m\kappa T)^{1/2} + 1 + \log(v-a)$$

which is analytic for $v > a$. Therefore no phase transitions are found in the physical region $v > a$.

The situation do no change if we add a short range finite interaction; no phase transitions in one dimensional gases, so that for finding them one has to go to higher dimensions.

1.6.3 *van der Waals approximation*

Assume that the interaction potential has the form

$$U(r) = U_{hc}(r) + U_{att}(r) \tag{1.97}$$

where U_{hc} is the hard-core potential, equal to ∞ for $r \leq r_0$ and 0 for $r \geq r_0$, while $U_{att}(r) < 0$ is the attractive potential. The canonical partition function can be therefore written as

$$Z = \frac{1}{N!} \int_{-\infty}^{\infty} dp_1 dq_1 ... dp_{dN} dq_1 ... dq_{dN} e^{-\beta \sum p_i^2/2m - \beta \sum_{1\leq j<i\leq N} U_{hc}(|\vec{q}_i - \vec{q}_j|) + U_{att}(|\vec{q}_i - \vec{q}_j|)} \tag{1.98}$$

Note that the integrals over the momenta can be performed as in the case of the perfect gas so that

$$Z = \frac{1}{N!}(2\pi m\kappa T)^{dN/2} \int dq_1 ... dq_{dN} \prod_{1\leq j<i\leq N} S(|\vec{q}_i - \vec{q}_j|) e^{-\beta \sum_{1\leq j<i\leq N} U_{att}(|\vec{q}_i - \vec{q}_j|)} \tag{1.99}$$

where $S(|r|)$ is $= 0$ for $|r| < r_0$ and $= 1$ for $r > r_0$. Up to this point the analysis is exact but analytically very difficult to treat.

Ornstein (1908) argued that the above integral can be well approximated by the following expression

$$Z \sim \frac{1}{N!}(2\pi m\kappa T)^{dN/2}(V - Nbr_0^d)^N e^{-\beta a \frac{N^2}{2V}} \tag{1.100}$$

where

$$a = -\int U_{att}(|\vec{r}|) d\vec{r} \tag{1.101}$$

In order to derive such expression, one can imagine that the interaction is very long ranged, so that we can replace $U_{att}(|\vec{q}_i - \vec{q}_j|)$ with its average over the volume $-a/V$ times $N^2/2$ taking into account the sums. After this replacememt we have to integrate $\int dq_1 ... dq_{dN} \prod_{1\leq j<i\leq N} S(|\vec{q}_i - \vec{q}_j|)$; in one dimension this integral can be explicitely done and one can guess that something similar is valid in any dimensions.

Assuming the validity of this approximation, we get that

$$\beta p = \frac{\partial}{\partial V} \log Z = \frac{N}{V - Nbr_0^d} - \beta a \frac{N^2}{2V^2} \tag{1.102}$$

or equivalently

$$\beta p = \frac{1}{v - br_0^d} - \beta a \frac{1}{v^2} \tag{1.103}$$

which is usually written in the form of the van der Waals equation

$$(v - br_0^d)(p + a\frac{1}{v^2}) - \kappa T = 0 \tag{1.104}$$

Write this equation $\psi_\beta(p, v) = 0$ and we verify that there exist (p_c, v_c) such that $\psi = (v - v_c)^3$; this means that the curve $p(v)$ at $v = v_c$ for $\beta = \beta_c$ has the first and second derivative in v vanishing. Therefore

$$(v - v_c)^3 = v^3 - 3v^2 v_c + 3v v_c^2 - v_c^3 = 0 \tag{1.105}$$

which should be compared with, if $v_0 = br_0^d$

$$v^3 - (v_0 + \kappa T_c/p_c)v^2 + a/p_c v - a v_0/p_c = 0 \tag{1.106}$$

from which $av_0/p_c = v_c^3$ and $3v_c^2 = a/p_c = v_c^3/v_0$ hence $v_c = 3v_0$; moreover from $av_0/p_c = v_c^3$ we get $p_c = a/27v_0^2$ and finally from $(v_0 + \kappa T_c/p_c) = 3v_c$ we get $\kappa T_c = a8v_0/27$. Consider an isotherm curve $T = T_c + \delta$ and consider the values of v for $p = p_c$; we get

$$v^3 - (v_0 + \kappa T/p_c)v^2 + a/p_c v - av_0/p_c = (v - v_c)^3 - \delta v^2/p_c \qquad (1.107)$$

whose derivative is

$$3(v - v_c)^2 + 3\delta v/p_c \qquad (1.108)$$

which can be vanishing only for $\delta < 0$; therefore for $\delta < 0$ has a maximum and a minimum and there are three v at the same p, while for $\delta > 0$ only one interesction.

It would be tempting to interprete this as a sign of a phase transition; note that $p(v)$ cannot have a maximum and a minimum (it violates convexity) but one can follow the maxwell prescription to replace the curve with a line so that the areas below and above are the same.

Despite found with an approximation, the result are in qualitatively agreement with data, provided that the phenomenological Maxwell construction is done. Subsequently, it has been proved by Lebowitz and Penrose (1966) that if one consider a potential with range γ and γ-independent average, then in the limit $\gamma \to \infty$ (after the thermodynamic limit) one get the Wan der waal equation with the Maxwell construction. This important result leaves still however open the possibility of phase transitions for physical finite range potentials.

It should also noted that difference the difference of the two minima from $3(v - v_c)^2 + 3\delta v/p_c = 0$ is $O(\beta - \beta_c)^{1/2}$; this measure the length of the plateaux and says that the difference vanishes as $\beta \to \beta_c^-$ as a power law with critical exponent $1/2$. Critical exponents plays an impotant role in statistical mechanics, as thy can be measured in experiments; it should be noticed that this exponent is quite different with respect to the, measured one. Not surprisingly, even clever approximations fails in getting quantitative results, and one should go to rxact computation (which at the moment no one has been able to do in this case).

At the end, tt is not obvious at all that statistical mechanics can really explain phase transitions, like the vapour liquid condensation. It was felt that extra conditions should be added to say to which phase the system has to go. It was only in 1940 with the example found by Onsager that it was shown that the artituon function alone contains all informations necessary for phase transition.

Chapter 2

2.1 The Ising model

A central problem in statistical mechanics is the understanding of *phase transitions*. It is well known in fact that macroscopic bodies can present themselves in different states, and that they can pass from one state to another changing for instance temperature and pressure (without of course any change in their microscopic composition). The more familiar example of phase transitions is probably the transition from ice to liquid of water when temperature is greater than 0 degree (at sea level pressure). Another example is given by magnetic materials; in presence of the external magnetic field, such materials are *ferromagnetic*, which means that they can be magnetized; if one applies an external magnetic field and remove it, the materials acquire a magnetization which remains also when the magnetic field is removed (*spontaneous magnetization*). The ferromagnetic state depends however from the temperature; there is usually a critical temperature T_c such that for $T \leq T_c$ the material is ferromagnetic (there is spontaneous magnetization) while for $T \geq T_c$ the material is paramagnetic, that is there is no spontaneous magnetization.

From a microscopic point of view, each of the molecules composing the magnet is associated a vector called magnetic dipole which tends to be aligned with the external magnetic field; moreover the molecules interact with the closes ones through exchange forces, favoring parallel dipoles. Therefore we can imagine that are the exchange forces which favor the ordering of the dipoles (responsible of the magnetism), an effect which is however contrasted by the thermal fluctuations. Can the law of statistical mechanics provide an explanation of a phenomenon like this? A good starting point, starting from Lenz (1920) is to start from an idealized de-

scription, named *Ising model* from the name of Lenz's student Ising.

We consider a d-dimensional square lattice $\Lambda \subset \mathbb{Z}^d$, where Λ is centered around the origin and contains $|\Lambda| = L^d$ lattice sites. On each site $\mathbf{x}$ is located a spin variable $\sigma_{\mathbf{x}}$ with values 1 or -1. The "configurations" of the system consist of a set $\sigma = (\sigma_{\mathbf{x}_1}, ..., \sigma_{\mathbf{x}_{|\Lambda|}})$ of $|\Lambda|$ numbers such that $\sigma_{\mathbf{x}} = \pm$. The number of these configurations is $2^{|\Lambda|}$ and to each spin configuration a certain energy $H(\sigma)$ is assigned. To each configuration an *energy* is associated

$$H_J = -J \sum_{j=1,..d} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - h \sum_{\mathbf{x}} \sigma_{\mathbf{x}} \tag{2.1}$$

where $\sigma_{\mathbf{x}} = \pm$ are spin variables, Λ is a square lattice, $\mathbf{x} \in \Lambda \subset R^d$, $\mathbf{e}_0 = (1, 0, 0, ..), \mathbf{e}_1 = (0, 1, 0, ..)$ and $J > 0$.

The postulates of statistical mechanics allow to compute macroscopic quantities appearing in the thermodynamical theory of the system starting from the microscopic energy $H(\sigma)$; indeed the *partition function* is

$$Z = \sum_{\sigma} e^{-\beta H(\sigma)} \tag{2.2}$$

where $\beta = (\kappa T)^{-1}$, T is the temperature and κ is the Boltzmann constant. The *free energy* per site is defined as $f_{\Lambda,\beta} = -\beta^{-1}|\Lambda|^{-1} \log Z$; it can be proved under rather general conditions on the energy, see *e.g.* Theorem 2.4.1. of [Ruelle], that the limit

$$f_\beta = -\beta^{-1} \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log Z \tag{2.3}$$

exists and is convex. Phase transitions, which can be present only in the *thermodynamic limit* $|\Lambda| \to \infty$, appear as non-analyticity points of f_β. The derivatives of f_β correspond to physical observables; for instance the *specific heat* is defined (when exists) as

$$C_v = -\frac{\partial}{\partial T}\frac{\partial}{\partial \beta} \log Z \tag{2.4}$$

If O is some observable property of the system with value $O(\sigma)$ in the spin configuration σ, than its observed average thermodynamic value is

$$\langle O \rangle_{\Lambda,\beta} = \sum_{\sigma} O(\sigma) \frac{e^{-\beta H}}{Z} \tag{2.5}$$

and if $O_{\mathbf{x}}$ is a local monomial in the spin variables, like $\sigma_{\mathbf{x}}$ or $\sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}$ (where $\mathbf{x}'$ is a nearest-neighbor of $\mathbf{x}$), its truncated correlations are

$$\lim_{\Lambda\to\infty} [\langle O_{\mathbf{x}} O_{\mathbf{y}} \rangle_{\Lambda,\beta} - \langle O_{\mathbf{x}} \rangle_{\Lambda,\beta} \langle O_{\mathbf{y}} \rangle_{\Lambda,\beta}] \tag{2.6}$$

The correlations measure, roughly speaking, the influence in a point $\mathbf{y}$ of a perturbation located at a certain point $\mathbf{x}$. Finally the *magnetization* is defned as

$$m(h) = \frac{1}{|\Lambda|}\frac{\partial \log Z}{\partial h} =< \sigma_{\mathbf{x}} > \tag{2.7}$$

and the *spontaneous magnetization* is $m(0)$. We assume periodic boundary conditions, that is $\sigma_{\mathbf{x}} = \sigma_{\mathbf{x}+\mathbf{e}_j L}$. Note that at L finte $m(0) = 0$ as a consequence of the symmetry $\sigma \to -\sigma$. If we consider $\lim_{L\to\infty} m(h)$ a spontaneous magnetization is possible (but not in $d = 1$). Indeed in $d = 2, 3$ and low temperatute $\beta > \beta_c$ then $\lim_{h\to 0^+} \lim_{L\to\infty} m(h)$ is non vanihing and opposite to $\lim_{h\to 0^-} \lim_{L\to\infty} m(h)$.

As we said, the Ising model gives a simplified description of a magnet, described as a lattice of molecules with a magnetic dipole pointing along a preferred axis, with two possible directions and interacting through exchange forces favoring their alignment; indeed given a couple of nearest molecules, their contributions to the total energy is $-J$ if the spins are in the same directions and $+J$ if they point in different directions. The last term in the hamiltonian describe the interaction with an external magnetic field. Of course the dipoles in reality do not point out in two preferred directions, there is no reason to consider only nearest neighbor interactions and so on. Despite these concerns, remarkably this so simplified model provide a rather effective description of a number of real systems, due to a property called universality. Indeed it would be very natural to add a next to nearest neigbor or a short range interaction, or to generalize the model to spins with more than two states, or see what happens if one include quantum mechanical effects.

Indeed the Ising model has a wide range of applicability is statistical mechanics, and it can be applied to a number of different situation. For instance the lattice gas interpretation; consider a gas of particles which can jump from a site to another. We assume that there is an hard core condition forbidding that two particles occupy the same site. This means that we can associate to each site of the lattice gas a variable $n_{\mathbf{x}} = 0, 1$, where 0 means an empty site and 1 an occupied site; the hard core condition forbids any other possibility. The Hamiltonian is

$$H_J = -\widetilde{J} \sum_{j=1,..d} \sum_{\mathbf{x}\in\Lambda} n_{\mathbf{x}} n_{\mathbf{x}+\mathbf{e}_j} \tag{2.8}$$

where $n_{\mathbf{x}} = 0, 1$ where the first term represents the (attractive)interactions between particles and the hard core repulsion is represented by the condi-

tion $n_x = 0, 1$. This lattice gas model is easily seen to be mapped in the Ising model setting $n_{\mathbf{x}} = \frac{1}{2}(1 - \sigma_{\mathbf{x}})$ so that

$$H_J = -\frac{J}{4} \sum_{j=1,..d} \sum_{\mathbf{x}\in\Lambda} (1-\sigma_{\mathbf{x}})(1-\sigma_{\mathbf{x}+\mathbf{e}_j}) = -\frac{J}{4} \sum_{j=1,..d} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - \frac{J}{2} z \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} + const \tag{2.9}$$

where z is the number of nearest neighbor. Therefore in the Grand-canonical partition function of the larttce gas

$$Z = \sum_{\sigma} e^{-\beta H - \mu N} \tag{2.10}$$

correspond to the partition function of the ising model in the canonical ensemble with $h = -\frac{\widetilde{J}}{2} - \mu$ and $J = \widetilde{J}/4$. The averaged density of particles is $\frac{\bar{N}}{|\Lambda|} = \frac{1}{|\Lambda|}\frac{\partial \log Z}{\partial \mu}$, that is $< n_{\mathbf{x}} >= \frac{1}{2}(1- < \sigma_{\mathbf{x}} >)$. Therefore for $d = 2, 3$ at $\beta > \beta_c$ for $\mu == -(\frac{\widetilde{J}}{2})^+$ or $\mu == -(\frac{\widetilde{J}}{2})^-$ one can have $< n_{\mathbf{x}} >= 0$ or $< n_{\mathbf{x}} >= 1$; that is there is a discontinuity in the averaged density as in the liquid-gas transition.

Note finally that is we are considering the Ising model in dimension d. Indeed not only $d = 3$ is important for physical applications, but also $d = 1, 2$; several real macroscopic systems are indeed so anisotropic to be described in terms of planar or linear systems. We well see moreover that is is very convenient to consider even $d \geq 4$, either for the mathematical and physical point of view.

2.2 The one dimensional Ising model

2.2.1 *The multipolygon representation*

Let us consider the Ising model in $d = 1$, with periodic boundary conditions

$$H_J = -J \sum_{x=1}^{L} \sigma_x \sigma_{x+1} - h \sum_{x=1}^{L} \sigma_x \tag{2.11}$$

where $\sigma_x = \pm$ and *periodic* boundary conditions are assumed

$$\sigma_1 = \sigma_{L+1} \tag{2.12}$$

Let us consider for definiteness the case $h = 0$. The partition function is given by

$$Z = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1} e^{\beta J \sum_{x=1}^{L} \sigma_x \sigma_{x+1}} = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1} \prod_{x=1}^{L} e^{\beta J \sigma_x \sigma_{x+1}} \tag{2.13}$$

As $\sigma_x = \pm$ and therefore $\sigma_x\sigma_{x+1} = \pm$ we can write

$$e^{\beta J\sigma_x\sigma_{x+1}} = \cosh\beta J + \sigma_x\sigma_{x+1}\sinh\beta J \tag{2.14}$$

so that

$$Z = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1}\prod_{x=1}^{L}[\cosh\beta J + \sigma_x\sigma_{x+1}\sinh\beta J] \tag{2.15}$$

Note that

$$\begin{aligned}&[\cosh\beta J + \sigma_1\sigma_2\sinh\beta J][\cosh\beta J + \sigma_2\sigma_3\sinh\beta J]...\\ &[\cosh\beta J + \sigma_{L-1}\sigma_L\sinh\beta J][\cosh\beta J + \sigma_L\sigma_1\sinh\beta J] =\\ &(\cosh\beta J)^L + (\sinh\beta J)^L(\sigma_1\sigma_2\sigma_2\sigma_3...\sigma_{L-1}\sigma_L\sigma_L\sigma_1) + R\end{aligned} \tag{2.16}$$

where R contains all the mixed terms, like $\sinh(\beta J)\sigma_1\sigma_2\cosh^{L-1}$; it is clear that the terms in R contains at least a single σ_x so that

$$\sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1} R = 0 \tag{2.17}$$

On the other hand $(\sigma_1\sigma_2\sigma_2\sigma_3...\sigma_{L-1}\sigma_L\sigma_L\sigma_1) = 1$ so that

$$Z = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1}((\cosh\beta J)^L + (\sinh\beta J)^L) = 2^L\cosh^L(\beta J)(1+(\tanh\beta J)^L] \tag{2.18}$$

and finally the free energy is given by

$$\beta f = \frac{1}{L}\log Z = \log[2\cosh(\beta J)] + \frac{1}{L}\log(1 + (\tanh\beta J)^L] \tag{2.19}$$

For $\beta J > 0$ then $\tanh\beta J < 1$ so that

$$\lim_{L\to\infty}\beta f = \log[2\cosh(\beta J)] \tag{2.20}$$

The free energy is analytic for βJ real so that no phase transition are present in the Ising model in one dimensions.

It is also instructive to see what happens with different boundary conditions, for instance open boundary conditions; in this case

$$Z = \sum_{\sigma_1=\pm1,\ldots,\sigma_{L+1}=\pm1} e^{\beta J\sum_{x=1}^{L}\sigma_x\sigma_{x+1}} = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1}\prod_{x=1}^{L} e^{\beta J\sigma_x\sigma_{x+1}} \tag{2.21}$$

and

$$Z = \sum_{\sigma_1=\pm1,\ldots,\sigma_L=\pm1}(\cosh\beta J)^L = 2^{L+1}\cosh^L(\beta J) \tag{2.22}$$

Therefore for finite L the partition function is slightly different but the $L\to\infty$ the free energy is the same with both boundary conditions.

2.2.2 *Transfer matrix method*

Let us consider the Ising model in $d = 1$, with periodic boundary conditions

$$H_J = -J \sum_{x=1}^{L} \sigma_x \sigma_{x+1} - h \sum_{x=1}^{L} \sigma_x \tag{2.23}$$

where $\sigma_x = \pm$ and *periodic* boundary conditions are assumed

$$\sigma_1 = \sigma_{L+1} \tag{2.24}$$

The partition function is given by

$$Z = \sum_{\sigma_1=\pm 1,...,\sigma_L=\pm 1} e^{\beta J \sum_{x=1}^{L} \sigma_x \sigma_{x+1} - h \sum_{x=1}^{L} \sigma_x} = \sum_{\sigma_1=\pm 1,...,\sigma_L=\pm 1} [e^{\beta J \sigma_1 \sigma_2 + \beta h \sigma_1}][e^{\beta J \sigma_2 \sigma_3 + \beta h \sigma_2}]...[e^{\beta J \sigma_L \sigma_1 + \beta h \sigma_L}] \tag{2.25}$$

We introduce now the matrix

$$V = \begin{pmatrix} e^{\beta(h+J)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(-h+J)} \end{pmatrix}$$

so that

$$Z = \sum_{\sigma_1=\pm 1,...,\sigma_L=\pm 1} V_{\sigma_1,\sigma_2} V_{\sigma_2,\sigma_3} ... V_{\sigma_L,\sigma_1} = \mathrm{Tr} V^L$$

where $V_{1,1}, V_{-1,-1}, V_{-1,1}, V_{1,-1}$ are the matrix elements of V.

The eigenvalues of V are

$$\lambda_\pm = e^{\beta J} \cosh \beta h \pm \sqrt{e^{2\beta J} \cosh^2(\beta h) + e^{-2\beta J} - e^{2\beta J}} \tag{2.26}$$

Therefore the free energy is given by, as $Z = \lambda_+^L + \lambda_-^L$

$$f_\beta = \lim_{L\to\infty} \frac{1}{L} \log Z = \lim_{L\to\infty} \frac{1}{L} \log \lambda_+^L + \lim_{L\to\infty} \frac{1}{L} \log(1 + \lim_{L\to\infty} \frac{1}{L} \log(1 + \frac{\lambda_-}{\lambda_+})^L) =$$

$$\log[e^{\beta J} \cosh \beta h + \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}] \tag{2.26}$$

The free energy is *analytic* for all $\beta > 0$ and all h; that is, *no phase transitions* are present in the one dimensional Ising model.

A similar computation for the spin-spin correlations gives in the $h = 0$ case, $y > x$

$$< \sigma_x \sigma_y >= \frac{\mathrm{Tr} V^x \sigma_3 V^{x-y} \sigma_3 V^{L-y}}{\mathrm{Tr} V^L} \tag{2.26}$$

Note that

$$AVA = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \tag{2.26}$$

where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.26}$$

and $A^2 = I$. Therefore

$$V^r = A\Lambda AA\Lambda A... = A\Lambda A = A\Lambda^r A \tag{2.26}$$

so that

$$< \sigma_x \sigma_y >= \frac{\mathrm{Tr} A\Lambda^x A\sigma_3 A\Lambda^{y-x} A\sigma_3 (A\Lambda^{L-y} A)}{\mathrm{Tr} V^L} = \frac{\mathrm{Tr} A\Lambda^x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda^{y-x} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda^{L-y} A}{\mathrm{Tr} V^L} = \tag{2.26}$$

$$\frac{\mathrm{Tr} \Lambda^x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda^{y-x} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda^{L-y}}{\mathrm{Tr} V^L} = \frac{\mathrm{Tr} \begin{pmatrix} 0 & \lambda_+^x \\ \lambda_-^x & 0 \end{pmatrix} \begin{pmatrix} \lambda_+^{y-x} & 0 \\ 0 & \lambda_-^{y-x} \end{pmatrix} \begin{pmatrix} 0 & \lambda_-^{L-y} \\ \lambda_+^{L-y} & 0 \end{pmatrix}}{\mathrm{Tr} V^L} = \tag{2.26}$$

$$\frac{\mathrm{Tr} \begin{pmatrix} 0 & \lambda_+^x \\ \lambda_-^x & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_+^{y-x} \lambda_-^{L-y} \\ \lambda_-^{y-x} \lambda_+^{L-y} & 0 \end{pmatrix}}{\mathrm{Tr} V^L} = \frac{\mathrm{Tr} \begin{pmatrix} \lambda_+^{L-y+x} \lambda_-^{y-x} & 0 \\ 0 & \lambda_-^{y-x} \lambda_+^{L-y+x} \end{pmatrix}}{\mathrm{Tr} V^L} \tag{2.26}$$

and finally if $r = y - x$

$$< \sigma_x \sigma_y >= \frac{\lambda_+^r \lambda_-^{L-r} + \lambda_-^r \lambda_+^{L-r}}{\lambda_-^L + \lambda_+^L} = \frac{\tanh \beta J^r + \tanh \beta J^{L-r}}{1 + \tanh \beta J^L} \tag{2.26}$$

so that in the limit $L \to \infty$, as $\tanh \beta J < 1$

$$\lim_{L \to \infty} < \sigma_x \sigma_y >= \tanh \beta J^r = e^{-r \log(1/\tanh \beta J)} \tag{2.26}$$

Therefore the spin-spin correlation decay exponentially with coerence length $a \log(1/\tanh \beta J)$ and the free energy is analytic; no phase transition in the Ising model in $d = 1$.

2.3 Mean field theory

2.3.1 *The Ising model in the mean field approximation*

The Ising model in one dimension is exactly solvable, but no phase transitions are present. One can hope that phase transitions are present in higher dimensions, but the analysis is of increasing difficulty. This has led people to develop approximate techniques which hopefully give some information on the real systems. The most celebrated one is the well known *mean field* approximation, consisting in the approximation that each spin behave as if it were an independent spin sitting in the mean field produced by all the other spins. Indeed we can rewrite the Ising model Hamiltonian as

$$H_J = -J \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} (\sum_{\mathbf{x}' n.n.to \mathbf{x}} \sigma_{\mathbf{x}'} - h) \tag{2.26}$$

where $\sum_{\mathbf{x}' n.n.to \mathbf{x}}$ means that the sum is over the nearest neighbor site of $\mathbf{x}$. The mean field approximation consists in replacing $\sum_{\mathbf{x}' n.n.to \mathbf{x}} \sigma_{\mathbf{x}'}$ by a sum over averaged values, that is

$$H_{MF} = -J \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} (\sum_{\mathbf{x}' n.n.to \mathbf{x}} < \sigma_{\mathbf{x}'} > -h) \tag{2.26}$$

As the system is translation invariant $< \sigma_{\mathbf{x}} >= m$ does not depend from $\mathbf{x}$ so that

$$H_{MF} = \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} [zm - h] \tag{2.26}$$

where z is the number of nearest neighbor ($z = 2d$ for square lattices); by self consistency

$$m = \frac{\sum_{\underline{\sigma}} e^{-\beta H_{MF}} \sigma_{\mathbf{x}}}{\sum_{\underline{\sigma}} e^{-\beta H_{MF}}} = \frac{\sum_{\sigma=\pm 1} \sigma e^{-\beta\sigma[zm-h]}}{\sum_{\sigma^{\pm}} e^{-\beta\sigma[Jzm-h]}} = \tanh \beta(zJm - h) \tag{2.26}$$

which in the case $h = 0$ it reduces to

$$m = \tanh(\beta J z m) \tag{2.26}$$

Note that $N < \sigma_{\mathbf{x}} >$ has the meaning of *spontaneous magnetization* of the system; spontaneous because there is no magnetic field and magnetization because the total magnetic dipole is proportional to the total spin.

The tangent of $\tanh(\beta J z m)$ in $m = 0$ has slope $\beta J z$ and when $\beta J z \leq 1$ there is only the intersection in $m = 0$ with the straight line with slope 1, while when $\beta J z < 1$ there are in addition two non zero intersection. This is a rather remarkable result: when $\beta J z \leq 1$, that is $T \geq T^c = Jz/\kappa$ the

spontaneous magnetization is vanishing, while for $T < T_c$ there is a spontaneous magnetization. We can expand close to the critical temperature and m small; note that

$$\tanh x = \frac{x + x^3/6 + ...}{1 + x^2/2 + ...} = x - x^3/3 + ... \tag{2.26}$$

so that for small m (that is for $\beta \to \beta_c^+$)

$$m = \frac{\beta}{\beta_c} m - \frac{1}{3} \frac{\beta}{\beta_c}^3 m^3 + .. \tag{2.26}$$

which gives for $\beta \to \beta_c^+$

$$m \sim (3(1 - \frac{\beta}{\beta_c}))^{\frac{1}{2}} \tag{2.26}$$

Therefore the spontaneous magnetization vanishes when $T \to T_c$ with a power law with exponent $1/2$.

Similarly at $\beta = \beta_c$ and small magnetic field then, remembering that $\beta_c J z = 1$

$$m = \tanh(\beta_c J z m + \beta_c h) \sim (m + \beta_c h) - (m + \beta_c h)^3/3 + ... \tag{2.26}$$

that is at $\beta = \beta_c$

$$m \sim h^{\frac{1}{3}} \tag{2.26}$$

The above analysis is known as the Curie-Weiss theory of magnetic transitions. From the above analysis we have learnt a lot. We have a simple theory which is able to capture the main qualitative features of the ferromagnetic transitions, namely when the temperature is high than T_c there is no magnetization when $h = 0$ while if $T < T_c$ there is spontaneous magnetization. Moreover, in agreement with experiments, close to the *critical point* $h = 0, T = T_c$ the behavior of the thermodynamic quantities are described in terms of *critical exponents* independent of J (that is from the material).

While apparently successful in predicting phase transitions, the above theory is however just an *approximation*, which we already know to be *wrong* in $d = 1$, in which our computations was exact. Moreover, the exponent do not fit well with experiments. Which is then the meaning of the above analysis?

2.3.2 *The Infinite range Ising model*

In the n.n. Ising model one assumes that the spins interact only with the nearest neighbor ones; the infinite-range model is very similar to the Ising model, with the difference that one assumes that each spin interact with all the others and that the forces do not decay with the distance. This very unrealistic assumption makes however the model exactly solvable. Note that as the forces are constant and involve all spins we can simply imagine to relabel all the spins with an index $j = 1, ..., N$. We will see that the result found in the mean field approxiamtions are true in an Ising model with infinite range forces.

The Hamiltonian of the Infinite range Ising model is

$$H = -\frac{J}{N} \sum_{1 \leq i,j \leq N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \tag{2.26}$$

which can be written as

$$H = -\frac{J}{2N} (\sum_i \sigma_i)^2 - h \sum_{i=1}^{N} \sigma_i + \frac{J}{2} \tag{2.26}$$

and the partition function is

$$Z = \sum_{\sigma_1, ..., \sigma_N} e^{\frac{\beta J}{N} (\sum_{1 \leq j \leq N} \sigma_i)^2 + \beta h \sum_{i=1}^{N} \sigma_i} \tag{2.26}$$

Using that

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx e^{-x^2/2 + \sqrt{a}x} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx e^{-1/2(x+\sqrt{a})^2 - a/2} = e^{a/2} \tag{2.26}$$

and setting $\sqrt{a} = (\frac{\beta J}{N})^{\frac{1}{2}} \sum_{1 \leq j \leq N} \sigma_i$ we get

$$Z = \sum_{\sigma_1, ..., \sigma_N} \frac{e^{-\frac{\beta J}{2}}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx e^{-x^2/2 + (\frac{\beta J}{N})^{\frac{1}{2}} [\sum_{i=1}^{N} \sigma_i] x + \beta h \sum_{i=1}^{N} \sigma_i} \tag{2.26}$$

Note that

$$\sum_{\sigma_1, ..., \sigma_N} e^{\frac{\beta J}{N})^{\frac{1}{2}} [\sum_{i=1}^{N} \sigma_i] x + \beta h \sum_{i=1}^{N} \sigma_i} = [\sum_{\sigma} e^{(\frac{\beta J}{N})^{\frac{1}{2}} \sigma x + \beta h \sigma}]^N = [2 \cosh[(\frac{\beta J}{N})^{\frac{1}{2}} x + \beta h)]^N \tag{2.26}$$

and finally

$$Z = \frac{e^{-\frac{\beta J}{2}}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx e^{-x^2/2} [2 \cosh[(\frac{\beta J}{N})^{\frac{1}{2}} x + \beta h)]^N \tag{2.26}$$

We perform now the change of variable

$$y = x(\frac{1}{\beta JN})^{\frac{1}{2}} \tag{2.26}$$

so that

$$Z = \frac{e^{-\frac{\beta J}{2}}}{(2\pi)^{\frac{1}{2}}}(N\beta J)^{\frac{1}{2}} \int_{-\infty}^{\infty} dy e^{-\frac{N\beta J}{2}y^2} [2\cosh[(\beta Jy + \beta h)]^N \tag{2.26}$$

The partition function can be explicitly evaluated in the limit $N \to \infty$. Indeed

$$Z = e^{-\frac{\beta J}{2}} \frac{1}{(2\pi)^{\frac{1}{2}}}(N\beta J)^{\frac{1}{2}} 2^N \int_{-\infty}^{\infty} dx e^{NF(x)} \tag{2.26}$$

where

$$F(x) = -\frac{\beta J}{2}x^2 + \log\cosh[(\beta Jx + \beta h) \tag{2.26}$$

Note that the second term behaves as x as $x \to \infty$.

We recall now a simplified version of Laplace method, saying that, in the sense of distributions, if $H(x)$ is a two times even differentiable function with a non degenerate global maximum 0 and such that $-c_1x^2 \leq H(x) \leq -c_2x^2$, then

$$\lim_{N\to\infty} \frac{e^{NH(x)}}{\int_{-\infty}^{\infty} dx e^{NH(x)}} = \delta(x - x_0) \tag{2.26}$$

This means that for any test function

$$\lim_{N\to\infty} \frac{\int_{-\infty}^{\infty} e^{NH(x)}(\phi(x) - \phi(x_0)))}{\int_{-\infty}^{\infty} dx e^{NH(x)}} = 0 \tag{2.26}$$

Let us choose for definiteness $x_0 = 0$ and $H(0) = 0$; We have to show that for any ε there is an N_ε such that for $N \geq N_\varepsilon$ then $\frac{\int_{-\infty}^{\infty} e^{NH(x)}(\phi(x)-\phi(0)))}{\int_{-\infty}^{\infty} dx e^{NH(x)}} \leq \varepsilon$. We write

$$\frac{\int_{|x|\leq\delta} e^{NH(x)}(\phi(x) - \phi(0)))}{\int_{-\infty}^{\infty} dx e^{NH(x)}} + \frac{\int_{|x|\geq\delta} e^{NH(x)}(\phi(x) - \phi(0)))}{\int_{-\infty}^{\infty} dx e^{NH(x)}}$$

Then by integrability there exists δ_ε such that the second addend is $< \frac{\varepsilon}{2}$; on the other hand

$$\sqrt{N}|\int_{|x|\leq\delta} e^{NH(x)}(\phi(x) - \phi(0))| \leq \int_{|y|\leq\sqrt{N}\delta} e^{NH(y/\sqrt{N})}(\phi(\frac{y}{\sqrt{N}}) - \phi(0))| \leq$$

$$C\frac{1}{\sqrt{N}}\int dx|x|e^{NH(x/\sqrt{N})} \leq C'\frac{1}{\sqrt{N}}$$

The extremal points of $F(x)$ are in correspondence of $\partial F(x) = 0$; Let us consider h very small and positive; then for $\frac{1}{2\beta J} > 1$ there is only one maximum $\bar{x}(h)$ such that $x(0) = 0$; for $\frac{1}{2\beta J} < 1$ there in one global maximum $\bar{x}(h)$ with the sign of h and $\bar{x}(h) = |\beta - \beta_c|^{\frac{1}{2}} + O(h)$.

Therefore for $\beta < \beta_c$ we apply the saddle point to

$$Z = \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{N}{\beta J}2^N e^{NF(x_0)}\int_{-\infty}^{\infty} dx e^{NF(x)-F(x_0)} \tag{2.26}$$

Note that $\lim_{N\to\infty} \log N/N = 0$; moreover

$$\lim_{N\to\infty} \frac{e^{NF(x)-F(x_0)}}{\int e^{NF(x)-F(x_0)}] = \delta(x - x_0)}$$

and the denominator is $O(\sqrt{N})$ for large N, so that the free energy is given, for $\beta \leq \beta_x$ by

$$\lim_{h\to 0}\lim_{N\to\infty} -\frac{\beta^{-1}}{N}\log Z = \log 2 \tag{2.26}$$

while for $\beta > \beta_c$

$$\lim_{h\to 0^+}\lim_{N\to\infty} -\frac{\beta^{-1}}{N}\log Z = \log 2 - \frac{bJ}{2}\bar{x}(0^+)^2 + \log\cosh[(\bar{\beta}Jx(0^+) \tag{2.26}$$

Therefore the specific heat $\partial_T\partial_\beta \log Z$ is zero for $T > T_c$ and non zero for $T < T_c$; that is has a discontinuity. Moreover the magnetization for $T < T_c$ is different from zero

$$\lim_{h\to o^\pm}\lim_{N\to\infty} < \sigma >= \lim_{h\to o^\pm}\lim_{N\to\infty} \frac{\int_{-\infty}^{\infty} dx x e^{NF(x)}}{\int_{-\infty}^{\infty} dx e^{NF(x)}} = \bar{x}(0^\pm) \tag{2.27}$$

Remark Note the order in which the limits are taken (in the opposite order one finds zero). Note also that F is approximately

$$F(x) \sim (\beta_c - \beta)x^2 + \frac{x^4}{8}$$

hence is a "mexican hat" for $\beta > \beta_c$.

2.3.3 *The long range Ising model*

2.4 The Ising model in two dimensions

2.4.1 *The multipolygon representation*

If Λ is a square subset of $\mathbb{Z}^2$ with side L and periodic boundary conditions and $\mathbf{x} = (x_0, x)$, the hamiltonian of the nearest-neighbor Ising model is

$$H_I = J \sum_{\mathbf{x}\in\Lambda} [\sigma_{x,x_0}\sigma_{x,x_0+1} + \sigma_{x,x_0}\sigma_{x+1,x_0}] = \sum_{\mathbf{x}\in\Lambda} H_{\mathbf{x}} \tag{2.28}$$

and periodic boundary conditions are imposed. The partition function at inverse temperature β is given by

$$Z_I = \sum_{\{\sigma_{\mathbf{x}}=\pm\}} e^{-\beta H} = \sum_{\underline{\sigma}} \prod_{\mathbf{x}} e^{\beta J[\sigma_{x,x_0}\sigma_{x,x_0+1}+\sigma_{x,x_0}\sigma_{x+1,x_0}]} \tag{2.29}$$

As $\sigma_{\mathbf{x}} = \pm$ we can write

$$\begin{aligned} e^{\beta J\sigma_{x,x_0}\sigma_{x,x_0+1}} &= \cosh(\beta J\sigma_{x,x_0}\sigma_{x,x_0+1}) + \sinh(\beta J\sigma_{x,x_0}\sigma_{x,x_0+1}) \\ &= cosh(\beta J) + \sinh \beta J\sigma_{x,x_0}\sigma_{x,x_0+1}) = \cosh(\beta J)(1 + t\sigma_{x,x_0}\sigma_{x,x_0+1}) \end{aligned} \tag{2.30}$$

where $t = \tanh \beta J$; similarly

$$e^{\beta J\sigma_{x+1,x_0}\sigma_{x,x_0}} = \cosh(\beta J)(1 + t\sigma_{x,x_0}\sigma_{x,x_0+1}) \tag{2.31}$$

We define b are the *bonds* linking nearest neighbor sited, and $\widetilde{\sigma}_b$ is the product of the spin variables over the two extermes of b; for instance if b is the bond between (x, x_0) and $x, x_0 + 1$ then $\widetilde{\sigma}_b = \sigma_{x,x_0}\sigma_{x,x_0+1}$. Therefore the partition function can be written as

$$Z = \sum_{\{\sigma_{\mathbf{x}}\}} \prod_b (\cosh \beta J + \widetilde{\sigma}_b \sinh \beta J) = (\cosh \beta J)^B \sum_{\{\sigma_{\mathbf{x}}\}} \prod_b (1 + \widetilde{\sigma}_b \tanh \beta J) \tag{2.32}$$

where B is the number of bonds of Λ. Developing the product, we are led to a sum of terms proportional to $\widetilde{\sigma}_{b_1} \cdots \widetilde{\sigma}_{b_k}$. To every term $\widetilde{\sigma}_{b_1} \cdots \widetilde{\sigma}_{b_k}$ we can associate a distinct figure on the lattice, given by the geometric set of lines $b_1, \ldots, b_k$. If $\widetilde{\sigma}_{b_1} \cdots \widetilde{\sigma}_{b_k}$ contains $\sigma_{\mathbf{x}}^n$ for some $\mathbf{x}$ and n an odd integer, such term gives a vanishing contribution to the sum in (2.32); hence the figures giving non vanishing contributions are just the closed multipolygons γ, that is polygons or union of polygons with a point (but not sides) in common.

To be more concrete, let us consider an example; for instance the term

$$(\sigma_{1,1}\sigma_{21})(\sigma_{2,1}\sigma_{22})(\sigma_{2,2}\sigma_{2,0}) \tag{2.33}$$

gives a zero contributions when summed, as is equal to $(\sigma_{1,1}\sigma_{2,0})$ and $\sum_\sigma \sigma = 0$; the three bonds form the three sides of a square. On the other hand

$$(\sigma_{1,1}\sigma_{21})(\sigma_{2,1}\sigma_{22})(\sigma_{2,2}\sigma_{2,0})\sigma_{2,0}\sigma_{1,1}) = 1 \tag{2.34}$$

and the bonds just represents the four sides of square.

The partition function can be then rewritten as:

$$Z_I = (\cosh \beta J)^B 2^S \sum_\gamma (\tanh \beta J)^{|\gamma|} \, . \tag{2.35}$$

where S is the total number of sites, the sum is over all the multipolygon γ with length $|\gamma|$. If open boundary conditions are assumed, only multipolygons *not* winding up the lattice are allowed. In the case of periodic boundary conditions the representation is the same, but the polygons are allowed to wind up the lattice.

2.4.2 *Gaussian integrals*

We recall well known formulas on gaussian integrals. If α_{ij} is a symmetric definite positive matrix

$$\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] e^{-\frac{1}{2} x_i \alpha_{ij} x_j} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{det\alpha}}$$

and

$$\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] e^{-\frac{1}{2} x_i \alpha_{ij} x_j + \sum_j h_j x_j} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{det\alpha}} e^{\sum_{ij} h_i (\alpha^{-1})_{ij} h_j}$$

Another important integral is

$$\frac{\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] x_i x_j e^{-\frac{1}{2} x_i \alpha_{ij} x_j}}{\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] e^{-\frac{1}{2} x_i \alpha_{ij} x_j}} = [\alpha^{-1}]_{ij} = < x_i x_j >$$

and

$$\frac{\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] x_1 x_2 x_3 x_4 e^{-\frac{1}{2} x_i \alpha_{ij} x_j}}{\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dx_i] e^{-\frac{1}{2} x_i \alpha_{ij} x_j}} = < x_1 x_2 >< x_3 x_4 > + < x_1 x_4 >< x_2 x_3 > + < x_1 x_3 >< x_2 x_4 >$$

Similarly one introduces complex integrals; if $z = x + iy$, $\bar{z} = x - iy$

$$\int_{-\infty}^{\infty} [\prod_{i=1}^{N} dz_i] e^{-\frac{1}{2} \bar{z}_i \alpha_{ij} z_j} = \frac{(2\pi)^N}{det\alpha}$$

and

2.4.3 *Critical exponents*

We have seen that the Ising model in one dimension do not exhibit phase transitions; they are on the other hand present in the infinite range Ising model, which is however a rather unphysical system. The question wether statistical mechanics can really predict phase transitions in physical models is then still open at this point. This therefore explains the great importance and role of the nearest neighbor Ising model in two dimensions: it was solved by Onssager in 1944 and was the first (and one of the few) models exhibiting a phase transition which can be studied in all details (the book McCoy Wu contains a detailed derivation of a number of important propertie). The original solution of Onsager was considered apparently quite different to follow, but much easier derivations have been found along the years, taking into account the mapping of the 2-dimensional Ising model with a gas of one dimensonal fermions, see Shultz, Lieb, Mattis 1964. As we have not still introduced quantum statistical mechanics, we pospone the exact solution and we simply describe the main results. It is however very important to stress already at this stage that there is a strict link between classical statistical mechanics in d-dimensions and quantum statistical mechanics in $d+1$ dimensions; this link is purely at a mathematical level, as the corresponding physical systems described are completely different. The exact solution of the 2d Ising model, and the exact solution of a few other models playes in statistical mechanics the role of integrable models, like two body or lagrange rotator, in clasical mechanics; of course, also here integrability is a very rare property and other methods must be developed.

According to the exact solution, the free energy is given by, in the limit $L \to \infty$, if $t = \tanh \beta J$

$$-\beta f = \log(2\cosh^2 \beta J) + \frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \log[(1+t^2)^2 - 2t(1-t^2)(\cos k + \cos k_0)] \quad (2.36)$$

The argument of the logarithm vanishes if

$$(1+t^2)^2 = 4t(1-t^2) \quad (2.37)$$

that is for

$$t_c = \tanh \beta J = \sqrt{2} - 1 \quad (2.38)$$

as $t_c^2 = 3-2\sqrt{2}$, $t_c(1-t_c^2) = 2(3-2\sqrt{2})$, $(1+t_c^2)^2 = 2(2-\sqrt{2})^2 = 8(3-2\sqrt{2})$. It is easy to verify that at $\beta = \beta_c$ the system undergoes a phase transition; indeed $\beta = \beta_c$ is not an analyticity point fpr the free energy and the second

derivative is logarithmically vanishing, or, in other words, the specific heat is logarithmically diverging. Indeed the specific heat is given by, calling $m^2(\beta) = (1+t^2)^2 - 4t(1-t^2)$ and taking the second derivatice weith respect to β

$$C_v = A(\beta) + B(\beta)\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\int_{-\pi}^{\pi}\frac{dk_0}{2\pi}m(\beta)^2 + (\cos k + \cos k_0 - 2)](2.39)$$

with suitable finite non vanishing regular functions $A(\beta), B(\beta)$; therefore, as $m(\beta) \sim |\beta - \beta_c|$ and noting that $(\cos k + \cos k_0 - 2)$ vanishes only at $k_0 = k = 0$ and close to taht point $(\cos k + \cos k_0 - 2) \sim k_0^2/2 + k^2/2$ we get

$$C_v \sim A + B\log|\beta - \beta_c| \tag{2.40}$$

Note that the specific heat is logarithmocally diverging while was discontinuous in the mean field approximation.

One can also compute the truncated spin-correlations; it turns out that at $\beta \not\beta_c$ they decay exponentially with decay rate

$$\xi \sim m(\beta) \sim C|\beta - \beta_c| \tag{2.41}$$

while at $\beta = \beta_c$ they decay as a power law; for istance the spin-spin correlation decay for large distance at $\beta = \beta_c$ as $|x - x'|^{-\frac{d1}{4}}$, again a value rather different from the one found in the mean field approximation (and in good agreement with experiments on layered magnetic materials). Actually the computation i quite involeved (see McCoy-Wu) and has been performed only when the points are on the same orizzontal or vertical line, or on a diagonal; it is believed that the same exponents is true in general but (surprisingly) a proof is still lacking. The analysis is based on the asymptotics of large matrices and the Tsego lemma.

Finally one can compute the magnetization, obtained by the limit when distances g to infinity of non truncated correlations; one gets a that for $\beta \geq \beta_c$ the magnetization is $m(\beta) \sim O(|\beta - \beta_c|^{1/8})$, going to zero as $\beta \to \beta_c^+$.

Chapter 3

3.1 Statistical mechanics for quantum gases

3.1.1 *First quantization formalism*

We are interested in understanding the macroscopic properties of a large set of quantum particles. A metal can be described as a lattice of ions and a set of conduction electrons moving in the lattice; this set of electrons, from which the conduction properties of matter depend, are described by a gas of fermions. Similarly the lattice excitations of the solid, which are responsible of the heat conduction, are effectively described by a gas of bosons. While at high temperature a description in terms of interacting particles is quite effective, at low temperatures quantum effects are important and have dramatic manifestation observed in experiments.

According to quantum mechanics, particles are described by a complex square integrable wave function wave-function $\psi(\vec{x}_1, ..., \vec{x}_N)$, $\vec{x} \in \Lambda$ and $\Lambda \subset \mathbb{R}^d$ with $|\Psi|^2$ representing the probability density of finding the N particles at position $\mathbf{x}_1, ..\mathbf{x}_N$. If the N particles are identical, then the probability density $|\psi(\vec{x}_1, ..., \vec{x}_N)|^2$ must be invariant under the exchange of coordinates, that is for any j, k

$$|\psi(\vec{x}_1, ..\vec{x}_i, .., \vec{x}_j., \vec{x}_N)|^2 = |\psi(\vec{x}_1, ..\vec{x}_j, .., \vec{x}_i., \vec{x}_N)|^2 \tag{3.1}$$

Regarding the symmetry properties of ψ itself, it is an experimental fact (which finds its theoretical explanation in the context of Relativistic Quantum Field theory) that only two possibilities can arise: either Ψ is symmetric or it is antisymmetric, which means that

$$\psi(\vec{x}_1, ..\vec{x}_i, .., \vec{x}_j., \vec{x}_N) = \varepsilon\psi(\vec{x}_1, ..\vec{x}_j, .., \vec{x}_i., \vec{x}_N) \tag{3.2}$$

with $\varepsilon = \pm$. Particles described by a symmetric wave function are called *bosons*, while particles with an antisymmetric Ψ are called *fermions*, from

Bose and Fermi who introduced these concepts. The fermionic wave function therefore vanishes if two coordinates are equal, a property called *Pauli exclusion principle*. Particles have an intrinsic quantized angular momentum called *spin* and particles with semi-integer spin are fermions, while particles with integer spin are bosons. Examples of fermions are electrons, protons or neutrons, with spin $\sigma = \pm\hbar/2$, where $\hbar$ is the Plank constant; examples of bosons are phonons or mesons with integer spin. The fact the quantum particles can be only fermions or bosons find its explanation in a deep result of relativistic quantum field theory which links the statistics with the spin of a particle.

The wave-functions, describing the state of a system, belongs to an Hilbert space

$$\mathcal{H}_{\varepsilon,N} = L^2_\epsilon(\Lambda^{3N}) \tag{3.3}$$

which is the subspace of $L^2(\Lambda^{3N})$ containing only symmetric $\varepsilon = +$ or antisymmetric functions $\varepsilon = -$ and Λ is a suitable susbset of $\mathbb{R}^{3N}$. For brevity, an element of $\mathcal{H}_{\varepsilon,\mathcal{N}}$ will be also called *state*. Sometimes it can be convenient to live on the space $\mathbb{R}^{dN}$. The *Hamiltonian* of N quantum particles is a linear operator on $\mathcal{H}_{\mathcal{N}}$, and the wave function evolves in time according to the *Schroedinger equation*

$$i\hbar\frac{\partial\psi}{\partial t} = H_N\psi \tag{3.4}$$

A typical form of H_N is the following

$$H_N = -\hbar^{2N}\sum_{i=1}^{N}\frac{\partial^2_{\vec{x}_i}}{2m} + \sum_{i=1}^{N}U(\mathbf{x}_i) + \lambda\sum_{1\le i<j\le N}^{N} v(\mathbf{x}-\mathbf{x}_j) \tag{3.5}$$

where the first term represents the kinetic energy, the second the interaction with an external field and the last one the interaction between particles.

A particularly important class of states belonging to $\mathcal{H}_{\varepsilon,\mathcal{N}}$ is obtained taking N functions $\psi_n(\vec{x}) \in L^2(\Lambda^3)$, considering the product and taking the symmmetrization (for bosons) or the antysymmetrization (for fermions). We define

$$|\psi_1,...\psi_N> \equiv \frac{1}{\sqrt{N!}}\sum_{\{\pi(1),..,\pi(n)\}}\varepsilon^P\psi_1(\vec{x}_{\pi(1)})...\psi_N(\vec{x}_{\pi(N)}). \tag{3.6}$$

where P is the parity of the permutation. For instance, if $N = 2$ and $\varepsilon = -$

$$|\psi_1,..,\psi_2> = \frac{1}{\sqrt{2}}[\psi_1(\vec{x}_1)\psi_2(\vec{x}_2) - \psi_2(\vec{x}_1)\psi_1(\vec{x}_2)] \tag{3.7}$$

and for N fermionic particles can be expressed by a determinant with elements $\psi_i(\vec{x}_j)$ (Slater determinant); for bosons the determinant is replaced by a permament.

The partition function in the canonical ensamble for a system of quantum particles is defined as

$$Z = \mathrm{Tr}_{\mathcal{H}_\mathcal{N}} e^{-\beta H_N} \tag{3.8}$$

where the trace is over the Hilbert space $\mathcal{H}_\mathcal{N}$. The free energy is given by $F = -\beta^{-1} \log Z$ and if A is an operator corresponding to an observable its statistical average is

$$< A >= \frac{\mathrm{Tr} e^{-\beta H_N} A}{\mathrm{Tr} e^{-\beta H_N}} \tag{3.9}$$

In the grand canonical ensemble it is convenient to consider states with any number of states, so we introduce the *Fock space* defined as

$$\mathcal{F}_\varepsilon = +_{N=0}^{\infty} L^2_\varepsilon(\Lambda^{3N}) \tag{3.10}$$

where $\mathcal{H}_\prime = \mathcal{C}$ by definition. where $\mathcal{H}_\prime = \mathcal{C}$. As Hamiltonian operator on the Fock space one takes

$$H = +_{N=0}^{\infty} H_N \tag{3.11}$$

so that the free energy in the grand-canonical ensamble is given by

$$F = -\beta^{-1} \log \mathrm{Tr}_{\mathcal{F}_\varepsilon} e^{-\beta(H-\mu N)} \equiv -\beta^{-1} \log \sum_{n=0}^{\infty} \mathrm{Tr}_{L^2_\varepsilon(\Lambda^{3n})} e^{-\beta(H_N - \mu n)} \tag{3.12}$$

where $N = +_{n=0}^{\infty} n$ is the number operator. Similarly the statistical average of an observable defined on the Fock space is

$$< A >= \frac{\mathrm{Tr}_\mathcal{F} e^{-\beta H} A}{\mathrm{Tr}_\mathcal{F} e^{-\beta H}} \tag{3.13}$$

3.1.2 *Second quantization formalism*

On the Fock space $\mathcal{F}_\varepsilon$ ($\varepsilon = +$ for bosons and $\varepsilon = -$ for fermions) we define the creation and annihilation operators $a^+(f), a^-(f)$ with $f \in L_1$; for $N \geq 1$, if $\psi_N \in L^2_\varepsilon(\Lambda^{3N})$, $\psi_N \equiv \psi_N(\vec{x}_1, ..., \vec{x}_N)$ then

$$(a(f)\psi_N)(\vec{x}_1, ..\vec{x}_{N-1}) = \sqrt{N} \int_\Lambda d\vec{x}_N \bar{f}(\vec{x}_N) \psi_N(\vec{x}_1, ..., \vec{x}_N)$$

$$(a^+(f)\psi_N)(\vec{x}_1, ..\vec{x}_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \varepsilon^{i-1} f(\vec{x}_i) \psi_N(\vec{x}_1, ., \not{\vec{x}}_i, .., \vec{x}_{N+1})$$

Therefore a^+ add a particle to ψ_N with state $f(\vec{x}_i)$ while $a(f)$ destroys a particle

$$a^+(f) : L^2_\varepsilon(\Lambda^{3N}) \to L^2_\varepsilon(\Lambda^{3(N+1)})$$
$$a(f) : L^2_\varepsilon(\Lambda^{3N}) \to L^2_\varepsilon(\Lambda^{3(N-1)}) \tag{3.14}$$

It follows by this definition that $a(f)$ is the adjoint of $a^+(f)$ and that the following canonical commutation relations are true

$$[a(f), a^+(g)]_\varepsilon =< f|g > \qquad [a^+(f), a^+(g)]_\varepsilon = [a(f), a(g)]_\varepsilon = 0 \tag{3.15}$$

where $< f|g >= \int d\vec{x} f(\vec{x})\gamma(\vec{x})$

$$[a, b]_\pm = ab \mp ba \tag{3.16}$$

In order to clarify the above definition, let us consider its action over the states of the form $|\psi_1, ...\psi_N >$. Then if

$$a(\psi)|\psi_1, ...\psi_N >=$$
$$\sqrt{N}\int_\Lambda d\vec{x}_N \bar{\psi}(\vec{x}_N)\frac{1}{\sqrt{N!}}\sum_{\{\pi(1),..,\pi(N)\}} \varepsilon^P \psi_1(\vec{x}_{\pi(1)})...\psi_N(\vec{x}_{\pi(N)}) =$$
$$\sum_i \varepsilon^{i-1} < \psi|\psi_i > |\psi_1, .., \not{\psi}_i, \psi_N >$$

where $< \psi|\psi_i >= \int_\Lambda d\vec{x}_N \bar{\psi}(\vec{x}_N)\psi_i(\vec{x}_N)$. If $\psi = \psi_1$ (say) then

$$a(\psi_1)|\psi_1, ...\psi_N >= |\psi_2, ...\psi_N > \tag{3.17}$$

that is the action of $a(\psi_1)$ is simply the one of erasing the particle ψ_1 from the n body state. Similarly

$$a^+(\psi)|\psi_1, ...\psi_N >= |\psi, \psi_1, ...\psi_N > \tag{3.18}$$

A particulary important class of ψ of the from $|\psi_1, ...\psi_N >$ is obtained choosing as single state particle the eigenfunctions of $-\hbar^2\frac{\partial^2_{\vec{x}}}{2m} + U(\vec{x})$; for instance if $U = 0$ then $\phi_{\vec{x}}(\mathbf{x}) = \frac{1}{L^{d/2}}e^{i\vec{k}\vec{x}}$ with $\hbar\vec{k}$ representing the *momentum*; due to periodic boundary conditions $\vec{k}$ has the form $\vec{k} = \frac{2\pi}{L}\vec{n}$, $\mathbf{n} = n_1, ..n_d$ with n_i integer and $-[L/2] \leq n_i \leq [(L-1)/2]$. We will denote by $|\vec{k}_1, n_1; , , \vec{k}_m, n_m >$ the element of $\mathcal{H}_{\varepsilon,\mathcal{N}}$, $N = n_1 + .. + n_m$ obtained by symmetrizing or antisymmetrizing $\psi_{\vec{k}_1}(\vec{x}_1)...\psi_{\vec{k}_1}(\vec{x}_{n_1})\psi_{\vec{k}_2}(\vec{x}_{n_1+1})...\psi_{\vec{k}_2}(\vec{x}_{n_1+n_2}...$; one usually says that $|\vec{k}_1, n_1; , , \vec{k}_m, n_m >$ describes a state with n_1 particles with quantum number $\vec{k}_1$, n_2 with $\vec{k}_2$ and so on. Note a crucial difference between fermions and bosons; in the case of fermions if $n = 0, 1$; indeed the antisymmetry

condition implies that $\psi_n(\vec{x}_1)\psi_n(\vec{x}_2)..=-\psi_n(\vec{x}_2)\psi_n(\vec{x}_2)..=0$. For bosons, instead, n_i can assume all possible values.

If we define $a_{\vec{k}}=a(f)$ with $f=\frac{1}{L^{d/2}}e^{i\vec{k}\vec{x}}$ then

$$a_{\vec{k}}|\vec{k},n:...>=|\vec{k},n-1;...> \qquad a^+_{\vec{k}}|\vec{k},n:...>=|\vec{k},n+1;...> \tag{3.19}$$

and

$$[a_{\vec{k}},a^+_{\vec{k}'}]_\varepsilon=\delta(\vec{k}-\vec{k}') \qquad [a^+_{\vec{k}},a^+_{\vec{k}'}]_\varepsilon=[a_{\vec{k}},a_{\vec{k}'}]_\varepsilon=0 \tag{3.20}$$

where $\delta(\vec{k}-\vec{k}')=|\Lambda|\delta_{\vec{k},\vec{k}'}$.

Calling $|0>$ the state such that $a_{\vec{k}}|0>=0$, then all states $|\vec{k}_1,n_1;,,\vec{k}_m,n_m>$ can be constructed by applying the operators $a^+_{\vec{k}}$.

Let us consider the the N particle hamiltonian

$$H^0_N=-\hbar^{2N}\sum_{i=1}^N\frac{\partial^2_{\vec{x}_i}}{2m} \tag{3.21}$$

Note that

$$-\hbar^{2N}\sum_{i=1}^N\frac{\partial^2_{\vec{x}_i}}{2m}|\vec{k}_1,n_1;,,\vec{k}_m,n_m>=\sum_{i=1}^m\frac{\hbar^2|\vec{k}|^2}{2m}n_i|\vec{k}_1,n_1;,,\vec{k}_m,n_m> \tag{3.22}$$

therefore the hamiltonian H^0 on the Fock space can be written in second quantization as

$$H^0=\frac{1}{L^d}\sum_{\vec{k}}\frac{\hbar^2|\vec{k}|^2}{2m}a^+_{\vec{k}}a_{\vec{k}} \tag{3.23}$$

as $|\vec{k}_1,n_1;,,\vec{k}_m,n_m\rangle = [a^+_{\vec{k}_1}]^{n_1}...|0\rangle$ and by the commutation rules $a^+_{\vec{k}_1}a_{\vec{k}_1}|\vec{k}_1,n_1;,,\vec{k}_m,n_m\rangle=n_1|\vec{k}_1,n_1;,,\vec{k}_m,n_m\rangle$.

In general if the Fock-space hamiltonian $H=+_{N\geq 0}H_N$ has the form

$$H_N=\sum_{i=1}^N h_i+\sum_{1\leq i,j\leq N}W_{ij} \tag{3.24}$$

where $h_i\equiv h(\vec{x}_i)$ is a single body operator on $\mathcal{H}_\infty$ and W a two body operator. If ϕ_i is a basis of $\mathcal{H}_\infty$, the Fock-space hamiltonian $H=+_{N\geq 0}H_N$ can be written as

$$H=\sum_{i,j}<\phi_i|h|\phi_j>a^+_ia_j+\frac{1}{2}\sum_{i,j,k,l}<\phi_i\otimes\phi_j|W|\phi_k\otimes\phi_l>a^+_ia^+_ja_ka_l \tag{3.25}$$

We introduce also the following operators

$$a_{\vec{x}} = \frac{1}{L^d} \sum_{\vec{k}} e^{i\vec{k}\vec{x}} a_{\vec{k}} \tag{3.26}$$

In terms of these operators the Hamiltonian on $\mathcal{F}$ is given by

$$H = \int_\Lambda d\vec{x} a^+(\vec{x}) [\frac{-\hbar^2\partial^2}{2m} + U(\vec{x})] a(\vec{x}) + \lambda \int d\vec{x} d\vec{y} v(\vec{x}-\vec{y}) a^+(\vec{x}) a^+(\vec{y}) a(\vec{y}) a(\vec{x}) \tag{3.27}$$

3.2 Gas of bosons

3.2.1 *Bose-Einstein condensation*

Let us consider the Hamiltonian operator for free bosons

$$H = \int_\Lambda d\vec{x} a^+(\vec{x}) [\frac{-\hbar^2\partial^2}{2m}] a(\vec{x}) \tag{3.28}$$

with $\vec{x}_i \in \Lambda$ where Λ is a square box $[0, L]^3$; periodic conditions are assumed. If we define new operators $a^\pm_{\vec{x}}$

$$a_{\vec{k}} = \frac{1}{\sqrt{L^3}} \sum_{\vec{k}} e^{i\vec{k}\vec{x}} a(x) \tag{3.29}$$

with $\vec{k} = \frac{2\pi}{L}\vec{n}$, $\vec{n} = (n_1, n_2, n_3)$ then

$$H = \frac{1}{L^3} \sum_{\vec{k}} \varepsilon(\vec{k}) a^+_{\vec{k}} a_{\vec{k}} \tag{3.30}$$

and $\varepsilon(\vec{k}) = \frac{\hbar^2|\vec{k}|^2}{2m}$. The spectrum of H is then of the form $\sum_{\vec{k}} \varepsilon(\vec{k}) n_{\vec{k}}$ with $n_{\vec{k}} = 0, 1, 2, ..,$; the partition function in the Grand canonical ensamble is then obtained by summing over all possible n_k and assuming $e_{\vec{k}} - \mu > 0$ for any $\vec{k}$ (hence $\mu < 0$)

$$Z = \sum_{\{n_{\vec{k}}\}} e^{-\sum_{\vec{k}} \beta n_{\vec{k}} (e_{\vec{k}} - \mu)} = \prod_{\vec{k}} [\sum_{n_{\vec{k}}=0}^{\infty} e^{-\beta n_{\vec{k}} (e_{\vec{k}} - \mu)}] = \prod_{\vec{k}} \frac{1}{1 - e^{-\beta(\varepsilon(\vec{k}) - \mu)}} \tag{3.31}$$

and the free energy is therefore

$$F = \beta^{-1} \sum_{\vec{k}} \log(1 - e^{-\beta(\varepsilon(\vec{k}) - \mu)}) \tag{3.32}$$

The averaged number of particles is

$$\langle n_{\vec{k}} \rangle = \frac{\sum_{n_{\vec{k}}=0}^{\infty} e^{-\beta(e_{\vec{k}}-\mu)} n_k}{\sum_{n_{\vec{k}}=0}^{\infty} e^{-\beta(e_{\vec{k}}-\mu)}} \tag{3.33}$$

and using the formula

$$\frac{\sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n} = x \frac{\partial}{\partial x} \log \sum_{n=0}^{\infty} x^n = \frac{x}{1-x} \tag{3.34}$$

we finally get

$$\langle n_{\vec{k}} \rangle = \frac{e^{-\beta(e_{\vec{k}}-\mu)}}{1 - e^{-\beta(\varepsilon(\vec{k})-\mu)}} = \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)} - 1} \tag{3.35}$$

which is known as Bose-Einstein distribution. The density in the infinite volume limit is given by

$$\rho = \lim_{L\to\infty} \frac{< N >}{L^3} = \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)} - 1} \tag{3.36}$$

which can be rewritten as

$$\rho = \lim_{L\to\infty} \frac{< N >}{L^3} = \lim_{L\to\infty} \frac{1}{L^3} \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{L^3} \sum_{\vec{k}\neq 0} \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)} - 1} \tag{3.37}$$

where the first term represent the averaged number of particles with $\vec{k} = 0$ and the second the contribution to the density coming from momenta with $\vec{k} \neq 0$. The chemical potential must be chosen so that the density has a fixed value. From the above expression we see that for any $\mu > 0$ and L-independent the first term vanishes in the limit $L \to \infty$ while the second converges to

$$\frac{1}{(2\pi)^3} \int d\vec{k} \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)} - 1} \tag{3.38}$$

which is a decreasing function of $\mu > 0$ with a finite maximum at $\mu = 0$ which defines a critical density

$$\rho_c(\beta) \equiv \frac{1}{(2\pi)^3} \int d\vec{k} \frac{1}{e^{\beta\varepsilon(\vec{k})} - 1} \tag{3.39}$$

which is finite in $d = 3$ as $e^{\beta\varepsilon(\vec{k})} - 1 \sim \beta p^2$ for small p. Therefore *if the limit $L \to \infty$ is taken at finite μ, so that $< a_0^+ a_0 >$ vanishes in the limit, the density is necessarily smaller than $\rho_c(\beta)$.*

If we want to have larger densities then μ must be taken L dependent; physically what happens is that for $\rho \leq \rho_c(\beta)$ fraction of density of particles

carrying zero momentum is macroscopic, that is non vanishing as $L \to \infty$. Indeed we have to choose μ vanishing as $L \to \infty$ so that

$$\mu = (-\beta L^3(\rho - \rho_c(\beta)))^{-1} \tag{3.40}$$

so that

$$\frac{n_0}{L^3} = \frac{1}{L^3}\frac{1}{e^{-\beta\mu-1}} \sim_{L\to\infty} \rho - \rho_c \tag{3.41}$$

Therefore for $\rho \geq \rho_c(\beta)$ the fraction of bosons with $\vec{k} = 0$ is macroscopically occupied (and no other state have this property; the smallest non-zero energy is $O(L^{-2})$ and $\frac{n_1}{L^3} \sim O(L^{-1})$) , while for $\rho \leq \rho_c(\beta)$ is not; this is a net example of phase transition. Indeed we can see that the free energy is non analytic, as the free energy in the $L \to \infty$ depends on ρ only through μ; for $\rho \leq \rho_c$ $\mu = 0$, so the free energy is constant in ρ, so cannot be analytic in ρ.

The macroscopic occupancy of the $\vec{k} = 0$ level is called *Bose-Einstein* condensation. The condition $\rho \geq \rho_c(\beta)$ for having condensation can be expressed as a condition on the critical temperature: using that

$$\rho_c(\beta) = 2\pi(2m\beta^{-1})^{\frac{4}{2}} \int_0^\infty d\varepsilon \frac{\sqrt{\varepsilon}}{e^\varepsilon - 1} \equiv I(2m\beta^{-1})^{\frac{3}{2}} \tag{3.42}$$

the condition $\rho \geq I(2m\beta^{-1})^{\frac{4}{2}}$ becomes

$$\beta > \beta_c(\rho) = C\rho^{-2/3} \tag{3.43}$$

It is believed that the λ-point transition obseved in liquid helium at $2.18K$ is essentially an Einstein condensation; remarkable physical properties described as superfluidity are exhibited by the low temperature phase.

All the above consideration have been done for an ideal gas, that is not considering the interaction between bosons; one would like to known if bose einstein condensation happens also for an hamiltonian like (3.27) with a short range interaction. The proof of BE condensation in such systems is still an open problem, but BC has been proved in presence of trapped potentials by Lieb, Seiringer (2002) in the low density and long ranged interaction limit.

3.2.2 *Black body radiation and specific heat of solids*

Let us consider a cubic region filled with electromagnetic radiation and in thermodynamical equilibrium. This system can be described as a gas of

photons, which are relativistic particles with zero mass, that is $\varepsilon(\vec{k}) = \hbar c|\vec{k}|$ so that the Hamiltonian is given by

$$H = \frac{1}{L^3}\sum_{\vec{k}} \hbar c|\vec{k}| a^+_{\vec{k}} a_{\vec{k}} \tag{3.44}$$

The total averaged energy, setting $\mu = 0$ is given by

$$\frac{< E >}{L^3} = \frac{1}{L^3}\sum_{\vec{k}} \hbar c|\vec{k}| n_{\vec{k}} = \frac{1}{L^3}\sum_{\vec{k}} \hbar c|\vec{k}| \frac{1}{e^{\beta\hbar hc|\vec{k}|} - 1} \tag{3.45}$$

which is given by, in the $L \to \infty$ limit

$$\lim_{L\to\infty} \frac{< E >}{L^3} = \int d\vec{k}\hbar c|\vec{k}| \frac{1}{e^{\beta\hbar c|\vec{k}|} - 1} = \beta^{-4}\frac{2\pi}{(\hbar c)^3}\int_0^\infty d\rho \frac{\rho^3}{e^\rho - 1} \tag{3.46}$$

The energy is proportional to T^4, what is the content of the Stefan-Boltzmann law.

The quantized vibration of a solid can be described in terms of phonons, with dispersion relation $c_s|\vec{k}|$ where c_s is the velocity of sound; a similar computation says that the energy is $\sim T^4$ so that the spefific heat is $O(T^3)$ for small T.

3.3 Gas of fermions

3.3.1 *The free Fermi gas*

Let us consider a gas of fermions with hamiltonian

$$H = \int_\Lambda d\vec{x} a^+(\vec{x})[\frac{-\hbar^2\partial^2}{2m}]a(\vec{x}) \tag{3.47}$$

with $\vec{x}_i \in \Lambda$ where Λ is a square box $[0, L]^3$; periodic conditions are assumed, or

$$H = \frac{1}{L^3}\sum_{\vec{k}} \varepsilon(\vec{k}) a^+_{\vec{k}} a_{\vec{k}} \tag{3.48}$$

and $\varepsilon(\vec{k}) = \frac{\hbar^2|\vec{k}|^2}{2m}$. The spectrum of H is then of the form $\sum_{\vec{k}} \varepsilon(\vec{k}) n_{\vec{k}}$ with $n_{\vec{k}} = 0, 1$; the partition function in the Grand canonical ensamble is then obtained by summing over all possible n_k

$$Z = \sum_{\{n_{\vec{k}}\}} e^{-\sum_{\vec{k}} \beta n_{\vec{k}}(e_{\vec{k}} - \mu)} = \prod_{\vec{k}}[\sum_{n_{\vec{k}}=0}^{1} e^{-\beta n_{\vec{k}}(e_{\vec{k}} - \mu)}] = \prod_{\vec{k}}(1 + e^{-\beta(\varepsilon(\vec{k}) - \mu)} \tag{3.49}$$

and the free energy is therefore

$$F = -\beta^{-1} \sum_{\vec{k}} \log(1 + e^{-\beta(\varepsilon(\vec{k})-\mu)}) \tag{3.50}$$

The averaged number of particles is

$$\langle n_{\vec{k}} \rangle = \frac{\sum_{n_{\vec{k}}=0}^{1} e^{-\beta(e_{\vec{k}}-\mu)} n_k}{\sum_{n_{\vec{k}}=0}^{1} e^{-\beta(e_{\vec{k}}-\mu)}} = \frac{1}{e^{-\beta(e_{\vec{k}}-\mu)}+1} \tag{3.51}$$

If $\phi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{L^{d/2}} e^{i\mathbf{k}\mathbf{x}}$ then $|\mathbf{k}_1, \sigma_1, ..\mathbf{k}_N \sigma_N\rangle$ are eigenfunctions of H with eigenvalue $\sum_{\mathbf{k},\sigma} \varepsilon(\mathbf{k}) n_{\mathbf{k},\sigma}$, where $\varepsilon(\mathbf{k}) = \frac{\hbar^2 |\mathbf{k}|^2}{2m}$ and $n_{\mathbf{k},\sigma} = 0, 1$, the *occupation number*, is the eigenvalue $a^+_{\mathbf{k},\sigma} a^-_{\mathbf{k},\sigma}$; $n_{\mathbf{k},\sigma} = 1$ if in the state there is a fermion with momentum $\mathbf{k}$ and spin σ and it is zero otherwise. The eigenfunction $|\Omega\rangle$ of H with lowest energy is called *ground state*, and it determines the low temperatures properties of the system. In order to find the ground state $|\Omega\rangle$, one has to minimize

$$\sum_{\mathbf{k},\sigma} \varepsilon(\mathbf{k}) n_{\mathbf{k},\sigma} \tag{3.52}$$

with the constraint that $n_{\mathbf{k},\sigma}$ can take only the values 0 or 1 and $\sum_{\mathbf{k},\sigma} n_{\mathbf{k},\sigma} = N$; if there are many solutions to this problem one says that the ground state is *degenerate*. An approximate solution is the following one: if $d = 3$ one can consider a state such that $n_{\mathbf{k},\sigma} = 1$ if $\mathbf{k}$ is in a *sphere* of radius k_F and zero otherwise; since the number of momenta $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ in the sphere is approximately given by

$$2\frac{4\pi k_F^2}{3}\frac{L^3}{8\pi^2} \tag{3.53}$$

, we can choose

$$k_F = (3\pi^2\rho)^{\frac{1}{3}} \tag{3.54}$$

, with $\rho = NL^{-3}$. The state $\prod_{|\mathbf{k}|\le k_F} a^+_{1/2,\mathbf{k}} a^+_{-1/2,\mathbf{k}} |0\rangle$ is not the true ground state when N, L are finite, but it is a very good approximation of it and converges to it (in a suitable sense) in the limit $N, L \to \infty$, ρ fixed. The boundary of the sphere with radius k_F in the space of momenta is called *Fermi surface* and it is a key notion in the theory of Fermi systems; if $d = 2$ it is replaced by a circle and in $d = 1$ by two points.

Coming to the thermodynamical properties, the partition function is given by

$$Z = \prod_{\mathbf{k}} \sum_{n_{\mathbf{k}}=0,1} e^{-\beta(\varepsilon(\mathbf{k})-\mu)n_{\mathbf{k}}} = \prod_{\mathbf{k}} (1 + e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)}) \tag{3.55}$$

and the specific heat by $C_v = -\frac{\partial}{\partial T}\frac{\partial}{\partial \beta}\log Z$; one finds, by expressing μ in terms of β through the relation

$$N = -\frac{\partial f}{\partial \mu} \tag{3.56}$$

, that if $d = 3$ in the $L \to \infty$ limit

$$C_v = \frac{\pi^2}{2}\rho\kappa(\frac{\kappa T}{\varepsilon_F}) + O((\frac{\kappa T}{\varepsilon_F})^2) \tag{3.57}$$

, where $\varepsilon_F = \hbar^2 k_F^2/2m$. This linear T dependence follows very easily from dimensional considerations.

Early models for metals described the electrons as classical particles; however in such a case a well known result of classical statistical mechanics says that they should contribute to the specific heat by $\frac{3}{2}\rho\kappa$, while experimentally their contribution is much smaller. The solution of this puzzle was provided by the above formula for C_v; the classical value is in fact depressed by a factor $\frac{\pi^2}{3}\frac{\kappa T}{\varepsilon_F}$ which at room temperatures is $O(10^{-2})$ in agreement with experimental data. The averaged number of electrons with momentum $\hbar k$ is given, in the infinite volume limit, by $< a^+_{\mathbf{k},\sigma} a^-_{\mathbf{k},\sigma} >= (1 + e^{\beta(\varepsilon(\mathbf{k})-\mu)})^{-1}$; at zero temperature it reduces to $\vartheta(|\mathbf{k}| \leq k_F)$, *i.e.* it has a discontinuity at the Fermi surface, while at high temperature it is very close to the Maxwell distribution $\simeq e^{-\beta(\varepsilon(\mathbf{k})-\mu))}$.

The above is is a model for a gas of fermions in the continuum. It can be convenient to consider lattice fermionic models on a lattice.

3.3.2 *Schwinger functions for the free Fermi gas*

3.4 Solvable models and exact mapping

3.4.1 *The XY model*

The XY model is a generalization of the Ising model to quantum spins. Remarkably t is closely related to the Ising model in two dimensions, as we will see below. The spectrum was computed by Bethe (1931) and Jordan and Wigner (1928), but here we will follow the analysis of Lieb, Schultz and Mattis, Ann. of. Phys. 16, 407 (1961).

If $(S^1_x, S^2_x, S^3_x) = \frac{1}{2}(\sigma^1_x, \sigma^2_x, \sigma^3_x)$, for $i = 1, 2, ..., L$, σ^α_i, $\alpha = 1, 2, 3$, and consider the tensor product $\mathcal{H}$ of N bidimensional linear spaces E and

$$\sigma^j_x = I \otimes I ... \sigma^j \otimes I ... \tag{3.58}$$

where σ^j being the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.59}$$

so that

$$[S_x^i, S_y^j] = i2\varepsilon^{i,j,k}\delta_{x,y}S_x^k \tag{3.60}$$

The Hamiltonian of the *Heisenberg-Ising XY chain* is given by

$$H = -\sum_{x=1}^{L-1}[J_1 S_x^1 S_{x+1}^1 + J_2 S_x^2 S_{x+1}^2 - hS_x^3] + U_L \tag{3.61}$$

where U_L takes care of boundary conditions (to be specified later). We find convenient to fix unity so that $J_1 + J_2 = 2$ and we introduce the *anisotropy* $\gamma = (J_1 - J_2)/(J_1 + J_2)$

$$H = -\sum_{x=1}^{L-1}\{[S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2] + \gamma[S_x^1 S_{x+1}^1 - S_x^2 S_{x+1}^2] - hS_x^3\} + U_L \tag{3.62}$$

with $-1 \leq \gamma \leq 1$. When $\gamma = 0$ the model is known as XX model and when $\gamma = 1$ the model reduces to the one dimensional Ising model.

3.4.2 *The Jordan-Wigner transformation*

We introduce the operators $S^{\pm} = (S^1 \pm iS^2)$ which for $x \neq y$ verify

$$[S_x^+, S_y^+]_- = [S_x^-, S_y^-]_- = [S_x^+, S_y^-]_- = 0 \tag{3.63}$$

while for $x = y$

$$[S_x^+, S_x^-]_+ = 1 \quad [S_x^+, S_+^-]_- = [S_x^-, S_y^-]_- = 0 \tag{3.64}$$

The *Jordan-Wigner transformation* allows to represent $S_x^{\pm}, S_x^3$ in terms of fermionic operators

$$S_x^- = e^{i\pi\sum_{k=1}^{x-1} a_k^+ a_k} a_x \qquad S_x^+ = a_x^+ e^{-i\pi\sum_{k=1}^{x-1} a_k^+ a_k}$$

$$S_x^3 = -a_x^+ a_x + 1/2 \tag{3.65}$$

where a_x, a_x^+ verifies the fermionic anticommutation relations

$$[a_x^+, a_y]_+ = \delta_{x,y} \qquad [a_x^+, a_y^+]_+ = [a_x, a_y]_+ = 0 \tag{3.66}$$

The above transformation can be easily inverted noting that $S_x^+ S_x^- = a_x^+ a_x$ we get

$$a_x = e^{-i\pi\sum_{k=1}^{x-1} S_k^+ S_k^-} S_x^- \qquad a_x^+ = S_x^+ e^{i\pi\sum_{k=1}^{x-1} S_k^+ S_k^-}$$

The proof of such relations consists in showing that a_x^+, a_x verify the anticommutation relations. First note that $S_x^+ S_x^-$ commutes with $S_y^+ S_y^-$ for $x \neq y$ so that

$$e^{-i\pi \sum_{k=1}^{x-1} S_k^+ S_k^-} = \prod_{k=1}^{x-1} e^{-i\pi S_k^+ S_k^-} \tag{3.67}$$

Moreover

$$S^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad S^+ S^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.68}$$

so that $(S_k^+ S_k^-)^n = (S_k^+ S_k^-)$; therefore

$$e^{-i\pi S_k^+ S_k^-} = \sum_{n=0}^{\infty} (-i\pi)/n! (S_k^+ S_k^-)^n = 1 + (e^{-i\pi} - 1) S_k^+ S_k^- = 1 - 2 S_k^+ S_k^- \tag{3.69}$$

Note that

$$[a_x^+, a_x]_+ = [S_x^+, S_x^-]_+ = 1 \tag{3.70}$$

where we have used that $e^{-i\pi \sum_{k=1}^{x-1} S_k^+ S_k^-} S_x^- = S_x^- e^{-i\pi \sum_{k=1}^{x-1} S_k^+ S_k^-}$. Moreover if $y > x$

$$[a_y^+, a_x]_+ = S_y^+ e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} S_x^- + S_x^- e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} S_y^+ \tag{3.71}$$

Note that

$$S_x^- e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} S_y^+ = S_x^- S_y^+ e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} \tag{3.72}$$

and

$$S_y^+ e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} S_x^- = S_y^+ e^{i\pi S_x^+ S_x^-} S_x^- e^{i\pi \sum_{k=x+1}^{y-1} S_k^+ S_k^-} \tag{3.73}$$

On the other hand, using that $[S_x^+, S_x^-]_- = 1$

$$\begin{aligned} & e^{i\pi S_x^+ S_x^-} S_x^- = (1 - 2 S_x^+ S_x^-) S_x^- \\ & = S_x^- - 2(1 - S_x^- S_x^+) S_x^- = -S_x^- (1 - 2 S_x^+ S_x^-) = -S_x^- e^{i\pi S_x^+ S_x^-} \end{aligned} \tag{3.74}$$

so that finally

$$[a_y^+, a_x]_+ = (S_y^+ S_x^- + S_x^- S_y^+) e^{i\pi \sum_{k=x}^{y-1} S_k^+ S_k^-} = 0 \tag{3.75}$$

In addition from the relation

$$S^+ S^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1/2 - S^3 \tag{3.76}$$

follows $S_x^3 = -a_x^+ a_x + 1/2$.

3.4.3 *Diagonalization of the XX hamiltonian*

Let us consider first the case $\gamma = 0$. We can express now the Hamiltonian in terms of fermionic operators. We start writing

$$H_0 = -\sum_{x=1}^{L}\{[S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2] + U_L = 2\sum_{x=1}^{L-1}\{[S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+] + U_L \tag{3.77}$$

Moreover as $a_x^+ a_x^+ = 0$

$$S_x^+ S_{x+1}^- = a_x^+ e^{i\pi a_x^+ a_x} a_{x+1} = a_x^+(1 - 2a_x^+ a_x)a_{x+1} = a_x^+ a_{x+1} \tag{3.78}$$

and

$$\begin{aligned} S_x^- S_{x+1}^+ &= a_x e^{-i\pi a_x^+ a_x} a_{x+1}^+ = a_x(1 - 2a_x^+ a_x)a_{x+1}^+ = \\ & a_x a_{x+1} - 2a_x(1 - a^x a_x^+)a_{x+1}^+ = -a_x a_{x+1}^+ = a_{x+1}^+ a_x \end{aligned} \tag{3.79}$$

so that

$$H_0 = \sum_{x=1}^{L}[a_x^+ a_{x+1} + a_{x+1}^+ a_x] + U_L \tag{3.80}$$

We find convenient to impose periodic boundary conditions on the fermions (called c-cycic conditions); therefore $a_{L+1} = a_1$ so that

$$H_0 = -\sum_{x=1}^{L}[a_x^+ a_{x+1} + a_{x+1}^+ a_x] \tag{3.81}$$

provided that

$$U_L = -(a_L^+ a_1 + a_L^+ a_1) = -(S_L^+ e^{i\pi \sum_{k=x}^{L} S_k^+ S_k^-} S_1 + S_L^- e^{i\pi \sum_{k=x}^{L} S_k^+ S_k^-} S_1^+] \tag{3.82}$$

Strictly speaking, with this choice U_L does not look really like a boundary term, because $\mathcal{N}$ depends on all the spins of the chain. However $[(-1)^{\mathcal{N}}, H] = 0$; hence the Hilbert space splits up in two subspaces on which $(-1)^{\mathcal{N}}$ is equal to 1 or to -1 and on each of these subspaces U_L^1 really depends only on the boundary spins.

It is now easy to diagonalize the Hamiltonian; we write

$$a_x = \sum_{k=2\pi n/L} e^{ikx} a_k \tag{3.83}$$

with $-L/2 \leq n \leq n/2$ so that

$$H_0 = -2\sum_k \cos k a_k^+ a_k \tag{3.84}$$

Therefore the eigenvalues are $-2cosk$ and the ground state energy is $\prod_{|k|\leq\pi/2} a_k^+|0>$ and the corresponding ground state energy is

$$E_0/L = -\frac{1}{L}2\sum_{|k|\leq\pi/2}\cos k \to_{L\to\infty} -\frac{1}{\pi}\int_{|k|\leq\pi/2}\cos k dk = -\frac{2}{\pi} \tag{3.85}$$

The total value of the magnetization in the ground state is

$$< S^3 >= \int_{|k|\leq\pi/2} dk - \frac{1}{2}\int_{|k|\leq\pi/2} dk = 0 \tag{3.86}$$

Finally we can compute the fermionic 2-point function in the $L\to\infty$ limit is

$$< a_{\mathbf{x}}^+ a_{\mathbf{y}} >= \int_{-\infty}^{\infty} dk_0 \int -\pi^{\pi} dk \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + 2\cos k} \tag{3.87}$$

which for large $|\mathbf{x}-\mathbf{y}|$ behaves as

$$< a_{\mathbf{x}}^+ a_{\mathbf{y}} >\sim \frac{(-1)^{x-y}}{|\mathbf{x}-\mathbf{y}|} \tag{3.88}$$

By using the Wick rule we get, for large $\mathbf{x}-\mathbf{y}$

$$< S_{\mathbf{x}}^3 S_0^3 >= \frac{(-1)^{x-y}}{(2\pi^2)}\frac{1}{x_0^2+x^2} + \frac{(-1)^{x-y}}{(2\pi^2)}\frac{x_0^2-x^2}{(x_0^2+x^2)^2} \tag{3.89}$$

Therefore $< S_{\mathbf{x}}^3 S_0^3 >$ decays as a power law with exponent 2. The computation of $< S_{\mathbf{x}}^+ S_0^- >$ is much harder; again one finds a power law decay with exponent $\frac{1}{4}$. While the exact computation is quite hard, a bound can be obtained quite easily using Hadamard bounds.

3.4.4 *Diagonalization of the XY hamiltonian*

Let us consider now the case $\gamma \neq 0$; the Hamiltonian in terms of fermionic operators is

$$H_0 = -\sum_{x=1}^{L}[a_x^+ a_{x+1} + a_{x+1}^+ a_x] + \gamma\sum_{x=1}^{L}[a_x^+ a_{x+1}^+ + a_{x+1}a_x] \tag{3.90}$$

and equivalently

$$H_0 = \sum_k [-\cos k a_k^+ a_k - \cos k a_{-k}^+ a_{-k} + \gamma i \sin k [a_k^+ a_{-k}^+ - a_k a_{-k}] \tag{3.91}$$

We perform then the transformation

$$\begin{aligned} a_k^+ &= b_k^+ \cos\vartheta_k + b_{-k}\sin\vartheta_k \\ a_{-k} &= -b_k^+ \sin\vartheta_k + b_{-k}\cos\vartheta_k \end{aligned} \tag{3.92}$$

and we assume ϑ_k odd so that so that

$$\begin{aligned} H_0 = \sum_k [& -\cos k(b_k^+ \cos\vartheta_k + b_{-k} \sin\vartheta_k)(b_k \cos\vartheta_k + b_{-k}^+ \sin\vartheta_k) + \\ & -\cos k(b_{-k}^+ \cos\vartheta_k - b_k \sin\vartheta_k)(b_{-k} \cos\vartheta_k - b_k^+ \sin\vartheta_k) \\ & +\gamma i \sin k(b_k^+ \cos\vartheta_k + b_{-k} \sin\vartheta_k)(b_{-k}^+ \cos\vartheta_k - b_k \sin\vartheta_k) + \\ & -\gamma i \sin k(-b_k^+ \sin\vartheta_k + b_{-k} \cos\vartheta_k)(b_{-k}^+ \sin\vartheta_k + b_k \cos\vartheta_k) \end{aligned} \tag{3.93}$$

By choosing ϑ_k so that

$$\sin 2\vartheta_k = i\gamma \tan k \tag{3.94}$$

the terms b^+b^+ and bb disappear, while the terms b^+b are multiplied by

$$[\cos^2\vartheta_k - \sin^2\vartheta_k]\cos k = \cos 2\vartheta_k \sin k = \sqrt{1+\gamma^2\tan^2 k}\cos k \tag{3.95}$$

so that

$$H_0 = \sum_k \sqrt{\cos^2 k + \gamma^2 \sin^2 k} b_k^+ b_k \tag{3.96}$$

The ground state is again given by $\prod_{|k|\leq\pi/2} a_k^+|0>$ and the corresponding ground state energy is

$$E_0/L = -\frac{1}{L} 2 \sum_{|k|\leq\pi/2} \sqrt{\cos^2 k + \gamma^2 \sin^2 k} \tag{3.97}$$

The crucial difference with respect to the previous case is that the energy of the first excited state is greate than γ with respect to the ground state energy, while in the previous case was $O(1/L)$; there is a gap in the spectrum, and the correlations decays exponentially instead of as power law.

3.5 The LMS solution of the 2D Ising model

3.5.1 *The transfer matrix*

The second quantization formaism is rather useful for providing a simple solution of the (classical) Ising model in two dimensions, as discovered by Schultz, Mattis and Lieb Rev. Mod. Phys. 36, 856 (1964).

The partition function of the 2D Ising model is

$$Z = \sum_\sigma \prod_{x=1}^{L} \prod_{y=1}^{L} e^{\beta J(\sigma_{x,y}\sigma_{x+1,y} + \sigma_{x,y}\sigma_{x,y+1})} \tag{3.98}$$

Note that by using periodic boundary conditions

$$\sum_{x,y}\sigma_{x,y}\sigma_{x,y+1}=\frac{1}{2}\sum_{x,y}\sigma_{x,y}\sigma_{x,y+1}+\frac{1}{2}\sum_{x,y}\sigma_{x+1,y}\sigma_{x+1,y+1} \tag{3.99}$$

so that

$$Z=\sum_{\sigma}\prod_{x=1}^{L}\prod_{y=1}^{L}e^{\beta J(\frac{1}{2}\sigma_{x,y}\sigma_{x,y+1}+\sigma_{x,y}\sigma_{x+1,y}+\frac{1}{2}\sigma_{x+1,y}\sigma_{x+1,y+1})} \tag{3.100}$$

We proceed now as in the 1D case introducing a $2L\times 2L$ matrix V with elements $V_{\underline{\sigma},\underline{\sigma}'}$; it is clear that, if $\underline{\sigma}_1=(\sigma_{1,1},\sigma_{1,2},...,\sigma_{1,L})$

$$Z=\sum_{\sigma}V_{\underline{\sigma}_1,\underline{\sigma}_2}V_{\underline{\sigma}_2,\underline{\sigma}_3}...=\sum_{\sigma}TrV^L \tag{3.101}$$

with

$$V_{\underline{\sigma},\underline{\sigma}'}=e^{\sum_y\beta J(\frac{1}{2}\sigma_y\sigma_{y+1}+\sigma_y\sigma'_y+\frac{1}{2}\sigma'_y\sigma'_{y+1})} \tag{3.102}$$

We can write

$$V=V_2^{\frac{1}{2}}V_1V_2^{\frac{1}{2}} \tag{3.103}$$

where

$$V_1=w\times w\times .... \tag{3.104}$$

with

$$w=\begin{pmatrix}e^{\beta J} & e^{-\beta J}\\ e^{-\beta J} & e^{\beta J}\end{pmatrix}=e^{\beta J}I+e^{-\beta J}\sigma^1 \tag{3.105}$$

We set $\beta J=J$ and

$$\tanh J^*=e^{-2J} \tag{3.106}$$

so that

$$\tanh J=e^{-2J^*}\qquad \sinh 2J\sinh 2J^*=1 \tag{3.107}$$

By using the above definitions we get

$$e^J(I+e^{-2\beta J}\sigma^1)=e^J(I+\tanh J^*\sigma^1)=$$
$$\frac{e^J}{\cosh J^*}(\cosh J^*+\sigma^1\sinh J^*)=\frac{e^J}{\cosh J^*}e^{J^*\sigma^1} \tag{3.108}$$

and using that $(\frac{\cosh J^*}{\sinh J^*})^{\frac{1}{2}}=e^J$ we get

$$V_1=(2\sinh 2J)^{\frac{1}{2}}e^{J^*\sum_x S_x^1} \tag{3.109}$$

and

$$V_2=e^{\sum_y\beta JS_y^3S_{y+1}^3} \tag{3.110}$$

Finally we perform an unitary transformation $\sigma^1\to\sigma^3$, $\sigma^3\to-\sigma_1$ so that

$$Z=\mathrm{Tr}(V_2^{\frac{1}{2}}V_1V_2^{\frac{1}{2}})^L \tag{3.111}$$

with

$$V_1=(2\sinh 2J)^{\frac{1}{2}}e^{J^*\sum_x S_x^3}=(2\sinh 2J)^{\frac{1}{2}}e^{J^*\sum_x(S_x^+S_x^--1/2)}= \tag{3.112}$$

$$V_2=e^{\sum_y\beta JS_y^1S_{y+1}^1} \tag{3.113}$$

The trace has the form $\lambda_1^L+\lambda_2^L+...$ so it is clear that in order to get the free energy per volume in the thermodynamic limit it is sufficient to the the largest eigenvalue.

3.5.2 *Introduction of fermion operators*

By using the Jordan-Wigner transformation we can rewrite the transfer matrix in terms of fermionic operatos a_x, a_x^+

$$V_1 = (2\sinh 2J)^{\frac{1}{2}} e^{J^* \sum_x \sigma_x^3} = (2\sinh 2J)^{\frac{1}{2}} e^{J^* \sum_x (a_x^+ a_x^- - 1/2)} = \tag{3.114}$$

$$V_2 = e^{\sum_x J(a_x^+ + a_x)(a_{x+1}^+ + a_{x+1})} \tag{3.115}$$

and in Fourier space

$$V_1 = (2\sinh 2J)^{\frac{1}{2}} e^{J^* \sum_{\pi \geq k \geq 0} 2(a_k^+ a_k^- + a_{-k}^+ a_{-k}^- - 1)} \tag{3.116}$$

$$V_2 = e^{J \sum_{\pi \geq k \geq 0} \cos k (a_k^+ a_k + a_{-k}^+ a_{-k}) + \sin k (a_k^+ a_{-k}^+ + a_{-k} a_k)} \tag{3.117}$$

We can write

$$V = \prod_{\pi \geq k \geq 0} V_k \tag{3.118}$$

as a consequence of the fact that the V_k with different k have vanishing commutator, as they are quadratic in anticommuting operators. Now we introduce the operators

$$\tau^+ = a_k^+ a_{-k}^+ \quad \tau^- = a_{-k} a_k \quad \tau^3 = a_k^+ a_k + a_{-k}^+ a_{-k} - 1 \tag{3.119}$$

and

$$\tau^1 = \tau^+ + \tau^- \qquad \tau^2 = \tau^+ - \tau^- \tag{3.120}$$

and we can check that they verify the rule $[\tau^i, \tau^j] = i2\varepsilon^{i,j,k}\tau^k$; for instance

$[\tau^+, \tau^-] = [a_k^+ a_{-k}^+, a_{-k} a_k] = a_k^+ a_{-k}^+ a_{-k} a_k - a_{-k} a_k a_k^+ a_{-k}^+ =$
$a_k^+ a_k a_{-k}^+ a_{-k} - a_k a_k^+ a_{-k} a_{-k}^+ = a_k^+ a_k a_{-k}^+ a_{-k} - a_k a_k^+ (1 - a_{-k}^+ a_{-k}) =$
$a_{-k}^+ a_{-k} - a_k a_k^+ = -1 + a_{-k}^+ a_{-k} + a_k^+ a_k$

Therefore

$$V_{1,k} = (2\sinh 2J)^{\frac{1}{2}} e^{2J^* \tau^3} = \begin{pmatrix} e^{2J^*} & 0 \\ 0 & e^{-2J^*} \end{pmatrix} \tag{3.121}$$

Moreover noting that

$$\tau' \equiv \tau_3 \cos k + \tau_1 \sin k = \begin{pmatrix} \cos k & \sin k \\ \sin k & -\cos k \end{pmatrix} \tag{3.122}$$

has eigenvalues ± 1 we can write

$$V_{2,k}^{\frac{1}{2}} = e^{J \cos k (\tau^3 + 1) + \sin k \tau_1} = e^{J \cos k} e^{J\tau'} = e^{J \cos k} [\cosh J + \tau' \sinh J] \tag{3.123}$$

where in the last step we have used that τ' has eigenvalues $\pm$ so that

$$V_{2,k}^{\frac{1}{2}} = e^{J\cos k}\begin{pmatrix}\cosh J + \cos k \sinh J & \sin k \sinh J \\ \sin k \sinh J & \cosh J - \cos k \sinh J\end{pmatrix} \tag{3.124}$$

We can finally write

$$e^{J\cos k} V_{2,k}^{\frac{1}{2}} V_{1,k} V_{2,k}^{\frac{1}{2}} \tag{3.125}$$

whose eigenvalues are $e^{J\cos k} e^{\pm\varepsilon_k}$ with

$$\cosh\varepsilon_k = \cosh 2J \cosh 2J^* - \sinh 2J \sinh 2J^* \cos k \tag{3.126}$$

Therefore the partition free energy in the thermodynamic limit is simply given by the largest eigenvalue, that is

$$-\beta f = \lim_{L\to\infty}\frac{1}{L}\log Z = \lim_{L\to\infty}\frac{1}{L}\log 2\sinh 2J)^L e^{\sum_k \cos k} e^{\sum_k \varepsilon_k} \tag{3.127}$$

and noting that $\sum_k \cos k = 0$

$$-\beta f = \lim_{L\to\infty}\frac{1}{L}\log Z =$$
$$\log(2\sinh 2J) + \frac{1}{2L}\sum_k \varepsilon(k)$$

By an explicit computation we can get the largest eigenvalues from which he free energy is obtained.

3.6 The Luttinger model exact solution

The Luttinger model hamiltonian is

$$H = H_0 + V = \int_0^L dx \sum_{\varepsilon=\pm} : \widetilde{\psi}^+_{x,\varepsilon}\varepsilon\partial\widetilde{\psi}^-_{x,\varepsilon} : + \tag{3.128}$$
$$\lambda\int dxdyv(x-y)[\sum_{\varepsilon=\pm} q_{1,\varepsilon} : \widetilde{\psi}^+_{\vec{x},\varepsilon}\widetilde{\psi}^-_{\vec{x},\varepsilon} :][\sum_{\varepsilon=\pm} q_{2,\varepsilon} : \widetilde{\psi}^+_{\vec{y},\varepsilon}\widetilde{\psi}^-_{\vec{y},\varepsilon} :]$$

where if $\varepsilon = \pm$

$$\psi^\pm_{x,\varepsilon} = e^{\pm i\varepsilon p_F x}\widetilde{\psi}^\pm_{x,\varepsilon} = \frac{e^{\pm i\varepsilon p_F x}}{\sqrt{L}}\sum_k e^{\pm ikx}\widehat{a}^\pm_{k,\varepsilon} \tag{3.129}$$

and $\widetilde{\psi}^\pm_{x,\varepsilon}$ is a fermionic field with periodic boundary conditions, p_F is the Fermi momentum, $v(x-y)$ is a short range potential $|\widehat{v}(p)| \le Ce^{-\zeta|p|}$ with $\zeta > 0$ a constant, and the Wick ordering is defined rearranging the order so that $a^+_{-k,+}, a^-_{k,+}, a^+_{k,-}, a^-_{-k,-}$, $k \ge 0$ are always to the right of the other

operators, and the new product is multiplied by the parity sign necessary to produce it. Finally $q_{i,\varepsilon}$ are charges, and for definiteness we will follow the original choice of [**?**]

$$q_{1,+} = q_{2,-} = 1 \tag{3.130}$$

and zero otherwise; this choice is done only for definiteness and the model is solvable for a wide class of charges $q_{i,\varepsilon}$. With this choice of the charges the Hamiltonian can be rewritten as

$$H_0 = \sum_{\varepsilon,k>0} k(\widehat{a}^+_{\varepsilon k,\varepsilon}\widehat{a}^-_{\varepsilon k,\varepsilon} + \widehat{a}^-_{-\varepsilon k,\varepsilon}\widehat{a}^+_{-\varepsilon k,\varepsilon}) \tag{3.131}$$

$$V = \frac{\lambda}{L}\sum_{p>0}\widehat{v}(p)[\rho_+(p)\rho_-(-p) + \rho_+(-p)\rho_-(p)] + \frac{\lambda\widehat{v}(0)}{L}N_1N_{-1}$$

where

$$\rho_\varepsilon(p) = \sum_k \widehat{a}^+_{k+p,\varepsilon}\widehat{a}_{k,\varepsilon} \quad N_\varepsilon = \sum_{k>0}(\widehat{a}^+_{\varepsilon k,\varepsilon}\widehat{a}^-_{\varepsilon k,\varepsilon} - \widehat{a}^-_{-\varepsilon k,\varepsilon}\widehat{a}^+_{-\varepsilon k,\varepsilon}) \tag{3.132}$$

The regularization which is implicit in the above Hamiltonian is the suppression of the modes $|k| \geq \Lambda$ for the fermion with momentum k, where Λ is a momentum cut-off to be removed $\Lambda \to \infty$; that is $\rho_\varepsilon(p)$ has be thought as $\sum_k \chi_\Lambda(k+p)\chi_\Lambda(k)\widehat{a}^+_{k+p,\varepsilon}\widehat{a}^-_{k,\varepsilon}$ where $\chi_\Lambda(k) = 1$ for $|k| \leq \Lambda$ and zero otherwise.

The Hamiltonian (3.129) can be regarded as an operator defined on the Hilbert space $\mathcal{H}$ constructed as follows. If $|0>$ is a state (the "Dirac sea" state) such that $\widehat{a}^+_{k,-}|0>=0$ and $\widehat{a}^+_{-k,+}|0>=0$ for $k \geq 0$, $\mathcal{H}_0$ is the linear span of all the states obtained applying finitely many creation or annihilation operators on $|0>$ and $\mathcal{H}$ is the completion of $\mathcal{H}_0$. Note that H is a bounded operators over $\mathcal{H}$ provided that $\widehat{v}(p)$ verifies the stability condition $|\lambda\widehat{v}(p)| \leq 2\pi$. The crucial observation of Mattis and Lieb was that the *density operators* verify the following *anomalous commutation relations*

$$[\rho_\varepsilon(-p), \rho_\varepsilon(p')] = \varepsilon\frac{pL}{2\pi}\delta_{p,p'} \quad p > 0 \tag{3.133}$$

This follows noting that the commutator when $p = p'$ is equal to

$$-\sum_{k=-\Lambda+p}^{\Lambda}\widehat{a}^+_{k,\varepsilon}\widehat{a}^-_{k,\varepsilon} + \sum_{k=-\Lambda}^{\Lambda-p}\widehat{a}^+_{k,\varepsilon}\widehat{a}^-_{k,\varepsilon} = \sum_{k=-\Lambda}^{-\Lambda+p}\widehat{a}^+_{k,\varepsilon}\widehat{a}^-_{k,\varepsilon} - \sum_{k=\Lambda-p}^{\Lambda}\widehat{a}^+_{k,\varepsilon}\widehat{a}^-_{k,\varepsilon} \tag{3.134}$$

which on any state in $\mathcal{H}$ is, in the limit $\Lambda \to \infty$, equal to $\varepsilon\frac{pL}{2\pi}$. Other important relations are, for $p > 0$

$$[H_0, \rho_\varepsilon(p)] = -\varepsilon p\rho_\varepsilon(p) \quad [\rho_\varepsilon(p), \sum_{\varepsilon,p>0}\rho_\varepsilon(\varepsilon p)\rho_\varepsilon(-\varepsilon p)] = -\varepsilon p\frac{L}{2\pi}\rho_\varepsilon(p) \tag{3.135}$$

and $\rho_\varepsilon(-\varepsilon p)|0>=0$. We can write

$$H = H_1 + H_2 \tag{3.136}$$

where, if $T = \frac{2\pi}{L}\sum_{\varepsilon,p>0}\rho_\varepsilon(\varepsilon p)\rho_\varepsilon(-\varepsilon p)$

$$H_1 = H_0 - T + \frac{\lambda\widehat{v}(0)}{L}N_+N_- \tag{3.137}$$
$$H_2 = T + \frac{\lambda}{L}\sum_{p>0}\widehat{v}(p)[\rho_+(p)\rho_+(-p) + \rho_-(-p)\rho_-(p)]$$

and $[H_1, \rho_\varepsilon(\pm p)] = 0$ for $p > 0$, while H_2 can be easily diagonalized; if $S = \frac{2\pi}{L}\sum_{p\neq 0} p^{-1}\rho_+(p)\rho_-(-p)$

$$e^{iS}H_2e^{-iS} = \frac{2\pi}{L}\sum_{p>0}\varepsilon(p)[\rho_+(p)\rho_+(-p) + \rho_-(p)\rho_-(-p)] + E_0 \tag{3.138}$$

where E_0 is a constant and

$$\varepsilon(p) = \mathrm{sech}(2\phi(p)) \qquad \tanh 2\phi = -\frac{\lambda\widehat{v}(p)}{2\pi} \tag{3.139}$$

The above formula (3.139) refers to the specific choice (3.130) of the $q_{i,\varepsilon}$, but a general formula can be easily obtained. The set of states $|j, n_1, n_2>$ obtained applying operators $\rho_+(p), \rho_-(-p)$ an arbitrary number of times on the state in which all the levels are filled up to n_1 with fermions of type + and down to level n_2 is *complete* [**?**; **?**] and H_1 is a constant in the subspace with fixed n_1, n_2, from the fact that $[H_1, \rho_\varepsilon(\pm p)] = 0$. Therefore $e^{-iS}|0>$ is the ground state of H as H_1 is constant on the subspace with $n_1 = n_2 = 0$ while

$$H_2e^{-iS}|0>= e^{-iS}(e^{iS}H_2e^{-iS})0>= E_0e^{-iS}|0> \tag{3.140}$$

Defining $\widetilde{\psi}_{\mathbf{x},\varepsilon} = e^{Ht}\widetilde{\psi}_{x,\varepsilon}e^{-Ht}$, $\mathbf{x} = (t, x)$ the n-point function is given by

$$<\widetilde{\psi}^{\sigma_1}_{\mathbf{x}_1,\varepsilon_1}...\widetilde{\psi}^{\sigma_n}_{\mathbf{x}_n,\varepsilon_n}>\equiv<0|e^{iS}\mathcal{T}(\widetilde{\psi}^{\sigma_1}_{\mathbf{x}_1,\varepsilon_1}...\widetilde{\psi}^{\sigma_n}_{\mathbf{x}_n,\varepsilon_n})e^{-iS}|0> \tag{3.141}$$

where $\mathcal{T}$ is the time ordering. In [**?**] only the case $n = 2, t_1 = t_2 = 0$ was studied, but it is only a matter of algebra to deduce from the exact solution the explicit form of the time-dependent n-point correlation; this was done in [**?**] for the 2-point and in [**?**], [**?**] for the n-point function. It is found

$$<\widetilde{\psi}^{\sigma_1}_{\mathbf{x}_1,\varepsilon_1}...\widetilde{\psi}^{\sigma_n}_{\mathbf{x}_n,\varepsilon_n}>=<\widetilde{\psi}^{\sigma_1}_{\mathbf{x}_1,\varepsilon_1}...\widetilde{\psi}^{\sigma_n}_{\mathbf{x}_n,\varepsilon_n}>_0\, e^{-Q_n} \tag{3.142}$$

where $<\widetilde{\psi}^{\sigma_1}_{\mathbf{x}_1,\varepsilon_1}...\widetilde{\psi}^{\sigma_n}_{\mathbf{x}_n,\varepsilon_n}>_0$ is the non-interacting $\lambda = 0$ (imaginary time) n-point function (expressed by the Wick rule in terms of the 2-point free

function) and

$$Q_n = \frac{2\pi}{L}\sum_{p>0}\sum_{\varepsilon=\pm}\{s^2(p)[\frac{n}{2} + \sum_{i,j\in I_\varepsilon, i<j}\sigma_i\sigma_j e^{-p|t_j-t_j|\varepsilon(p)}\cos p(x_i-x_j)]$$
$$-c(p)s(p)\sum_{i\in I_\varepsilon, j\in I_{-\varepsilon}}\sigma_i\sigma_j e^{-p|t_j-t_j|\varepsilon(p)}\cos p(x_i-x_j)$$
$$-\sum_{i,j\in I_\varepsilon, i<j}[e^{-|t_i-t_j|} - e^{-p|t_i-t_j|\varepsilon(p)}]\cos p(x_i-x_j)$$
$$-i\varepsilon\sum_{i,j\in I_\varepsilon, i<j}\sigma_i\sigma_j\frac{t_j-t_j}{|t_i-t_j|}[e^{-|t_i-t_j|} - e^{-p|t_i-t_j|\varepsilon(p)}]\sin p(x_i-x_j)\} \quad (3.143)$$

where

$$s(p) = \sinh\phi(p) \quad c(p) = \cosh\phi(p) \quad (3.144)$$

In particular the 2-point function is given by

$$< \widetilde{\psi}^-_{\mathbf{x},\varepsilon}\widetilde{\psi}^+_{0,\varepsilon} >= \frac{1}{2\pi}\frac{1}{i\varepsilon x + t}e^{-Q_2(\mathbf{x})} \quad (3.145)$$

$$Q_2(\mathbf{x}) = \int_0^\infty \frac{dp}{p}[2s^2(p)(1 - e^{-p\varepsilon(p)|t|}\cos px) +$$
$$(\cos px - i\varepsilon\frac{t}{|t|}\sin px)(e^{-p|t|} - e^{-p\varepsilon(p)|t|})] \quad (3.146)$$

From the above expression is easy to see that the non locality of the two body potential is essential to have a finite 2-point function; if we consider a local potential then $\widehat{v}(p) = 1$ and the above expression is diverging. Note also that (3.146) has *the same* ultraviolet divergence at $\mathbf{x} = \mathbf{0}$ than the *free* 2-point function as $Q_2(\mathbf{0}) = 0$. On the contrary the large distance behavior is different with respect to the non interacting case and given by

$$< \widetilde{\psi}^-_{\mathbf{x},\varepsilon}\widetilde{\psi}^+_{0,\varepsilon} >\sim_{|\mathbf{x}|\to\infty} \frac{1}{i\varepsilon x + v_F t}\frac{A(\lambda)}{|x^2 + v_F^2 t^2|^{\eta/2}} \quad (3.147)$$

with $A(\lambda)$ a suitable constant and

$$v_F = \varepsilon(0) = \sqrt{1 - (\lambda\widehat{v}(0)/2\pi)^2} \quad (3.148)$$

$$\eta = 2\sinh^2\phi(0) = \frac{1}{2}[K + K^{-1} - 2] \qquad K = e^{2\phi} = \sqrt{\frac{1+\lambda\widehat{v}(0)/2\pi}{1-\lambda\widehat{v}(0)/2\pi}}$$

From (3.149) we see that the interaction not only modifies the Fermi velocity but also changes qualitatively the asymptotic infrared behavior of the 2-point function, producing an anomalous dimension with exponent $1 + \eta$, with $\eta > 0$ and non trivial function of the coupling. The presence

of anomalous exponents is one of the most interesting feature of the Luttinger model, and implies a rather different physical behavior with respect to the non interacting system; for instance the occupation number is not discontinuous as in the $\lambda = 0$ case, but it becomes continuous and such that $n_{k+p_F} - n_{p_F} \sim |k|^\eta$ for small k.

It is also important to compare the the 2-point function of the Luttinger model with the 2-point function $S_{th}(\mathbf{x})$ of the Thirring model, as computed in [**?**]

$$S_{th}(\mathbf{x}) = \frac{1}{2\pi} \frac{1}{i\varepsilon x + t} \frac{1}{|x^2 + t^2|^\eta} \tag{3.149}$$

Despite the similar long distance behavior, there are important differences; the *ultraviolet* short distance behavior is different with respect the the free one, and the light velocity is not changed due to Lorentz invariance. with

$$X_- = K^{-1} \tag{3.150}$$

Note that (3.149),(**??**),(3.150) relates the exponents by simple relations; while the exponents depend from the details of the model (3.129) (for instance from the charges $q_{i,\varepsilon}$) such relations are true for any choice of the charges for which the solvability holds. For instance if $q_{i,\varepsilon} = \frac{1}{2}$ then (3.149),(**??**),(3.150) are still true even if K has a different expression

$$K = \sqrt{1 + \frac{\lambda \widehat{v}(0)}{2\pi}} \tag{3.151}$$

As we will see in the following section such relations between exponents have been proposed to be true in a wide class of models, even when an exact solution is lacking.

Defining the *density* and *current* operators as

$$\rho_{\mathbf{x}} = \sum_{\varepsilon=\pm} \rho_{\mathbf{x},\varepsilon} \qquad j_{\mathbf{x}} = \sum_{\varepsilon=\pm} \varepsilon \rho_{\mathbf{x},\varepsilon} \tag{3.152}$$

we get, as an immediate consequence of the commutation rules (3.129)

$$\frac{\partial \widehat{\rho}_{t,p}}{dt} = e^{Ht}[H, \widehat{\rho}_p]e^{-Ht} = p v_N(p) \widehat{j}_{t,p} \qquad \frac{\partial \widehat{j}_{t,p}}{dt} = e^{Ht}[H, \widehat{j}_p]e^{-Ht} = p v_J(p) \widehat{\rho}_{t,p}$$

with

$$v_J(p) = (1 - \frac{\lambda \widehat{v}(p)}{2\pi}) \qquad v_N(p) = (1 + \frac{\lambda \widehat{v}(p)}{2\pi}) \tag{3.153}$$

and we have used that, from the commutation rules (3.129), $[V,\rho_p] = \frac{\lambda\widehat{v}(p)}{2\pi}\rho_p$, $[V,\rho_p] = -\frac{\lambda\widehat{v}(p)}{2\pi}\rho_p$. As a consequence, the following Ward Identities [**?**] can be derived

$$-i\omega\langle\widehat{\rho}_{\mathbf{p}}\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle_T + \varepsilon v_J\langle\widehat{j}_{\mathbf{p}}\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle_T = \langle\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k},\varepsilon}\rangle - \langle\widehat{\psi}^{-}_{\mathbf{k}+\mathbf{p},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle \tag{3.154}$$

and

$$-i\omega\langle\widehat{j}_{\mathbf{p}}\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle_T + \varepsilon v_N p\langle\widehat{\rho}_{\mathbf{p}}\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle_T = \langle\widehat{\psi}^{-}_{\mathbf{k},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k},\varepsilon}\rangle - \langle\widehat{\psi}^{-}_{\mathbf{k}+\mathbf{p},\varepsilon}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{p},\varepsilon}\rangle \tag{3.155}$$

The quantities $v_N(p)$ and $v_J(p)$ are velocities associated with the charge and current excitations; they appear also related to the Drude weight D and the susceptibility κ [**?**] (see below) by the relations $D = \frac{v_J(0)}{\pi}$ and $\kappa = \frac{\pi}{v_N(0)}$ so that

$$\frac{D}{\kappa} = v_F^2 \tag{3.156}$$

Chapter 4

4.1 Grassmann variables and functional integrals

In the previous analysis of quantum gases we have only considered non interacting systems. Of course a number of important physical properties (either at the front of current research in theretical and mathematical physics) include the necessity of including the interaction. Among phenomena which can be understood only in interaction is included, there is meganetism or superconductivity.

Let us consider for definiteness the case of fermions. We have seen that the eigenfuctions can be constructed starting from the single particle eigenfunctions; this means that the properties of the N particle system can be understood starting from the single particle description. In presence of interaction this is not true; interacting fermions can be described in terms of completely different objects (like bosons). Indeed in one dimension there is a small number of models which can be solved (like the XXZ chain solved by Yang and Yang (1966), the XYZ solved by Baxter, the Hubbard model solved by Lieb and Wu and the Luttinger model solved by Mattis and Lieb.

In the general case one uses a functional integral representation; such integral are usually defined with a number of cut-offs necessary to transform them in finite dimansional integral, and then one has to prove that the integrals are still defined in the limit of removed cut-off. There are two kind of functional integral depending if they describe bosons or fermions; in this second case they are called Grassmann integrals. We will define them in this section: despite their utility in studying interacting fermionic models, they are a very useful tool for analyzing lattice classical spin models in two dimension.

A finite dimensional *Grassman algebra*, see also Refs.[**?**],[**?**], is a set

of *Grassman variables* ψ_α, with α an index belonging to some finite set $A = (1, .., 2n)$ which are anticommuting, that is

$$\{\psi_\alpha, \psi_{\alpha'}\} = \psi_\alpha \psi_{\alpha'} + \psi_{\alpha'} \psi_\alpha = 0 \tag{4.1}$$

and commuting with numbers.

A *Grassmann integral* $\int d\psi_\alpha$ is a linear operation defined as

$$\int d\psi_\alpha = 0 \qquad \int d\psi_\alpha \psi_\alpha = 1 \tag{4.2}$$

The Grassmann integral of any analytic function can be obtained by linearity ; for instance

$$\int d\psi_\alpha e^{\psi_\alpha} = \int d\psi_\alpha (1 + \psi_\alpha) = 1 \tag{4.3}$$

$d\psi_\alpha$ is also a Grassmann variable, anticommuting with ψ_α, so that $\int \psi_\alpha d\psi_\alpha = -1$. A slight generalization of (4.3) is

$$\int \prod_\alpha d\psi_\alpha e^{\frac{1}{2} \sum_{\alpha,\beta} \psi_\alpha A_{\alpha,\beta} \psi_\beta} = \mathrm{Pf} A \tag{4.4}$$

where A is an even antisymmetric $2n$-matrix and $\mathrm{Pf}A$ denotes the *Pfaffian*. It holds

$$\int d\psi_{2n} ... d\psi_1 \exp \frac{1}{2} \sum_{\alpha,\beta} \psi_\alpha A_{\alpha,\beta} \psi_\beta = \int d\psi_{2n} ... d\psi_1 \prod_{\alpha<\beta} (1 + A_{\alpha,\beta} \psi_\alpha \psi_\beta) =$$

$$\frac{1}{2^n n!} \sum_p (-1)^p A_{p_1,p_2} A_{p_3,p_4} ... A_{p_{2n-1},p_{2n}} \equiv \mathrm{Pf} A \tag{4.5}$$

where the sum is over all the permutations.

Suppose that we can split the variables ψ in two sets $\psi_1, ..., \psi_{2n} = \psi_1^+, .., \psi_n^+, \psi_1^- ..., \psi_n^-$ and that A has the form

$$A = \begin{pmatrix} 0 & -K \\ K^t & 0 \end{pmatrix} \tag{4.6}$$

then

$$\mathrm{Pf} A = \int d\psi_1^+ ... d\psi_n^+ \psi_1^- ... d\psi_n^- e^{\frac{1}{2} \sum_{i,j} \psi_\alpha^+ A_{i,j} \psi_j^-} = \det K \tag{4.7}$$

Moreover by defining

$$\langle O \rangle = \frac{\int d\psi_1 ... d\psi_{2n} e^{\frac{1}{2} \sum_{\alpha,\beta} \psi_\alpha A_{\alpha,\beta} \psi_\beta} e^{\sum_\alpha \psi_\alpha h_\alpha}}{\int d\psi_1 ... d\psi_{2n} e^{\frac{1}{2} \sum_{\alpha,\beta} \psi_\alpha A_{\alpha,\beta} \psi_\beta}} \tag{4.8}$$

we get

$$\langle e^{\sum_\alpha \psi_\alpha \eta_\alpha} \rangle = e^{\frac{1}{2} \sum_{\alpha,\beta} h_\alpha (A^{-1})_{\alpha,\beta} h_\beta} \tag{4.9}$$

and differentiating

$$\langle h_1 h_2 ... h_{2p} \rangle = \mathrm{Pf}(A^{-1})_{1,..,2p} \tag{4.10}$$

where $(A^{-1})_{1,..,2p}$ means the restriction of A^{-1} to the subspace $1, .., 2p$.

If A is of the form (4.6) and

$$h_1, h_2, ..., h_{2p} = \eta_1^+, \eta_2^+ ..., \eta_p^+, \eta_1^-, \eta_2^-, ..., \eta_p^- \tag{4.11}$$

we have

$$\langle \eta_1^+ \eta_2^+ ... \eta_p^+ \eta_1^- \eta_2^- ... \eta_p^- \rangle = \mathrm{Det}(K^{-1})_{1,..,p} \tag{4.12}$$

In the very simple case in which $n = 1$

$$\int d\psi_\alpha^+ d\psi_\alpha^- e^{-a\psi_\alpha^+ \psi_\alpha^-} = -a \int d\psi_\alpha^+ d\psi_\alpha^- \psi_\alpha^+ \psi_\alpha^- = a \tag{4.13}$$

and

$$\frac{\int d\psi_\alpha^+ d\psi_\alpha^- \psi_\alpha^+ \psi_\alpha^- e^{-a\psi_\alpha^+ \psi_\alpha^-}}{\int d\psi_\alpha^+ d\psi_\alpha^- e^{-a\psi_\alpha^+ \psi_\alpha^-}} = \frac{1}{a} \tag{4.14}$$

The above formula is strongly reminescent of the formulas for gaussian integrals

$$\frac{\int_{-\infty}^{\infty} dx x^2 e^{-ax^2}}{\int_{-\infty}^{\infty} dx e^{-ax^2}} = \frac{1}{2a} \tag{4.15}$$

and this analogy motivates the name "integrals" given to the above operations.

The well known relation $(\mathrm{Pf} A)^2 = \det A$ can be quite easily deduced by the above Grassmann integrals; it can be written as

$$\int \prod_\alpha d\psi_\alpha^- d\psi_\alpha^+ e^{\psi_\alpha^- A_{\alpha,\beta} \psi_\beta^+} = \int \prod_\alpha d\psi_\alpha^{(1)} e^{\frac{1}{2}\psi_\alpha^{(1)} A_{\alpha,\beta} \psi_\beta^{(1)}} \int \prod_\alpha d\psi_\alpha^{(2)} e^{\frac{1}{2}\psi_\alpha^{(2)} A_{\alpha,\beta} \psi_\beta^{(2)}} \tag{4.16}$$

which can be proved by the change of variables

$$\psi_\alpha^+ = \frac{1}{\sqrt{2}}(\psi_\alpha^{(1)} + i\psi_\alpha^{(2)}) \quad \psi_\alpha^- = \frac{1}{\sqrt{2}}(\psi_\alpha^{(1)} - i\psi_\alpha^{(2)}) \tag{4.17}$$

in $\int d\psi_\alpha^- d\psi_\alpha^+ e^{\psi_\alpha A_{\alpha,\beta} \psi_\beta^+}$. Then $d\psi_\alpha^- d\psi_\alpha^+ = i d\psi_\alpha^{(1)} d\psi_\alpha^{(2)}$ and

$$\psi_\alpha A_{\alpha,\beta} \psi_\beta^+ = \frac{1}{2}\psi_\alpha^{(1)} A_{\alpha,\beta} \psi_\beta^{(1)} + \frac{1}{2}\psi_\alpha^{(2)} A_{\alpha,\beta} \psi_\beta^{(2)} \tag{4.18}$$

as

$$\psi_\alpha^{(1)} A_{\alpha,\beta} \psi_\beta^{(2)} - \psi_\alpha^{(2)} A_{\alpha,\beta} \psi_\beta^{(1)} = \psi_\alpha^{(1)} A_{\alpha,\beta} \psi_\beta^{(2)} - \psi_\beta^{(1)} A_{\beta,\alpha} \psi_\alpha^{(2)} = 0 \tag{4.19}$$

4.1.1 *Grassmann integral representation of the 2D Ising model*

Starting from the sum of multipolygons,, Z can be written as the sum of four Grassmann integrals with different boundary conditions

$$Z_I = -Z_{+,+} + Z_{+,-} + Z_{-,+} + Z_{-,-} \tag{4.20}$$

$$Z_{\varepsilon,\varepsilon'} = (\cosh \beta J)^B 2^S \frac{1}{2} \int \prod_{\mathbf{x}\in\Lambda} dH_{\mathbf{x}} d\bar{H}_{\mathbf{x}} dV_{\mathbf{x}} d\bar{V}_{\mathbf{x}} e^{S_{\varepsilon,\varepsilon'}} \tag{4.21}$$

where

$$S_{\varepsilon,\varepsilon'} = \sum_{\mathbf{x}\in\Lambda} \tanh \beta J[\bar{H}_{x,x_0} H_{x+1,x_0} + \bar{V}_{x,x_0} V_{x,x_0+1}] + \tag{4.22}$$

$$\sum_{\mathbf{x}\in\Lambda} [\bar{H}_{x,x_0} H_{x,x_0} + \bar{V}_{x,x_0} V_{x,x_0} + \bar{V}_{x,x_0} \bar{H}_{x,x_0} + V_{x,x_0} \bar{H}_{x,x_0} + H_{x,x_0} \bar{V}_{x,x_0} + V_{x,x_0} H_{x,x_0}] \tag{4.23}$$

and $H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}}$ are *Grassmann variables* such that

$$\begin{aligned} \bar{H}_{x,x_0+L} &= \varepsilon \bar{H}_{x,x_0} \qquad \bar{H}_{x+L,x_0} = \varepsilon' \bar{H}_{x,x_0} \\ H_{x,x_0+L} &= \varepsilon H_{x,x_0} \quad H_{x+L,x_0} = \varepsilon' H_{x,x_0} \end{aligned} \tag{4.24}$$

and identical relations hold for the variables $V, \bar{V}$.

By using this formula and the Grassman integral rules we can get the partition function; by performing the Fourier transform can be rewritten in Fourier space as:

$$S_{\varepsilon,\varepsilon'} = \sum_{\mathbf{k}} \tanh \beta J[\bar{H}_{\mathbf{k}} H_{-\mathbf{k}} e^{i\mathbf{k}} + \bar{V}_{x,x_0} V_{x,x_0+1}] + \tag{4.25}$$

$$\sum_{\mathbf{x}\in\Lambda} [\bar{H}_{x,x_0} H_{x,x_0} + \bar{V}_{x,x_0} V_{x,x_0} + \bar{V}_{x,x_0} \bar{H}_{x,x_0} + V_{x,x_0} \bar{H}_{x,x_0} + H_{x,x_0} \bar{V}_{x,x_0} + V_{x,x_0} H_{x,x_0}] \tag{4.26}$$

and $H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}}$ are *Grassmann variables* such that

$$S_t(\Phi) = \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \widehat{\Phi}^T_{-\mathbf{k}} C_{\mathbf{k}} \widehat{\Phi}_{\mathbf{k}} \,, \tag{4.27}$$

where

$$C_{\mathbf{k}} := \frac{a^{-1}}{2} \begin{pmatrix} 0 & 1 + te^{-iak_1} & -1 & -1 \\ -1 - te^{iak_1} & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 + te^{-iak_2} \\ 1 & 1 & -1 - te^{iak_2} & 0 \end{pmatrix} \,, \tag{4.28}$$

Then, unless for a sign

$$\int \prod_{\mathbf{x}\in\Lambda} \mathrm{d}\overline{H}_{\mathbf{x}}\mathrm{d}H_{\mathbf{x}}\mathrm{d}\overline{V}_{\mathbf{x}}\mathrm{d}V_{\mathbf{x}} e^{S_{\varepsilon,\varepsilon'}(t)} = \tag{4.29}$$
$$\prod_{\mathbf{k}>0}\Big[\int \mathrm{d}\widehat{\overline{H}}_{\mathbf{k}}\mathrm{d}\widehat{\overline{H}}_{-\mathbf{k}}\mathrm{d}\widehat{H}_{\mathbf{k}}\mathrm{d}\widehat{H}_{-\mathbf{k}}\mathrm{d}\widehat{\overline{V}}_{\mathbf{k}}\mathrm{d}\widehat{\overline{V}}_{-\mathbf{k}}\mathrm{d}\widehat{V}_{\mathbf{k}}\mathrm{d}\widehat{V}_{-\mathbf{k}}\cdot e^{\Psi^T_{\mathbf{k}}M_{\mathbf{k}}\Psi_{-\mathbf{k}}}\Big]$$

and we see that the r.h.s. of (4.30) is equal, by (2.12), to $\prod_{\mathbf{k}>0}\det M_{\mathbf{k}}$. Now, developing the determinant of $M_{\mathbf{k}}$ along its first row we find:

$$\det M_{\mathbf{k}} = -(1+te^{ik})\det\begin{pmatrix} -(1+te^{-ik}) & 1 & -1 \\ 1 & 0 & 1+te^{ik_0} \\ 1 & -(1+te^{-ik_0}) & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} -(1+te^{-ik}) & 0 & -1 \\ 1 & -1 & 1+te^{ik_0} \\ 1 & 1 & 0 \end{pmatrix}+\det\begin{pmatrix} -(1+te^{-ik}) & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -(1+te^{-ik_0}) \end{pmatrix}$$

More explicitely the last expression is equal to

$$-(1+te^{ik})\Big[-(1+te^{-ik})\big|1+te^{ik_0}\big|^2+(1+te^{ik_0})+(1+te^{-ik_0})\Big]-$$
$$\Big[-(1+te^{-ik})(1+te^{ik_0})-2\Big]+\Big[-(1+te^{-ik})(1+te^{-ik_0})+2\Big] \tag{4.30}$$

so that

$$\det M_{\mathbf{k}} = \Big[1+t^2+2t\cos k\Big]\Big[1+t^2+2t\cos k_0\Big]-4t(\cos k+\cos k_0)$$
$$-4t^2\cos k\cos k_0 = (1+t^2)^2-2t(1-t^2)(\cos k+\cos k_0) \tag{4.31}$$

In the limit $L\to\infty$, the four Grassmann integrals in (4.20) have the same limit so that the free energy f is given by

$$-\beta f = \lim_{L\to\infty}\frac{1}{L^2}\log Z_I = \log(2\cosh^2\beta J)+$$
$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\int_{-\pi}^{\pi}\frac{dk_0}{2\pi}\log\{(1+t^2)^2-2t(1-t^2)(\cos k+\cos k_0)\} \tag{4.32}$$
$$=\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\int_{-\pi}^{\pi}\frac{dk_0}{2\pi}\log\Big\{4\big[\cosh^2 2\beta J-\sinh 2\beta J(\cos k+\cos k_0)\big]\Big\}$$

Note that the argument of the logarithm in the last expression is always ≥ 0 and it vanishes if and only if $\beta=\beta_c$ with $\sinh 2\beta_c J=1$, that is the equation for the critical temperature. In the following we shall also write this condition in the equivalent form

$$\tanh\beta_c J = \sqrt{2}-1 \tag{4.33}$$

The specific heat has the form

$$C_v \simeq -C \log|\beta - \beta_c| + C_2 \tag{4.34}$$

that is it has a logarithmic singularity at β_c, which is a signal of the occurring of a phase transition at β_c. It is also possible to compute the correlations and one sees that they are exponentially vanishing for large distances except at $\beta = \beta_c$ when they have a weaker power law decay. For instance for large distances at $\beta \neq \beta_c$

$$| < \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}; \sigma_{\mathbf{y}}\sigma_{\mathbf{y}'} >_T | \leq C \frac{e^{-\kappa m(\beta)|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2} \tag{4.35}$$

where $m(\beta) = O(|\beta - \beta_c|)$ is the correlation legnth, while if $\beta = \beta_c$ and $\mathbf{x}, \mathbf{x}'$ nearest neighbor

$$< \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}; \sigma_{\mathbf{y}}\sigma_{\mathbf{y}'} >_T | \simeq \frac{1}{|\mathbf{x}-\mathbf{y}|^2} \tag{4.36}$$

4.1.2 *Majorana form of the action*

As we have shown, the non-integrable Ising model under consideration can be expressed in the form of an interacting fermionic system, described by the action Eq.(**??**), which consists of a leading term, $S_t(\Phi) + \sum_b \big[t + (1 - t^2)E_b\big]A_b$, plus an *interaction*, which vanishes as $\lambda \to 0$ and, in this respect, is "subdominant". Since in the following we want to treat this subdominant term as a perturbation in the vicinity of the critical point, it is convenient to use coordinates adapted to the critical modes of the leading term, as described in the following.

The Fourier transforms of the Grassmann variables are defined as:

$$\widehat{H}_{\mathbf{k}} := a^2 \sum_{\mathbf{x}\in a\mathbb{Z}^2_M} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{x}} \,, \qquad \widehat{\overline{H}}_{\mathbf{k}} := a^2 \sum_{\mathbf{x}\in a\mathbb{Z}^2_M} e^{i\mathbf{k}\cdot\mathbf{x}} \overline{H}_{\mathbf{x}} \,, \tag{4.37}$$

$$\widehat{V}_{\mathbf{k}} := a^2 \sum_{\mathbf{x}\in a\mathbb{Z}^2_M} e^{i\mathbf{k}\cdot\mathbf{x}} V_{\mathbf{x}} \,, \qquad \widehat{\overline{V}}_{\mathbf{k}} := a^2 \sum_{\mathbf{x}\in a\mathbb{Z}^2_M} e^{i\mathbf{k}\cdot\mathbf{x}} \overline{V}_{\mathbf{x}} \,, \tag{4.38}$$

where, if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, then $\mathbf{k} \in \mathcal{D}^{\boldsymbol{\alpha}}_M := \{\frac{2\pi}{L}(+\frac{1}{2}\boldsymbol{\alpha}) :\in \mathbb{Z}^2/M\mathbb{Z}^2\}$. The inverse transformation reads:

$$H_{\mathbf{x}} := \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}^{\boldsymbol{\alpha}}_M} e^{-i\mathbf{k}\cdot\mathbf{x}} \widehat{H}_{\mathbf{k}} \,, \qquad \overline{H}_{\mathbf{x}} := \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}^{\boldsymbol{\alpha}}_M} e^{-i\mathbf{k}\cdot\mathbf{x}} \widehat{\overline{H}}_{\mathbf{k}} \,, \tag{4.39}$$

$$V_{\mathbf{x}} := \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}^{\boldsymbol{\alpha}}_M} e^{-i\mathbf{k}\cdot\mathbf{x}} \widehat{V}_{\mathbf{k}} \,, \qquad \overline{V}_{\mathbf{x}} := \frac{1}{L^2} \sum_{\mathbf{k}\in\mathcal{D}^{\boldsymbol{\alpha}}_M} e^{-i\mathbf{k}\cdot\mathbf{x}} \widehat{\overline{V}}_{\mathbf{k}} \,. \tag{4.40}$$

If $\widehat{\Phi}_{\mathbf{k}}$ is the column vector with components $\widehat{\overline{H}}_{\mathbf{k}}, \widehat{H}_{\mathbf{k}}, \widehat{\overline{V}}_{\mathbf{k}}, \widehat{V}_{\mathbf{k}}$, respectively, the leading quadratic part of the action, $S_t(\Phi)$, can be rewritten in Fourier space as:

$$S_t(\Phi) = \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}_M^{\boldsymbol{\alpha}}} \widehat{\Phi}^T_{-\mathbf{k}} C_{\mathbf{k}} \widehat{\Phi}_{\mathbf{k}} \ , \tag{4.41}$$

where

$$C_{\mathbf{k}} := \frac{a^{-1}}{2} \begin{pmatrix} 0 & 1+te^{-iak_1} & -1 & -1 \\ -1-te^{iak_1} & 0 & 1 & -1 \\ 1 & -1 & 0 & 1+te^{-iak_2} \\ 1 & 1 & -1-te^{iak_2} & 0 \end{pmatrix} \ , \tag{4.42}$$

the inverse of which has the meaning of free propagator of the Grassmann field Φ. Note that $C_{\mathbf{k}}^{-1}$ is singular only at $\mathbf{k} = \mathbf{0}$ and $t = \sqrt{2} - 1$ (criticality condition). In this case, $C_{\mathbf{0}}$ has two vanishing eigenvalues and two purely imaginary eigenvalues $\pm i a^{-1}\sqrt{2}$, and the corresponding eigenmodes read:

$$\begin{pmatrix} \widehat{\psi}'_{\mathbf{0},+} \\ \widehat{\psi}'_{\mathbf{0},-} \\ \widehat{\chi}'_{\mathbf{0},+} \\ \widehat{\chi}'_{\mathbf{0},-} \end{pmatrix} = U \widehat{\Phi}_{\mathbf{0}} \ , \qquad U = \frac{1}{2} \begin{pmatrix} e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & 1 & -i \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 1 & i \\ -e^{i\frac{\pi}{4}} & -e^{-i\frac{\pi}{4}} & 1 & -i \\ -e^{-i\frac{\pi}{4}} & -e^{i\frac{\pi}{4}} & 1 & i \end{pmatrix} \ . \tag{4.43}$$

The natural variables at the critical point are the "critical eigenmodes" defined by the unitary transformation U in Eq.(4.43), namely

$$\begin{pmatrix} \widehat{\psi}'_{\mathbf{k},+} \\ \widehat{\psi}'_{\mathbf{k},-} \\ \widehat{\chi}'_{\mathbf{k},+} \\ \widehat{\chi}'_{\mathbf{k},-} \end{pmatrix} = U \widehat{\Phi}_{\mathbf{k}} \ . \tag{4.44}$$

For later convenience, we rescale these variables as (where $\omega = \pm$)

$$\widehat{\psi}'_{\mathbf{k},\omega} = \frac{i\omega}{\sqrt{\pi t}} \widehat{\psi}_{\mathbf{k},\omega} \ , \qquad \widehat{\chi}'_{\mathbf{k},\omega} = \frac{i\omega}{\sqrt{\pi t}} \widehat{\chi}_{\mathbf{k},\omega} \ , \tag{4.45}$$

so that, defining $\widehat{\psi}_{\mathbf{k}}$ (resp. $\widehat{\chi}_{\mathbf{k}}$) as the column vector with components $\widehat{\psi}_{\mathbf{k},+}, \widehat{\psi}_{\mathbf{k},-}$ (resp. $\widehat{\chi}_{\mathbf{k},+}, \widehat{\chi}_{\mathbf{k},-}$), we can rewrite:

$$S_t(\Phi) = -\frac{1}{4\pi L^2} \sum_{\mathbf{k} \in \mathcal{D}_M^{\boldsymbol{\alpha}}} \left(\widehat{\psi}^T_{-\mathbf{k}} C_\psi(\mathbf{k}) \widehat{\psi}_{\mathbf{k}} + \widehat{\chi}^T_{-\mathbf{k}} C_\chi(\mathbf{k}) \widehat{\chi}_{\mathbf{k}} \right) + Q(\psi, \chi) \ , \tag{4.46}$$

where, if $\# = \psi, \chi$:

$$C_\#(\mathbf{k}) = \begin{pmatrix} a^{-1}(-i \sin ak_1 + \sin ak_2) & i\sigma_\#(\mathbf{k}) \\ -i\sigma_\#(\mathbf{k}) & a^{-1}(-i \sin ak_1 - \sin ak_2) \end{pmatrix} \tag{4.47}$$

and

$$\sigma_\psi(\mathbf{k}) = \frac{1}{a}\Big(\cos ak_1 + \cos ak_2 - 2\frac{\sqrt{2}-1}{t}\Big) , \tag{4.48}$$

$$\sigma_\chi(\mathbf{k}) = \frac{1}{a}\Big(\cos ak_1 + \cos ak_2 + 2\frac{\sqrt{2}+1}{t}\Big) . \tag{4.49}$$

Moreover,

$$Q(\psi,\chi) = \frac{1}{4\pi L^2}\sum_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \big(\widehat{\psi}^T_{-\mathbf{k}} Q(\mathbf{k}) \widehat{\chi}_{\mathbf{k}} + \widehat{\chi}^T_{-\mathbf{k}} Q(\mathbf{k}) \widehat{\psi}_{\mathbf{k}}\big) , \tag{4.50}$$

$$Q(\mathbf{k}) = \begin{pmatrix} a^{-1}(-i\sin ak_1 - \sin ak_2) & a^{-1}(i\cos ak_1 - i\cos ak_2) \\ a^{-1}(-i\cos ak_1 + i\cos ak_2) & a^{-1}(-i\sin ak_1 + \sin ak_2)\end{pmatrix} .$$

In terms of this notation, we can rewrite Eq.(**??**) as

$$\Xi() = \sum_{\boldsymbol{\alpha}\in\{\pm\}^2} C_{M,\boldsymbol{\alpha}} \int P_{\boldsymbol{\alpha}}(d\psi) P_{\boldsymbol{\alpha}}(d\chi) e^{Q(\psi,\chi)+\mathcal{B}(\psi,\chi,)+\mathcal{V}(\psi,\chi,)} , \tag{4.51}$$

where:

- The normalization constant $C_{M,\boldsymbol{\alpha}}$ is defined as

  $$C_{M,\boldsymbol{\alpha}} = \frac{C_M}{2}(a^2M)^{4M^2} \tau_{\boldsymbol{\alpha}} \mathcal{N}_{\psi,\boldsymbol{\alpha}} \mathcal{N}_{\chi,\boldsymbol{\alpha}} ,$$

  where: C_M was defined in Eq.(**??**); the factor $(a^2M)^{4M^2}$ takes into account the change of variable from the set of Grassmann variables $\{\overline{H}_{\mathbf{x}}, H_{\mathbf{x}}, \overline{V}_{\mathbf{x}}, V_{\mathbf{x}}\}$ to $\{\widehat{\psi}_{\mathbf{k},\omega}, \widehat{\chi}_{\mathbf{k},\omega}\}$: in fact a computation shows that

  $$\int \mathcal{D}\Phi \Big[\prod_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \widehat{\overline{H}}_{\mathbf{k}} \widehat{H}_{\mathbf{k}} \widehat{\overline{V}}_{\mathbf{k}} \widehat{V}_{\mathbf{k}}\Big] = a^{8M^2} \big[\det_{\mathbf{k},\mathbf{x}} (e^{i\mathbf{k}\cdot\mathbf{x}})\big]^4 = (a^2M)^{4M^2} ;$$

  $\tau_{\boldsymbol{\alpha}}$ was defined after Eq.(**??**); $\mathcal{N}_{\psi,\boldsymbol{\alpha}}$ and $\mathcal{N}_{\chi,\boldsymbol{\alpha}}$ are the normalization constants of the two Grassmann gaussian integrations $P_{\boldsymbol{\alpha}}(d\psi)$ and $P_{\boldsymbol{\alpha}}(d\chi)$, see next item.
- The Grassmann gaussian integrations $P_{\boldsymbol{\alpha}}(d\psi)$, $P_{\boldsymbol{\alpha}}(d\chi)$ are defined as

  $$P_{\boldsymbol{\alpha}}(d\psi) := \frac{1}{\mathcal{N}_{\psi,\boldsymbol{\alpha}}} \Big[\prod_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \prod_{\omega=\pm} d\widehat{\psi}_{\mathbf{k},\omega}\Big] \exp\Big\{ -\frac{1}{4\pi L^2} \sum_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \widehat{\psi}^T_{-\mathbf{k}} C_\psi(\mathbf{k}) \widehat{\psi}_{\mathbf{k}}\Big\} ,$$

  $$P_{\boldsymbol{\alpha}}(d\chi) := \frac{1}{\mathcal{N}_{\chi,\boldsymbol{\alpha}}} \Big[\prod_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \prod_{\omega=\pm} d\widehat{\chi}_{\mathbf{k},\omega}\Big] \exp\Big\{ -\frac{1}{4\pi L^2} \sum_{\mathbf{k}\in\mathcal{D}_M^{\boldsymbol{\alpha}}} \widehat{\chi}^T_{-\mathbf{k}} C_\chi(\mathbf{k}) \widehat{\chi}_{\mathbf{k}}\Big\} ,$$

  and $\mathcal{N}_{\psi,\boldsymbol{\alpha}}, \mathcal{N}_{\chi,\boldsymbol{\alpha}}$ are two normalization constants, fixed in such a way that $\int P_{\boldsymbol{\alpha}}(d\psi) = \int P_{\boldsymbol{\alpha}}(d\chi) = 1$.

- The source term $\mathcal{B}(\psi,\chi,)$ is the rewriting of $(1-t^2)(\boldsymbol{E},)$ in terms of the new variables, namely

$$\mathcal{B}(\psi,\chi,) = -i\,\frac{1-t^2}{2t}\int\frac{d\mathbf{x}}{2\pi}\cdot \tag{4.52}$$
$$\Big[A_{\mathbf{x},1}(\psi_{\mathbf{x},+}-i\psi_{\mathbf{x},-}-\chi_{\mathbf{x},+}+i\chi_{\mathbf{x},-})\tau_1(-i\psi_{\mathbf{x},+}+\psi_{\mathbf{x},-}+i\chi_{\mathbf{x},+}-\chi_{\mathbf{x},-})$$
$$+A_{\mathbf{x},2}(\psi_{\mathbf{x},+}-\psi_{\mathbf{x},-}+\chi_{\mathbf{x},+}-\chi_{\mathbf{x},-})\tau_2(\psi_{\mathbf{x},+}+\psi_{\mathbf{x},-}+\chi_{\mathbf{x},+}+\chi_{\mathbf{x},-})\Big]\,,$$

  where τ_j is the translation operator that shifts by one lattice step the argument of the field which it acts on: $\tau_j\psi_{\mathbf{x},\omega}=\psi_{\mathbf{x}+a\widehat{\mathbf{e}}_j,\omega}$ and similarly for χ.
- $\mathcal{V}(\psi,\chi,)$ is the rewriting of $\mathcal{V}(\Phi,)$ in terms of the new variables. It is easy to check that its kernels satisfy the same decay estimates as those of $\mathcal{V}(\Phi,)$, see Eq.(**??**) in Proposition **??**.

As mentioned in the introduction, we are concerned with a scaling limit such that we take the thermodynamic limit $M\to\infty$ first, keeping $\beta=\beta(a)\neq\beta_c(\lambda)$, and then (simultaneously) $a\to 0$ and $\beta\to\beta_c(\lambda)$. In doing so, the resulting multi-point energy correlations are insensitive to the Grassmann boundary conditions, labeled by the four possible values of $\boldsymbol{\alpha}$; this is true both in the $\lambda=0$ and in the $\lambda\neq 0$ case, see [**?**, Appendix G]. Therefore, the multi-point energy correlation functions in the specific scaling limit that we consider are the same as those computed from the following generating function:

$$\Xi_{-,-}() = \int P(d\psi)P(d\chi)e^{Q(\psi,\chi)+\mathcal{B}(\psi,\chi,)+\mathcal{V}(\psi,\chi,)}\,, \tag{4.53}$$

where $P(d\psi)$ is a shorthand for $P_{-,-}(d\psi)$ and similarly for $P(d\chi)$. For future reference, let us note that the propagator of the ψ and χ fields associated with the gaussian integrations $P(d\psi)$ and $P(d\chi)$ are given by, if $D_a^{\pm}(\mathbf{k})=a^{-1}(i\sin ak_1\pm\sin ak_2)$ and $|D_a(\mathbf{k})|^2=-D_a^+(\mathbf{k})D_a^-(\mathbf{k})$,

$$g^{\psi}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \int P(d\psi)\psi_{\mathbf{x},\omega}\psi_{\mathbf{y},\omega'} \tag{4.54}$$
$$= \frac{2\pi}{L^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{|D_a(\mathbf{k})|^2+[\sigma_\psi(\mathbf{k})]^2}\begin{pmatrix} D_a^+(\mathbf{k}) & i\sigma_\psi(\mathbf{k})\\ -i\sigma_\psi(\mathbf{k}) & D_a^-(\mathbf{k})\end{pmatrix}_{\omega,\omega'}$$

and

$$g^{\chi}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \int P(d\chi)\chi_{\mathbf{x},\omega}\chi_{\mathbf{y},\omega'} \tag{4.55}$$
$$= \frac{2\pi}{L^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{|D_a(\mathbf{k})|^2+[\sigma_\chi(\mathbf{k})]^2}\begin{pmatrix} D_a^+(\mathbf{k}) & i\sigma_\chi(\mathbf{k})\\ -i\sigma_\chi(\mathbf{k}) & D_a^-(\mathbf{k})\end{pmatrix}_{\omega,\omega'}$$

where $\mathcal{D}_M$ is a shorthand for $\mathcal{D}_M^{-,-}$.

Remark. We define the *unperturbed scaling limit* as follows. Let $\beta(a)$ be fixed in such a way that, if $t = t(a) = \tanh\big(\beta(a)J\big)$ and $t_c(0) = \sqrt{2}-1$,

$$\frac{t(a) - t_c(0)}{t(a)} = \frac{a\sigma^0(a)}{2} \,, \tag{4.56}$$

where $\sigma^0(a) \neq 0$ for all $a \neq 0$ and $\lim_{a\to 0} \sigma^0(a) = m^*$. The limit as $a \to 0$ with $t(a)$ fixed in this way will be referred to as the unperturbed scaling limit. The explicit form of the propagator of the ψ field shows that in the unperturbed scaling limit it behaves as:

$$g^\psi(\mathbf{x}) \to \mathfrak{g}^0(\mathbf{x}) := \int \frac{d\mathbf{k}}{2\pi} \frac{e^{-i\mathbf{k}\,\mathbf{x}}}{\mathbf{k}^2 + (m^*)^2} \begin{pmatrix} ik_1 + k_2 & im^* \\ -im^* & ik_1 - k_2 \end{pmatrix} . \tag{4.57}$$

Note that the limiting propagator $\mathfrak{g}^0(\mathbf{x})$ is normalized in such a way that in the massless case, $m^* = 0$, it reduces to Eq.(**??**). In the same limit, the propagator of the χ field $g^\chi(\mathbf{x})$ tends to zero for every fixed $\mathbf{x} \in \mathbb{R}^2$; moreover, the combination $a^{-1}\int d\mathbf{x}\, g^\chi(\mathbf{x})$ tends to $-c_\chi\sigma_2$, where $c_\chi = \lim_{a\to 0} a\sigma_\chi(\mathbf{0}) = \frac{2}{t_c(0)}(t_c(0)+\sqrt{2}+1)$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix. In this sense, as a tends to zero:

$$g^\chi(\mathbf{x}) \simeq -ac_\chi\delta(\mathbf{x})\sigma_2 \,, \tag{4.58}$$

where $\delta(\mathbf{x})$ is the Dirac delta function.

The quadratic coupling $Q(\psi,\chi)$ in Eq.(4.53) can be eliminated by the linear change of variables

$$\widehat{\chi}_{\mathbf{k}} \to \widehat{\chi}_{\mathbf{k}} + C_\chi^{-1}(\mathbf{k})Q(\mathbf{k})\widehat{\psi}_{\mathbf{k}} \,, \tag{4.59}$$

which leaves $\Xi_{-,-}()$ invariant. Note that for small $\mathbf{k}$, the kernel $C_\chi^{-1}(\mathbf{k})Q(\mathbf{k})$ associated to this transformation is small, namely $C_\chi^{-1}(\mathbf{k})Q(\mathbf{k}) = O(a|\mathbf{k}|)$; that is, from a dimensional point of view, the action of $C_\chi^{-1}(\mathbf{k})Q(\mathbf{k})$ on $\widehat{\psi}_{\mathbf{k}}$ is the same as the action of the differential operator $a\partial_{\mathbf{x}}$. After the transformation Eq.(4.59) we can rewrite:

$$\Xi_{-,-}() = \frac{\overline{\mathcal{N}}_\psi}{\mathcal{N}_\psi} \int \overline{P}(d\psi)P(d\chi) e^{\overline{\mathcal{B}}(\psi,\chi,)+\overline{\mathcal{V}}(\psi,\chi,)} \,, \tag{4.60}$$

where $\overline{\mathcal{B}}(\psi,\chi,)$ and $\overline{\mathcal{V}}(\psi,\chi,)$ are the rewritings of $\mathcal{B}(\psi,\chi,)$ and $\mathcal{V}(\psi,\chi,)$, respectively, in terms of the new variables; moreover, the gaussian integration $\overline{P}(d\psi)$ is defined as

$$\overline{P}(d\psi) = \frac{1}{\overline{\mathcal{N}}_\psi}\Big[\prod_{\mathbf{k}\in\mathcal{D}_M}\prod_{\omega=\pm} d\widehat{\psi}_{\mathbf{k},\omega}\Big] \exp\Big\{-\frac{1}{4\pi L^2}\sum_{\mathbf{k}\in\mathcal{D}_M} \widehat{\psi}^T_{-\mathbf{k}}\overline{C}_\psi(\mathbf{k})\widehat{\psi}_{\mathbf{k}}\Big\} \,, \tag{4.61}$$

where the normalization constant $\overline{\mathcal{N}}_\psi$ is chosen in such a way that $\int \overline{P}(d\psi) = 1$ and

$$\overline{C}_\psi(\mathbf{k}) = C_\psi(\mathbf{k}) - Q(\mathbf{k})C_\chi^{-1}(\mathbf{k})Q(\mathbf{k}). \tag{4.62}$$

4.1.3 *Generalized Ising models*

From a physical point of view, there is no reason for which only nearest-neighbor spins should interact; much more reasonable is to assume that the interaction is short ranged, in the sense that it becomes weaker and weaker as more distant spins are considered. In the same way, it is also not very natural to exclude interactions involving four or a greater number of spins. Such considerations suggest to consider a more general Ising model with hamiltonian

$$H = H_I + \bar{\lambda}V \tag{4.63}$$

where $\bar{\lambda}$ is the coupling and V have the form

$$\sum_{m=2}^{\infty} \int d\mathbf{x}_1 d\mathbf{x}_m v_m(\mathbf{x}_1, .., \mathbf{x}_m) \prod_{i=1}^{m} \sigma_{\mathbf{x}_i}\sigma_{\mathbf{x}'_i} \tag{4.64}$$

where $\mathbf{x}, \mathbf{x}'$ are neraest neighbor and $v(\mathbf{x}_1, ..., \mathbf{x}_m)$ short ranged. We will consider for definiteness an interaction of the form

$$V = \sum_{\mathbf{x}} [\sigma_{x+1,x_0}\sigma_{x,x_0+1} + \sigma_{x+1,x_0-1}\sigma_{x,x_0}] \tag{4.65}$$

which can be rewritten as

$$V = \sum_{\mathbf{x}} [\sigma_{x,x_0}\sigma_{x+1,x_0}\sigma_{x,x_0}\sigma_{x,x_0+1} + \sigma_{x+1,x_0-1}\sigma_{x+1,x_0}\sigma_{x,x_0}\sigma_{x+1,x_0}] \tag{4.66}$$

The rest of this chapter is devoted to the proof of the following theorem, proved in [**?**],[**?**] to which we refer for more details.

Theorem 4.1. *For $\bar{\lambda}$ small enough there exists a bounded function $\bar{\nu(\lambda)}$ such that, if*

$$\tanh \beta_c J = \sqrt{2} - 1 + \nu(\bar{\lambda}) \tag{4.67}$$

then for suitable constants C_1, C_2

$$-C_1 \log|\beta - \beta_c| \leq |C_v| \leq -C_2 \log|\beta - \beta_c| \tag{4.68}$$

and, if $\mathbf{x}, \mathbf{x}'$ are nearest neighbor

$$| < \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}; \sigma_{\mathbf{y}}\sigma_{\mathbf{y}'} >_T | \leq C \frac{e^{-\kappa m(\beta)|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2} \tag{4.69}$$

where $m(\beta) = O(|\beta - \beta_c|)$ is the correlation length.

The above result establishes a form of universality for the Ising model; the critical indices for the specific heat and the correlation length are insensitive to the perturbation. On the contrary the value of the critical temperature is not universal but it depends from the detail of the perturbation.

4.1.4 *Fermionic representation of the generalized Ising model*

Also the partition function of the model (4.63),(4.64) can be written as a Grassman integral; we show this for definiteness only in the case (4.66) but the same analysis can be repeated for the general case (4.65).

Noting that

$$\sigma_{x+1,x_0}\sigma_{x,x_0+1} = \sigma_{x,x_0}\sigma_{x+1,x_0}\sigma_{x,x_0}\sigma_{x,x_0+1} \tag{4.70}$$

the partition function of (4.63) can be written as

$$Z = \sum_{\substack{\sigma=\pm 1 \\ \mathbf{x}\in\Lambda}} e^{-H_I(\sigma)} \prod_{\mathbf{x}} \{[1+\tanh\bar{\lambda}\sigma_{x,x_0}\sigma_{x+1,x_0}\sigma_{x,x_0}\sigma_{x,x_0+1}]$$
$$[1+\tanh\bar{\lambda}\sigma_{x,x_0}\sigma_{x+1,x_0}\sigma_{x+1,x_0-1}\sigma_{x+1,x_0})]\} \tag{4.71}$$

If Z_I is the partition function of the Ising model, we note that

$$\sigma_{x,x_0}\sigma_{x+1,x_0}e^{-H_I(\sigma)} = \frac{\partial}{\partial\beta J_{1;x,x_0;x+1,x_0}} Z_I(J_{\mathbf{x},\mathbf{x}'})|_{\{J_{\mathbf{x},\mathbf{x}'}\}=\{J\}} \tag{4.72}$$

and from (**??**) this derivative gives an extra factor $\tanh\beta J + \mathrm{sech}^2\beta J\bar{H}_{x,x_0}H_{x+1,x_0}$ in (4.21). In the same way

$$\sigma_{x,x_0}\sigma_{x,x_0+1}e^{-H_I(\sigma)} = \frac{\partial}{\partial\beta J_{2;x,x_0;x,x_0+1}} Z_I(J_{\mathbf{x},\mathbf{x}'})|_{\{J_{\mathbf{x},\mathbf{x}'}\}=\{J\}} \tag{4.73}$$

and this derivative gives a factor $\tanh\beta J + \mathrm{sech}^2\beta J\bar{V}_{x,x_0}V_{x,x_0+1}$. We can write than, if $\delta_{+,+}=1$ and $\delta_{+,-}=\delta_{-,+}=\delta_{-,-}=2$

$$Z = \sum_{\varepsilon,\varepsilon'} (-1)^{\delta_{\varepsilon,\varepsilon'}} Z^{\varepsilon,\varepsilon'} \tag{4.74}$$

where

$$Z^{\varepsilon,\varepsilon'} = (\cosh\beta J)^B 2^S \frac{1}{2}\int [\prod_{\mathbf{x}} dH_{\mathbf{x}}d\bar{H}_{\mathbf{x}}dV_{\mathbf{x}}d\bar{V}_{\mathbf{x}}] e^{S_{\varepsilon,\varepsilon'}}$$
$$\prod_{\mathbf{x}} \{[1+\tanh\bar{\lambda}(t+\mathrm{sech}^2\beta J\bar{H}_{x,x_0}H_{x+1,x_0})(t+\mathrm{sech}^2\beta J\bar{V}_{x,x_0}V_{x,x_0+1})]$$
$$[1+\tanh\bar{\lambda}(t+\mathrm{sech}^2\beta J\bar{H}_{x,x_0}H_{x+1,x_0})(t+\mathrm{sech}^2 J\bar{V}_{x+1,x_0-1}V_{x+1,x_0})]\}$$

The above expression can be rewritten as

$$\widehat{Z}^{\varepsilon,\varepsilon}_{2I} = (\cosh J)^B 2^S \frac{1}{2} \int [\prod_{\mathbf{x}} dH_{\mathbf{x}} d\bar{H}_{\mathbf{x}} dV_{\mathbf{x}} d\bar{V}_{\mathbf{x}} e^{S_{\varepsilon,\varepsilon}}] e^{\mathcal{V}} \tag{4.75}$$

and, if $f = log(1 + \lambda \tanh^2 \beta J)$

$$\mathcal{V} = \sum_{\mathbf{x}} [f + \widetilde{\lambda}[\bar{H}_{x,x_0} H_{x+1,x_0} + \bar{V}_{x,x_0} V_{x,x_0+1}] + \lambda \bar{H}_{x,x_0} H_{x+1,x_0} \bar{V}_{\mathbf{x}} V_{x,x_0+1}] \tag{4.76}$$
$$+ \sum_{\mathbf{x}} [f + \widetilde{\lambda}[\bar{V}_{x+1,x_0-1} V_{x+1,x_0} + \bar{H}_{x,x_0} H_{x+1,x_0}] + \lambda \bar{V}_{x+1,x_0-1} V_{x+1,x_0} \bar{H}_{\mathbf{x}} H_{x+1,x_0}]$$

It is easy in fact to verify that

$$e^{f+\widetilde{\lambda}[\bar{H}_{\mathbf{x}} H_{x+1,x_0} + \bar{V}_{\mathbf{x}} V_{x,x_0+1}] + \lambda \bar{H}_{\mathbf{x}} H_{x+1,x_0} \bar{V}_{\mathbf{x}} V_{x,x_0+1}} = (1 + \tag{4.77}$$
$$\lambda \tanh^2 \beta J)[1 + \widetilde{\lambda}[\bar{H}_{\mathbf{x}} H_{x+1,x_0} + \bar{V}_{\mathbf{x}} V_{x,x_0+1}] + (\lambda + (\widetilde{\lambda})^2) \bar{H}_{\mathbf{x}} H_{x+1,x_0} \bar{V}_{\mathbf{x}} V_{x,x_0+1}]$$

hence the equality between (4.76) and (4.75) holds with the identification

$$\begin{aligned} \widetilde{\lambda}(1 + \tanh \bar{\lambda} \tanh^2 \beta J) &= \tanh \bar{\lambda} \mathrm{sech}^2 \beta J \tanh \beta J \\ (1 + \tanh \bar{\lambda} \tanh^2 \beta J)(\lambda + (\widetilde{\lambda})^2) &= \tanh \bar{\lambda} \mathrm{sech}^4 \beta J \end{aligned} \tag{4.78}$$

The conclusion of the above computations is that even the "physically harmless" inclusion of next to nearest interactions has the effect that the Grassmann integral is not Gaussian, so that exact solvability is lost. The exact solvability is then a rather delicaty property, related to certain simplifying assmption in the model and which is immediately lost when they are removed.

We shall consider for simplicity the partition function $Z^{-,-}$, *i.e.* the partition function in which the Grassmannian variables verify antiperiodic boundary conditions. The other partition functions in (4.74) admit similar expressions. Furthermore it will appear that the logarithm of $Z^{\varepsilon,\varepsilon'}$ divided by its expression for $\lambda = 0$ is insensitive to boundary conditions up to corrections which are exponentially small in the size L of the system in the thermodynamic limit in which $L \to \infty$ (and if the limit $\beta \to \beta_c$ is performed after the thermodinamic limit).

4.1.5 *Integration of the χ-variables*

We integrate the heavy χ fields

$$\int \bar{P}(d\psi) e^{L^2 \mathcal{N} + \mathcal{V}^{(1)}(\psi)} = \int P(d\psi) \int P(d\chi) e^{Q(\chi,\psi) + \mathcal{V}(\psi,\chi)} \tag{4.79}$$

where $\mathcal{N}$ is a constant, $\bar{P}(d\psi)$ is given by (**??**) and

$$\mathcal{V}^{(1)} = \sum_{n\geq 1} \sum_{\underline{\alpha},\underline{\omega},\underline{\varepsilon}} \sum_{\mathbf{x}_1,..,\mathbf{x}_{2n}} W_{\underline{\alpha},\underline{\varepsilon},2n}(\mathbf{x}_1,..,\mathbf{x}_{2n}) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} ... \partial^{\alpha_{2n}} \psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}}$$

$$|\widehat{W}_{\underline{\alpha},\underline{\varepsilon},n}(\mathbf{k}_1,...\mathbf{k}_{n-1})| \leq L^2 C^n |\lambda|^{n/2}, \qquad n \geq 2 \tag{4.80}$$

The term with $n = 1$ can be written as

$$\sum_{\omega} \sum_{\mathbf{x}} [i\omega \bar{\nu}_1 \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{x}} + \psi_{\mathbf{x}}(a_1 \partial_0 + a_2 \partial_1)\psi_{\mathbf{x}} + \bar{\psi}_{\mathbf{x}}(\bar{a}_1 \partial_0 + \bar{a}_2 \partial_1)\bar{\psi}_{\mathbf{x}}$$
$$+ \sum_{\mathbf{x}_1,\mathbf{x}_2} \sum_{\{\omega\}} \sum_{\alpha_1+\alpha_2\geq 2,\varepsilon_1,\varepsilon_2} W_{\underline{\alpha},\underline{\varepsilon}}(\mathbf{x}_1,\mathbf{x}_2) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1} \partial^{\alpha_2} \psi^{\varepsilon_2}_{\mathbf{x}_2} \tag{4.81}$$

with $\bar{\nu}_1, a_i, \bar{a}_i = O(\lambda)$ and $|\widehat{W}_{\underline{\alpha},\underline{\varepsilon}}(\mathbf{k})| \leq L^2 C|\lambda|$

Calling

$$-\overline{\mathcal{V}}(\psi,\chi) = Q(\psi,\chi) + \widetilde{\lambda} V(\psi,\chi)\ , \tag{4.82}$$

the above result can be obtained from

$$\int P(d\chi) e^{-\overline{\mathcal{V}}(\psi,\chi)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \mathcal{E}^T_{\chi}(\overline{\mathcal{V}}; n)\ . \tag{4.83}$$

where $\mathcal{E}^T_{\chi}$ is truncated expectation with respect to $P(d\chi)$. Also for $\mathcal{E}^T_{\chi}$ holds a formula similar to (2.118), namely

$$\mathcal{E}^T_{\chi}(\widetilde{\chi}(P_{v_1}),\ldots,\widetilde{\chi}(P_{v_n})) = \sum_{T} \prod_{\ell\in T} g^{\chi}(\mathbf{x}(f^1_\ell) - \mathbf{x}(f^2_\ell)) \int dP_T(\mathbf{t}) \mathrm{Pf} G^T(\mathbf{t}) \tag{4.84}$$

where T and $dP_T(\mathbf{t})$, $\ell \in T$ and f^1_ℓ, f^2_ℓ the field labels associated to the points connected by ℓ; finally if $2n = \sum_{i=1}^{s} |P_{v_i}|$, then $G^T(\mathbf{t})$ is a $(2n - 2s + 2) \times (2n - 2s + 2)$ antisymmetrix matrix, whose elements are given by $G^T_{f,f'} = t_{i(f),i(f')} g_{\chi}(f, f')$, where: $f, f' \notin F_T$ and $F_T = \cup_{\ell\in T}\{f^1_\ell, f^2_\ell\}$.

Formula (4.84) can be obtained exactly as in the proof of Lemma 3.1, starting from the formula (replacing (2.122))

$$\mathcal{E}_{\chi}\Big(\prod_{j=1}^{s} \widetilde{\chi}(P_j)\Big) = PfG = (-1)^n \int \mathcal{D}\chi \, \exp\Big[\frac{1}{2}(\chi, G\chi)\Big]\ , \tag{4.85}$$

where: the expectation $\mathcal{E}_{\chi}$ is w.r.t. $P(d\chi)$; if $2m = \sum_{j=1}^{s} |P_j|$, G is the $2m \times 2m$ antisymmetric matrix with entries $G_{f,f'} = g^{\chi}(\mathbf{x}(f) - \mathbf{x}(f'))$; and

$$\mathcal{D}\chi = \prod_{j=1}^{n} \prod_{f\in P_j} \mathrm{d}\chi^{\alpha(f)}_{\mathbf{x}(f)} \qquad (\chi, G\chi) = \sum_{f,f'\in\cup_i P_i} \chi^{\alpha(f)}_{\mathbf{x}(f)} G_{f,f'} \chi^{\alpha(f')}_{\mathbf{x}(f')}\ . \tag{4.86}$$

By (**??**), (4.86) and the fact that $m_\chi = O(1)$, the bound (4.81) can be obtained by proceeding as in the proof of Lemma 3.1, using that $|PfG^T| = \sqrt{|\det G^T|}$ and bounding $\det G^T$ by Gram inequality.

As in the case of Wilson fermions discussed in the previous chapter, in which the interaction modifies the mass, we expect in this case that the interaction changes the value of the critical temperature; in order to take into account this fact it is convenient to write

$$\bar{P}(d\psi) = \widetilde{P}(d\psi) e^{-i\nu F_\nu(\psi)} , \tag{4.87}$$

where $\widetilde{P}(d\psi)$ is equal to $\bar{P}(d\psi)$ with m_1 replaced by $\sigma = \frac{m_\psi + \bar{\nu}}{1 - m_\psi/2}$, $\nu = \frac{\bar{\nu}}{1 - m_\psi/2}$ and

$$F_\nu(\psi) = \sum_{\mathbf{x}} \bar{\psi}_{\mathbf{x}} \psi_{\mathbf{x}} \tag{4.88}$$

4.1.6 *Ashkin-Teller or Vertex models*

We have seen in the previous chapter that any perturbation of the nearest neighbor Ising model, obtained adding small short range perturbations in the spins, does not change the critical properties of the specific heat or the asymptotic behavior of energy correlations. The reason why universality (at least for such quantities) holds appears, in this approach, quite subtle; when mapped in a fermionic system, all the local monomials in the fermions with degree higher than two are irrelevant. The only possible non irrelevant term is $\psi_{\mathbf{x}} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{x}} \bar{\psi}_{\mathbf{x}}$ which is indeed vanishing by the anticommutativity properties of Grasssman variables.

We consider in this and in the following chapter other spin lattice models, which can be still considered as perturbations of the Ising model (more exactly, they can be mapped in two copies of the Ising model), in which however universality can be violated.

The first model we consider is the *Ashkin-Teller* model, see Ref.[**?**], introduced as a generalization of the Ising model to a four component system. The assumption that the spins have only two values is physically unrealistic, as a magnetic dipole can have a continuum of pointing directions. A step in this direction is provided by the Ashkin-Teller model, in which each site of a bidimensional lattice is occupied by one of four kinds of atoms: A, B, C, D. Two neighbouring atoms interact with an energy: ε_0 for AA, BB, CC, DD; ε_1 for AB, CD; ε_2 for AC, BD; and ε_3 for AD, BC.

This Ashkin-Teller model can be expressed in terms of Ising spins; one associates to each site of the square lattice two spins variables, $\sigma^{(1)}_{\mathbf{x}}$ and $\sigma^{(2)}_{\mathbf{x}}$.

If $(\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}}) = (+,+)$ there is an atom A associated to $\mathbf{x}$, if $(\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}}) = (+,-)$ there is an atom B, if $(\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}}) = (-,+)$ there is an atom C and if $(\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}}) = (-,-)$ there is an atom D.

The partition function is then given by $Z_\Lambda = \sum_{\sigma^{(1)},\sigma^{(2)}} e^{-H_\Lambda}$, where

$$H_\Lambda(\sigma^{(1)}, \sigma^{(2)}) = J^{(1)} H_I(\sigma^{(1)}) + J^{(2)} H_I(\sigma^{(2)}) - J^{(3)} V(\sigma^{(1)}, \sigma^{(2)}) + J^{(4)}$$

$$H_I(\sigma^{(j)}) = -\sum_{\mathbf{x}\in\Lambda} [y_1 \sigma^{(j)}_{\mathbf{x}} \sigma^{(j)}_{\mathbf{x}+\widehat{e}_1} + y_2 \sigma^{(j)}_{\mathbf{x}} \sigma^{(j)}_{\mathbf{x}+\widehat{e}_0}] \tag{4.89}$$

$$V_{AT}(\sigma^{(1)}, \sigma^{(2)}) = \sum_{\mathbf{x}\in\Lambda} [\sigma^{(1)}_{x,x_0} \sigma^{(1)}_{x+1,x_0} \sigma^{(2)}_{x,x_0} \sigma^{(2)}_{x+1,x_0} + \sigma^{(1)}_{x,x_0} \sigma^{(1)}_{x,x_0+1} \sigma^{(2)}_{x,x_0} \sigma^{(2)}_{x,x_0+1}]$$

where H_I is the Ising model hamiltonian, $y_1 = y_2 = 1$, Λ is a square subset of $\mathbb{Z}^2$ of side L and

$$\begin{aligned} -J^{(1)} &= (\varepsilon_0 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)/4 \quad & -J^{(2)} &= (\varepsilon_0 + \varepsilon_2 - \varepsilon_3 - \varepsilon_1)/4 \\ -J^{(3)} &= (\varepsilon_0 + \varepsilon_3 - \varepsilon_1 - \varepsilon_2)/4 \quad & -J^{(4)} &= (\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3)/4 \end{aligned} \tag{4.90}$$

The Ashkin-Teller model is not exactly solvable, except for some special choice of the parameters corrosponding to $J^{(3)} = 0$ in which reduces to two independent Ising models; in particular the specific heat has a log-singularity in corrispondence of the critical temperatures located at $\tanh J^{(1)}\beta_c = \sqrt{2} - 1$ and $\tanh J^{(2)}\beta_c = \sqrt{2} - 1$; in the case $J^{(1)} = J^{(2)}$ the Ashkin-Teller model is called isotropic and the two critical temperatures coincides. In this chapter we will discuss the isotropic model, and the anisotropic case will be discussed in the following chapter.

Another important lattice statistical mechanics model is the *8-Vertex model*, see Ref.[**?**], in which to each site of a bidimensional lattice one associates one among 8 possible vertices composed by four arrows pointing in or out the center. To each vertex is associated an energy, and ε_A is the common energy of the first and second vertex, ε_B the common energy of the third and fourth vertex and so on. Such a model has been introduced as a generalization of the "ice-type" models, describing crystals with hydrogen bounds which can be conveniently described by arrows placed on the bounds.

Also the 8V model can be exactly mapped, see Ref.[**?**], in two Ising models coupled by a four spin interaction bilinear in the energy densities of the two sublattices, with the following Hamiltonian

$$H_\Lambda(\sigma^{(1)}, \sigma^{(2)}) = J H_I(\sigma^{(1)}) + J H_I(\sigma^{(2)}) + \lambda V_{8V}(\sigma^{(1)}, \sigma^{(2)}) \tag{4.91}$$

where

$$V_{8V}(\sigma^{(1)}, \sigma^{(2)}) = -\lambda \sum_{\mathbf{x}} \Big\{ [\sigma^{(1)}_{x,x_0} \sigma^{(1)}_{x+1,x_0} \sigma^{(2)}_{x,x_0} \sigma^{(2)}_{x,x_0+1} + \sigma^{(1)}_{x,x_0} \sigma^{(1)}_{x,x_0+1} \sigma^{(2)}_{x-1,x_0+1} \sigma^{(2)}_{x,x_0+1}] \Big\} \tag{4.92}$$

and with the identifications, if $Jy^{(1)} = J^{(1)}, Jy^{(2)} = J^{(2)}$, $a = e^{-\beta\varepsilon_A}, b = e^{-\beta\varepsilon_B}, c = e^{-\beta\varepsilon_C}, d = e^{-\beta\varepsilon_D}$

$$a = e^{\beta(J^{(1)}+J^{(2)}+\lambda)} \quad b = e^{\beta(-J^{(1)}-J^{(2)}+\lambda)} \tag{4.93}$$

$$c = e^{\beta(-J^{(1)}+J^{(2)}-\lambda)} \quad b = e^{\beta(J^{(1)}-J^{(2)}-\lambda)} \tag{4.94}$$

Contrary to the Ashkin-Teller model, the 8V model can be exactly solved and some critical exponents can be computed.

Fig. 4.1 The spins involved in the interaction of the models in (4.95). The heavy dots and lines or the light dots and lines mark the Ising lattices and the nearest neighbors Ising couplings. The ellipses symbolize the Ashkin–Teller four spins interactions (λa–couplings) and the circles the Baxter four spins interactions (λb) couplings.

The methods introduced in the previous chapters allow a detailed analyis of such model, at least in certain region of the parameters; remarkably, such analysis donot require any property of exact solvability.

For fixing the ideas we consider in this chapter a model with hamiltonian

$$H = JH_I(\sigma^{(1)}) + JH_I(\sigma^{(2)}) - \lambda a V_{AT} - \lambda b V_{8V} \tag{4.95}$$

reducing to the 8V or AT model for $b = 0$ or $a = 0$; from the analysis it will appear clear that similar results holds in a rather general class of models.

4.1.7 *Fermionic representation*

The partition function of the model (4.95) is

$$Z_{2I} = \sum_{\substack{\sigma^{(1)}_{\mathbf{x}}=\pm 1 \\ \mathbf{x}\in\Lambda}} \sum_{\substack{\sigma^{(2)}_{\mathbf{x}}=\pm 1 \\ \mathbf{x}=\Lambda}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} e^{-V(\sigma^{(1)},\sigma^{(2)})} \tag{4.96}$$

Setting $\widehat{\lambda a} = \tanh(\lambda\, a), \widehat{\lambda b} = \tanh(\lambda\, b)$ we see that Z_{2I} becomes $(\cosh\, \lambda a \cosh\, \lambda b)^{2S}$ times $\widehat{Z}_{2I}$ with

$$\widehat{Z}_{2I} = \sum_{\substack{\sigma^{(1)}=\pm 1 \\ \mathbf{x}\in\Lambda}} \sum_{\substack{\sigma^{(2)}=\pm 1 \\ \mathbf{x}\in\Lambda}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} \tag{4.97}$$

$$\cdot \prod_{\mathbf{x}\in\Lambda} [1 + \langle \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0}] \prod_{\mathbf{x}\in\Lambda} [1 + \langle \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]$$

$$\sigma \prod_{\mathbf{x}\in\Lambda} [1 + {}^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}] \prod_{\mathbf{x}\in\Lambda} [1 + {}^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}]$$

Proceeding as in §9.4 we can express $\widehat{Z}_{2I}$ as a sum of sixteen partition functions labeled by $\gamma_1, \gamma_2 = (\varepsilon^{(1)}, \varepsilon'^{(1)}), (\varepsilon^{(2)}, \varepsilon'^{(2)})$ (corresponding to choosing each ε and ε' as $\pm$)

$$\widehat{Z}_{2I} = (\cosh\, \lambda a \cosh\, \lambda b)^{2S} \sum_{\gamma_1,\gamma_2} (-1)^{\delta_{\gamma_1}+\delta_{\gamma_2}} \widehat{Z}^{\gamma_1,\gamma_2}_{2I} \tag{4.98}$$

with $\widehat{Z}_{2I}$ can be written as

$$\widehat{Z}^{\gamma_1,\gamma_2}_{2I} = \frac{(\cosh \beta J)^{2B} 2^{2S}}{4} \int \prod_{j=1}^{2} \Big([\prod_{\mathbf{x}\in\Lambda} dH^{(\alpha_j)}_{\mathbf{x}} d\overline{H}^{(\alpha_j)}_{\mathbf{x}} dV^{(\alpha_j)}_{\mathbf{x}} d\overline{V}^{(\alpha_j)}_{\mathbf{x}}] \, e^{S^{(\alpha_j)}_{J,\gamma_j}} \Big) \, e^{-\mathcal{V}} \tag{4.99}$$

with

$$\mathcal{V} = \mathcal{V}_a + \mathcal{V}_b \tag{4.100}$$

and, if $f_i = \log(1 + \widehat{\lambda[i]} \tanh^2 \beta J)$ and $[i] = a, b$

$$-\mathcal{V}_a = \sum_{\mathbf{x}\in\Lambda} [2f_a + \widetilde{\lambda}_a [\overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} + \overline{H}^{(2)}_{x,x_0} H^{(2)}_{x+1,x_0}] + \lambda_a \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} \overline{H}^{(2)}_{\mathbf{x}} H^{(2)}_{x+1,x_0}$$

$$+\widetilde{\lambda}_a [\overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} + \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] + \lambda_a \overline{V}^{(1)}_{\mathbf{x}} V^{(1)}_{x,x_0+1} \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] \tag{4.101}$$

$$-\mathcal{V}_b = \sum_{\mathbf{x}\in\Lambda} [2f_b + \widetilde{\lambda}_b [\overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} + \overline{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1}] + \lambda_b \overline{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0} \overline{V}^{(2)}_{\mathbf{x}} V^{(2)}_{x,x_0+1}]$$

$$+\widetilde{\lambda}_b [\overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} + \overline{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1}] + \lambda_b \overline{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1} \overline{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1}]$$

where

$$\widetilde{\lambda}_i (1 + \widehat{\lambda[i]} \tanh^2 \beta J) = \widehat{\lambda[i]} \mathrm{sech}^2 \beta J \tanh \beta J$$

$$(1 + \widehat{\lambda[i]} \tanh^2 \beta J)(\lambda_i + (\widetilde{\lambda}_i)^2) = \widehat{\lambda[i]} \mathrm{sech}^4 \beta J \tag{4.102}$$

For small λ it is $\widetilde{\lambda}_i = \lambda[i](\tanh J \mathrm{sech}^2 \beta J + O(\lambda))$, $\lambda_i = \lambda[i](\mathrm{sech}^4 \beta J + O(\lambda))$.

We shall consider for simplicity the partition function $\widehat{Z}_{2I}^{-,-,-,-} = i\widehat{Z}_{2I}^{-}$, *i.e.* the partition function in which all Grassmannian variables verify antiperiodic boundary conditions . The other fifteen partition functions in (4.98) admit similar expressions. The logarithm of $Z_{2I}^{\gamma_1,\gamma_2}$ divided by its expression for $\lambda = 0$ is insensitive to boundary conditions up to corrections which are exponentially small in the size L of the system in the thermodynamic limit in which $L \to \infty$ so that it will turn out that it is sufficient to study just one of the sixteen partition functions and $\widehat{Z}_{2I}^{-,-,-,-}$ is chosen here (arbitrarily).

As in chap.9 we perform the following change of variables, $\alpha = 1, 2$

$$\begin{aligned} &\overline{H}_{\mathbf{x}}^{(\alpha)} + iH_{\mathbf{x}}^{(\alpha)} = e^{i\frac{\pi}{4}}\psi_{\mathbf{x}}^{(\alpha)} - e^{i\frac{\pi}{4}}\chi_{\mathbf{x}}^{(\alpha)} \qquad \overline{H}_{\mathbf{x}}^{(\alpha)} - iH_{\mathbf{x}}^{(\alpha)} = e^{-i\frac{\pi}{4}}\overline{\psi}_{\mathbf{x}}^{(\alpha)} - e^{-i\frac{\pi}{4}}\overline{\chi}_{\mathbf{x}}^{(\alpha)} \\ &\overline{V}_{\mathbf{x}}^{(\alpha)} + iV_{\mathbf{x}}^{(\alpha)} = \psi_{\mathbf{x}}^{(\alpha)} + \chi_{\mathbf{x}}^{(\alpha)} \qquad \overline{V}_{\mathbf{x}}^{(\alpha)} - iV_{\mathbf{x}}^{(\alpha)} = \overline{\psi}_{\mathbf{x}}^{(\alpha)} + \overline{\chi}_{\mathbf{x}}^{(\alpha)} \end{aligned} \tag{4.103}$$

which replaces the $H, V, \overline{H}, \overline{V}$ variables with "Majorana variables" $\psi^{(\alpha)}, \chi^{\alpha)}$.

Subsequently we replace the Majorana variables with Dirac variables by setting

$$\begin{aligned} \psi_{1,\mathbf{x}}^{\mp} &= \frac{1}{\sqrt{2}}(\psi_{\mathbf{x}}^{(1)} \pm i\psi_{\mathbf{x}}^{(2)}), \qquad \psi_{-1,\mathbf{x}}^{\mp} = \frac{1}{\sqrt{2}}(\overline{\psi}_{\mathbf{x}}^{(1)} \pm i\overline{\psi}_{\mathbf{x}}^{(2)}) \\ \chi_{1,\mathbf{x}}^{\mp} &= \frac{1}{\sqrt{2}}(\chi_{\mathbf{x}}^{(1)} \pm i\chi_{\mathbf{x}}^{(2)}), \qquad \chi_{-1,\mathbf{x}}^{\mp} = \frac{1}{\sqrt{2}}(\overline{\chi}_{\mathbf{x}}^{(1)} \pm i\overline{\chi}_{\mathbf{x}}^{(2)}) \ . \end{aligned} \tag{4.104}$$

The final expression is

$$\widehat{Z}_{2I}^{-} = \mathcal{N} \int P(d\psi)P(d\chi)e^{Q(\chi,\psi)-\mathcal{V}(\chi,\psi)}. \tag{4.105}$$

where if ϕ denotes either ψ or χ

$$P(d\phi) = \mathcal{N}_{\phi}^{-1} \prod_{\mathbf{k}} \prod_{\omega=\pm 1} d\phi_{\mathbf{k},\omega}^{+} d\phi_{\mathbf{k},\omega}^{-} \tag{4.106}$$

$$\exp[\frac{t}{2L^2} \sum_{\mathbf{k}} \xi_{\mathbf{k}}^{(-),\mathbf{T}} A_{\phi}(\mathbf{k}) \xi^{(+)}{}_{\mathbf{k}}] \tag{4.107}$$

where

$$A_{\phi}(\mathbf{k}) = \begin{pmatrix} i\sin k + \sin k_0 & -im_{\phi}(\mathbf{k}) \\ im_{\phi}(\mathbf{k}) & i\sin k - \sin k_0 \end{pmatrix} \xi^{\mathbf{T}}{}_{\mathbf{k}} = (\phi_{\mathbf{k},1}^{-}, \phi_{\mathbf{k},-1}^{-})$$

with m_{ϕ} defined, *differently* for $\phi = \psi$ (choose $-t$) and for $\phi = \chi$ (choose $+t$), by

$$\frac{t}{2} m_{\phi}(\mathbf{k}) = (\sqrt{2} - 1 \mp t) + \frac{t}{2}(\cos k_0 + \cos k - 2). \tag{4.108}$$

Note the remarkable similarity with $P(\psi)$ with the integartion of lattice relatistic fermions in $d = 2$ discussed in chapt. 9.

Finally $Q(\chi, \psi)$ and $\mathcal{V}(\chi, \psi)$ are obtained respectively from (**??**) through the change of variables (4.103) and (4.105).

4.1.8 *Anomalous behaviour*

The main difference with respect to the case treated in the previous chapter is that the system can be mapped in terms of Dirac instead of Majorana fermions; as a consequence there local quartic monomials in the fields that with non negative dimension, namely $\psi^+_{+,\mathbf{x}}\psi^-_{+,\mathbf{x}}\psi^+_{-,\mathbf{x}}\psi^-_{-,\mathbf{x}}$. It holds the following result (originally proved in Ref.[**?**; **?**], where more details can be found), if $H_{\mathbf{x}}(\sigma^{(1)},\sigma^{(2)})$ is the summand in (4.95)

Theorem 4.2. *If $a = 0$ or $b = 0$, for λ small enough one can uniquely define $\nu(\lambda)$, bounded function in λ, so that the model is critical at $t = t_c = \sqrt{2}-1+\nu(\lambda)$. This means that, for $|t-t_c|$ strictly positive and small enough*

$$\lim_{|\Lambda|\to\infty} \langle H_{\mathbf{x}}(\sigma^{(1)},\sigma^{(2)}) H_{\mathbf{y}}(\sigma^{(1)},\sigma^{(2)})\rangle_T = \Omega^a(\mathbf{x},\mathbf{y}) + \Omega^b(\mathbf{x},\mathbf{y}) \tag{4.109}$$

and the bounds, for any integer N

$$|\Omega^a(\mathbf{x},\mathbf{y})| \le \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+2\eta_1}} \frac{C_N}{1+(\Delta|\mathbf{x}-\mathbf{y}|)^N} \tag{4.110}$$

$$|\Omega^b(\mathbf{x},\mathbf{y})| \le \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+\vartheta}} \frac{C_N}{1+(\Delta|\mathbf{x}-\mathbf{y}|)^N} \tag{4.111}$$

hold, with $\vartheta > 0$ a constant and correlation length Δ^{-1} and critical indices η_1, η_2 given by

$$\Delta = |t-t_c|^{1+\eta_2},\ \eta_1(\lambda) = -a_1\,(a+b)\,\lambda+O(\lambda^2) \quad \eta_2(\lambda) = -a_2\,(a+b)\,\lambda+O(\lambda^2) \tag{4.112}$$

with $a_1 > 0, a_2 > 0$ constants. Furthermore if $1 \le |\mathbf{x}| \le \Delta^{-1}$ the correlation is asymptotic to Ω^a in the sense that Ω^b is neglegible because

$$\Omega^a(\mathbf{x},\mathbf{y}) = \frac{1+A(\mathbf{x}-\mathbf{y})}{(\mathbf{x}-\mathbf{y})^{2+2\eta_1}}, \qquad |A(\mathbf{x})| \le C\left[|\lambda| + (\Delta|\mathbf{x}|)^{\frac{1}{2}}\right] \tag{4.113}$$

Finally the specific heat C_v verifies

$$C_1 \frac{1}{2\eta_1}[1-|\Delta|^{2\eta_1}] \le C_v^\lambda \le C_2 \frac{1}{2\eta_1}[1-|\Delta|^{2\eta_1}] \tag{4.114}$$

where C_1, C_2 are positive constants.

Note that the logarithmic singularity of the specific heat of the Ising model is removed or changed in a power law (with a non universal critical index) depending on the sign of the interaction. Moreover also the critical index of the correlation length is changed; universality is violated in such models and the critical behaviour is expressed in term of non-universal critical indices.

4.1.9 *The Grassmann representation of the 2d Ising model with open boundary conditions.*

In order to represent the sum over multipolygons in (2.35) as a Grassmann integral, we first associate to each site $\mathbf{x} \in \Lambda$, a set of four Grassmann variables, $\overline{H}_{\mathbf{x}}, H_{\mathbf{x}}, \overline{V}_{\mathbf{x}}, V_{\mathbf{x}}$, that must be thought as associated to four new sites drawn very near to $\mathbf{x}$ and to its right, left, up side, down side respectively, see Fig A.1. We shall denote these sites by $R_{\mathbf{x}}, L_{\mathbf{x}}, U_{\mathbf{x}}, D_{\mathbf{x}}$ respectively.

Fig. 4.2 The four Grassmann fields associated to the sites $\mathbf{x}$ and $\mathbf{y}$.

If $t = \tanh \beta J$, we consider the action

$$S(t) = t \sum_{\mathbf{x} \in \Lambda} \left[\overline{H}_{\mathbf{x}} H_{\mathbf{x}+\widehat{e}_1} + \overline{V}_{\mathbf{x}} V_{\mathbf{x}+\widehat{e}_0} \right] + \sum_{\mathbf{x} \in \Lambda} \left[\overline{H}_{\mathbf{x}} H_{\mathbf{x}} + \overline{V}_{\mathbf{x}} V_{\mathbf{x}} + \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}} + V_{\mathbf{x}} \overline{H}_{\mathbf{x}} + H_{\mathbf{x}} \overline{V}_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}} \right] \quad (4.115)$$

where $\widehat{e}_1, \widehat{e}_0$ are the coordinate versors in the horizontal and vertical directions, respectively. Open boundary conditions are assumed.

The following identity holds:

$$\frac{Z}{2^{L^2} (\cosh \beta J)^B} = (-1)^{L^2} \int \prod_{\mathbf{x} \in \Lambda} \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}} e^{S(t)} \quad (4.116)$$

where Ξ_I in the l.h.s. is calculated using open boundary conditions; below is given the proof of this classical result (following the exposition in [**?**]).

In order to prove (4.116) we expand the exponential in the r.h.s., we integrate term by term the Grassmann variables, and we get a summation over terms that we want to put in correspondence with the terms in the summation over mutipolygons of (2.35).

We represent, as in Chapt.2, each Grassmann field as an half line so that every term obtained by the contraction of the Grassmann variables is represented by the union of the lines. The figure one obtaines (called a dimer)

exactly coincide with a multipolygon if one shrinks the sites $R_{\mathbf{x}}, L_{\mathbf{x}}, U_{\mathbf{x}}, D_{\mathbf{x}}$ to let them coincide with $\mathbf{x}$. This graphical construction allows to put in correspondence each dimer with a unique multipolygon. We then have to show that the total weight of the dimer corresponding to the same multipolygon γ is exactly $(-1)^{L^2} t^{|\gamma|}$, where $(-1)^{L^2}$ is the same factor appearing in the r.h.s. of (4.116) and, if $|\gamma|$ is the length of γ, $t^{|\gamma|}$ is the weight (2.35) assigns to γ.

We first note that the correspondence between dimers and multipolygons is not one to one, because an empty site $\mathbf{x}$ in the multipolygon representation corresponds to three different contractions of Grassmann fields, that is either to $\int \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}}\ \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} V_{\mathbf{x}}$, or to $\int \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}}\ V_{\mathbf{x}} \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}}$, or to $\int \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}}\ V_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}}$. The total contribution of these three contractions is:

$$\int \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}} \tag{4.117}$$
$$(\overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} V_{\mathbf{x}} + V_{\mathbf{x}} \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}}) = 1 - 1 - 1 = -1$$

as wanted.

It is easy to realize that, unless for the above ambiguity, the correspondence between dimers and multipolygons is unique. And, since each side of a dimer is weighted by a factor t and each empty site is weighted by (-1), the weights of the corresponding figures are the same, at least *in absolute value*. From now on we shall extract from the weight of γ the contribution of the empty sites together with the trivial factor $t^{|\gamma|}$ (that is we redefine the weight of γ by dividing it by $(-1)^{L^2 - n_\gamma} t^{|\gamma|}$, where n_γ is the number of sites belonging to γ, possibly different from $|\gamma|$, if γ has self intersections).

We are then left with proving that the weight of a dimer (as just redefined) is exactly $(-1)^{n_\gamma}$; in this way the sign of every configuration of dimers together with the minus signs of the empty sites, (4.118), would reproduce exactly the factor $(-1)^{L^2}$ in (4.116).

We start with considering the simplest dimer, that is the square with unit side. Let us denote its corner sites with $(0,0) \equiv \mathbf{x}_1$, $(1,0) \equiv \mathbf{x}_2$, $(1,1) \equiv \mathbf{x}_3$, $(0,1) \equiv \mathbf{x}_4$ and let us prove that its weight is $(-1)^4 = 1$. The explicit expression of its weight in terms of Grassmann integrals, as generated by the expansion of the exponent in (4.116) is:

$$\int \prod_{i=1}^{4} \mathrm{d}\overline{H}_{\mathbf{x}_i} \mathrm{d}H_{\mathbf{x}_i} \mathrm{d}\overline{V}_{\mathbf{x}_i} \mathrm{d}V_{\mathbf{x}_i} \Big[\overline{H}_{\mathbf{x}_1} H_{\mathbf{x}_2} \cdot \tag{4.118}$$
$$V_{\mathbf{x}_2} \overline{H}_{\mathbf{x}_2} \cdot \overline{V}_{\mathbf{x}_2} V_{\mathbf{x}_3} \cdot \overline{V}_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_3} \cdot (-H_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_4}) \cdot H_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_4} \cdot (-V_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_1}) \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \Big]$$

In the previous equation, we wrote the different binomials corresponding to the segments of the dimer following the anticlockwise order, starting from $\overline{H}_{\mathbf{x}_1}$. By collecting the minus signs and by permutating the position of $\overline{H}_{\mathbf{x}_1}$ from the first to the last position, we find that (4.118) is equal to

$$-\int \prod_{i=1}^{4} \mathrm{d}\overline{H}_{\mathbf{x}_i} \mathrm{d}H_{\mathbf{x}_i} \mathrm{d}\overline{V}_{\mathbf{x}_i} \mathrm{d}V_{\mathbf{x}_i} \cdot \tag{4.119}$$
$$\Big[H_{\mathbf{x}_2} V_{\mathbf{x}_2} \overline{H}_{\mathbf{x}_2} \overline{V}_{\mathbf{x}_2} \cdot V_{\mathbf{x}_3} \overline{V}_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_3} H_{\mathbf{x}_3} \cdot \overline{H}_{\mathbf{x}_4} H_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_4} V_{\mathbf{x}_4} \cdot \overline{V}_{\mathbf{x}_1} V_{\mathbf{x}_1} H_{\mathbf{x}_1} \overline{H}_{\mathbf{x}_1}\Big]$$

where now we wrote separated from a dot the contributions corresponding to the same site. The explicit computation of (4.119) gives $-[(-1)(-1)(+1)(-1)] = +1$, as desired.

Let us now consider a generic dimer γ not winding up the lattice and without self intersections, and let us prove by induction that its weight is $(-1)^{n_\gamma}$. We will then assume that the dimers with number of sites $k \le n_\gamma$ have weights $(-1)^k$. The first step from which the induction starts is the case $k = 4$, that we have just considered.

Let us consider the smallest rectangle R containing γ. Necessarely, each side of R has non empty intersection with γ. Let us enumerate the corners of γ which are also extremes of straight segments belonging to the sides of R, starting from the leftmost among the lowest of these points (possibly coinciding with the lower left corner of R) and proceeding in anticlockwise order; call $\mathbf{x}_j$ the site with label j. Note that two consecutive indeces $j, j+1$ could represent the same site $\mathbf{x}_j \equiv \mathbf{x}_{j+1} \in \Lambda_M$; in that case $\mathbf{x}$ would be a corner of R. Call $2N$ the cardinality of the set of the enumerated points (it is even by construction) and let us identify the label $2N+1$ with the label 1.

Let us denote with the symbol $(2j-1 \to 2j)$, $j = 1, \dots, N$, the product of Grassmann fields corresponding to the straight line connecting the point $2j-1$ with $2j$ (not including the fields located in $2j-1$ and in $2j$), written in the anticlockwise order and with the sign induced by the expansion of the exponential in (4.116). That is, if the two fields belonging to a binomial appearing in (4.115), written following the anticlockwise order, are in the same order as they appear in (4.115), we will assign a + sign to the second of those two fields (of course, second w.r.t. the anticlockwise order); otherwise a − sign. As an example, if $2j-1$ and $2j$ are two points on the upper horizontal side of R, $(2j-1 \to 2j)$ would be equal to

$$(-\overline{H}_{\mathbf{x}_{2j-1}-\widehat{e}_1}) \overline{V}_{\mathbf{x}_{2j-1}-\widehat{e}_1} V_{\mathbf{x}_{2j-1}-\widehat{e}_1} H_{\mathbf{x}_{2j-1}-\widehat{e}_1} \cdot \cdots \cdot (-\overline{H}_{\mathbf{x}_{2j}+\widehat{e}_1}) \overline{V}_{\mathbf{x}_{2j}+\widehat{e}_1} V_{\mathbf{x}_{2j}+\widehat{e}_1} H_{\mathbf{x}_{2j}+\widehat{e}_1} \tag{4.120}$$

With a small abuse of notation, in the following we shall also denote with the symbol $(2j-1 \to 2j)$ the straight line connecting $2j-1$ with $2j$ on the polygon (*i.e.* the geometric object, not only the algebraic one).

Moreover, let us denote with the symbol $[2j \to 2j+1]$, $j = 1, \dots, N$, the product of Grassmann fields corresponding to the *non* straight line connecting the point $2j$ with $2j+1$ (including the fields located in $2j$ and in $2j+1$) in the order induced by the choice of proceeding in anticlockwise order and with the sign induced by the expansion of the exponential in (4.116). With a small abuse of notation we shall also denote with the same symbol $[2j \to 2j+1]$ the corresponding line connecting $2j$ with $2j+1$ on the polygon γ. The sites $2j$ and $2j+1$ could either coincide (in that case $2j$ is a corner of R) or, if they do not, they could belong to the same side of R or to different adjacent sides of R. Let us denote with γ_j the union of $[2j \to 2j+1]$ with the shortest path on R connecting $2j$ with $2j+1$. The key remark is that $n_{\gamma_j} < n_\gamma$ so that, by the inductive hypothesis, the weight of γ_j is $(-1)^{n_{\gamma_j}}$.

With these notations and remarks, let us calculate the weight of γ. We write the weight in terms of a Grassmann integral as follows:

$$-\int \prod_{\mathbf{x}\in\gamma} \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}} \quad (1 \to 2)[2 \to 3] \cdots (2N\text{–}1 \to 2N)[2N \to 1] \tag{4.121}$$

The minus sign in front of the integral, appearing for the same reason why it appears in (4.121), is due to the permutation of the field $\overline{H}_{\mathbf{x}_1}$ from the first position (that is the one one gets by expanding the exponential in (4.116), writing the Grassmann binomials starting from site 1 and proceeding in anticlockwise order) to the last one (that is the position it appears into the product $[2N \to 1]$).

By a simple explicit calculation, it is straightforward to verify that the integral of the "straight line" $(2j-1 \to 2j)$ gives a contribution $(-1)^{\ell_{2j-1}-1}$, where ℓ_{2j-1} is the length of the segment $(2j-1 \to 2j)$ (note that $\ell_{2j-1}-1$ is the number of sites belonging to $(2j-1 \to 2j)$, excluding the extremes). We are left with computing the integral of the "non straight line" $[2j \to 2j+1]$. We must distinguish 12 different cases, which we shall now study in detail.

1) j and $j+1$ are distinct and they belong to the low side of R. In this case

$$[j \to j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \left\{ \overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}}) \right\} \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \overline{H}_{\mathbf{x}_{j+1}} \ , \tag{4.122}$$

as it follows from the rules explained above. In order to compute (4.122) we use the inductive hypothesis, telling us that the weight of γ_j is $(-1)^{n_{\gamma_j}}$, that is, explicitely:

$$(-1)^{D_j+d_j} = \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \overline{H}_{\mathbf{x}_j} (j \to j+1) H_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} \tag{4.123}$$

In the last equation we called D_j the length of the non straight line $[j \to j+1]$ (note that $D_j + 1$ is the number of sites belonging to $[j \to j+1]$, including both extremes), we denoted by the symbol $(j \to j+1)$ the product of Grassmanian fields corresponding to the straight line on R connecting $\mathbf{x}_j$ with $\mathbf{x}_{j+1}$ and by d_j its length (note that $d_j - 1$ is the number of sites belonging to $(j \to j+1)$, excluding both extremes). By performing the integration over the fields in $(j \to j+1)$, we find:

$$\begin{aligned}(-1)^{D_j+1} &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_{j+1}} V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} = \\ &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} H_{\mathbf{x}_{j+1}} V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}}\end{aligned} \tag{4.124}$$

and the last line is clearly equal to the r.h.s. of (4.122).

2) j and $j+1$ coincide with the low right corner of R. In this case

$$[j \to j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} = -1\,. \tag{4.125}$$

3) j and $j+1$ are distinct and they belong to the low and the rights sides of R, respectively. In this case

$$[j \to j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}}\,. \tag{4.126}$$

Calling $\mathbf{0}$ the lower right corner of R, the inductive hypothesis tells us that:

$$\begin{aligned}(-1)^{D_j+d_j} &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \overline{H}_{\mathbf{x}_j} (j \to \mathbf{0}) H_{\mathbf{0}} \cdot V_{\mathbf{0}} \overline{H}_{\mathbf{0}} \cdot \\ &\overline{V}_{\mathbf{0}} (\mathbf{0} \to j+1) V_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\}\,.\end{aligned} \tag{4.127}$$

In the last equation we called d_j the length of the shortest path on R connecting j with $j+1$ that is the sum of the lengths of $(j \to \mathbf{0})$ and $(\mathbf{0} \to j+1)$. By performing the integration over the fields in $(j \to \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \to j+1)$ we find:

$$\begin{aligned}(-1)^{D_j+1} &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} = \\ &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}}\end{aligned} \tag{4.128}$$

and the last line is clearly equal to the r.h.s. of (4.126).

4) j and $j+1$ are distinct and they belong to the right side of R. In this case

$$[j \to j{+}1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \big\{ H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}} \big\} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}} \ . \quad (4.129)$$

The inductive hypothesis tells us that:

$$(-1)^{D_j + d_j} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} (j \to j{+}1) V_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \big\{ H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}} \big\} \ . \quad (4.130)$$

By performing the integration over the fields in $(j \to j+1)$ we find:

$$(-1)^{D_j+1} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \big\{ H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}} \big\} =$$

$$= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \big\{ H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}} \big\} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \quad (4.131)$$

and the last line is clearly equal to the r.h.s. of (4.129).

5) j and $j+1$ coincide with the upper right corner of R. In this case

$$[j \to j+1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} = -1 \, . \quad (4.132)$$

6) j and $j+1$ are distinct and they belong to the right and upper sides of R, respectively. In this case

$$[j \to j{+}1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \big\{ H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \big\} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot H_{\mathbf{x}_{j+1}} \ . \quad (4.133)$$

Calling $\mathbf{0}$ the upper right corner of R, the inductive hypothesis tells us that:

$$(-1)^{D_j + d_j} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} (j \to \mathbf{0}) V_{\mathbf{0}} \cdot \overline{V}_{\mathbf{0}} \overline{H}_{\mathbf{0}} \cdot$$

$$H_{\mathbf{0}} (\mathbf{0} \to j+1) (-\overline{H}_{\mathbf{x}_{j+1}}) \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot \big\{ H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \big\} \ . \quad (4.134)$$

By performing the integration over the fields in $(j \to \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \to j+1)$ we find:

$$(-1)^{D_j+1} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \big\{ H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \big\} =$$

$$= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \big\{ H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \big\} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \quad (4.135)$$

and the last line is clearly equal to the r.h.s. of (4.133).

7) j and $j+1$ are distinct and they belong to the upper side of R. In this case

$$[j \to j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot \left\{ V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \right\} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot H_{\mathbf{x}_{j+1}} \tag{4.136}$$

The inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} (j \to j+1) (-\overline{H}_{\mathbf{x}_{j+1}}) \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot \left\{ V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} \right\} \tag{4.137}$$

By performing the integration over the fields in $(j \to j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} V_{\mathbf{x}_j} V_{\mathbf{x}_{j+1}} = \\ &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \end{aligned} \tag{4.138}$$

and the last line is clearly equal to the r.h.s. of (4.136).
8) j and $j+1$ coincide with the upper left corner of R. In this case

$$[j \to j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} = -1 \,. \tag{4.139}$$

9) j and $j+1$ are distinct and they belong to the upper and left sides of R, respectively. In this case

$$\begin{aligned} [j \to j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot \big\{ V_{\mathbf{x}_j} \cdots \\ (-\overline{H}_{\mathbf{x}_{j+1}}) \big\} \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \,. \end{aligned} \tag{4.140}$$

Calling $\mathbf{0}$ the upper left corner of R, the inductive hypothesis tells us that:

$$\begin{aligned} (-1)^{D_j+d_j} = \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} (j \to \mathbf{0}) (-\overline{H}_{\mathbf{0}}) \cdot H_{\mathbf{0}} \overline{V}_{\mathbf{0}} \cdot \\ V_{\mathbf{0}} (\mathbf{0} \to j+1) (-\overline{V}_{\mathbf{x}_{j+1}}) \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \left\{ V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} \,. \end{aligned} \tag{4.141}$$

By performing the integration over the fields in $(j \to \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \to j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \left\{ V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} = \\ &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} \left\{ V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \end{aligned} \tag{4.142}$$

and the last line is clearly equal to the r.h.s. of (4.140).

10) j and $j+1$ are distinct and they belong to the left side of R. In this case

$$[j \to j+1] = \int (-\overline{V}_{\mathbf{x}_j}) \cdot V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \left\{ \overline{H}_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \,. \tag{4.143}$$

The inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} (j \to j+1) (-\overline{V}_{\mathbf{x}_{j+1}}) \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \left\{ \overline{H}_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} \,. \tag{4.144}$$

By performing the integration over the fields in $(j \to j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} V_{\mathbf{x}_j} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \left\{ \overline{H}_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} = \\ &= \int H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} V_{\mathbf{x}_j} \left\{ \overline{H}_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}}) \right\} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \end{aligned} \tag{4.145}$$

and the last line is clearly equal to the r.h.s. of (4.143).

11) j and $j+1$ coincide with the lower left corner of R. In this case it is necessarely $j \equiv 2N$ and $j+1 \equiv 1$ and we have:

$$[2N \to 1] = \int (-\overline{V}_{\mathbf{x}_1}) \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \cdot \overline{H}_{\mathbf{x}_1} = +1 \,. \tag{4.146}$$

Note that this time the result is $+1$. This "wrong" sign exactly compensates the minus sign appearing in the r.h.s. of (4.121).

12) j and $j+1$ are distinct and they belong to the left and lower sides of R, respectively. In this case it is necessarely $j \equiv 2N$ and $j+1 \equiv 1$ and we have

$$[2N \to 1] = \int (-\overline{V}_{\mathbf{x}_{2N}}) \cdot V_{\mathbf{x}_{2N}} H_{\mathbf{x}_{2N}} \cdot \left\{ \overline{H}_{\mathbf{x}_{2N}} \cdots (-\overline{V}_{\mathbf{x}_1}) \right\} \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \cdot \overline{H}_{\mathbf{x}_1} \,. \tag{4.147}$$

Calling $\mathbf{0}$ the lower left corner of R, the inductive hypothesis tells us that:

$$\begin{aligned} &(-1)^{D_N+d_N} = \int H_{\mathbf{x}_{2N}} \overline{V}_{\mathbf{x}_{2N}} \cdot \\ &V_{\mathbf{x}_{2N}} (2N \to \mathbf{0}) (-\overline{V}_{\mathbf{0}}) \cdot V_{\mathbf{0}} H_{\mathbf{0}} \cdot \overline{H}_{\mathbf{0}} (\mathbf{0} \to 1) H_{\mathbf{x}_1} \cdot V_{\mathbf{x}_1} \overline{H}_{\mathbf{x}_1} \cdot \left\{ \overline{H}_{\mathbf{x}_{2N}} \cdots (-\overline{V}_{\mathbf{x}_1}) \right\} \end{aligned} \tag{4.148}$$

By performing the integration over the fields in $(2N \to \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \to 1)$ we find:

$$\begin{aligned} (-1)^{D_N} &= \int H_{\mathbf{x}_{2N}} \overline{V}_{\mathbf{x}_{2N}} V_{\mathbf{x}_{2N}} H_{\mathbf{x}_1} V_{\mathbf{x}_1} \overline{H}_{\mathbf{x}_1} \left\{ \overline{H}_{\mathbf{x}_{2N}} \cdots (-\overline{V}_{\mathbf{x}_1}) \right\} = \\ &= \int H_{\mathbf{x}_{2N}} \overline{V}_{\mathbf{x}_{2N}} V_{\mathbf{x}_{2N}} \left\{ \overline{H}_{\mathbf{x}_{2N}} \cdots (-\overline{V}_{\mathbf{x}_1}) \right\} H_{\mathbf{x}_1} V_{\mathbf{x}_1} \overline{H}_{\mathbf{x}_1} \end{aligned} \tag{4.149}$$

and the last line is clearly equal to the r.h.s. of (4.140). It follows that $[2N \to 1] = -(-1)^{D_N+1}$, consistently with the result in item (11) above. Also in this case, the appearently "wrong" sign exactly compensates the minus sign appearing in the r.h.s. of (4.121).
Combining the results of previous items, we can simply say that the integration of $(2j-1 \to 2j)$ contributes to the weight of γ with $(-1)^{\ell_{2j-1}-1}$; the integration of $[2j \to 2j+1]$, with $j < N$, contributes with $(-1)^{L_{2j}+1}$ (here we defined L_{2j} to be the length of $[2j \to 2j+1]$), while $[2N \to 1]$ with $(-1)^{L_{2N}}$. Substituting these results into (4.121), we find that the weight of γ is equal to $(-1)^{n_\gamma}$, as desired.

The above discussion concludes the proof in the case of polygons without self intersections. Let us call *simple* a polygon without self intersections. If γ is not simple, calling ν_γ the number of its self intersections, we can easily prove that its weight is equal to $(-1)^{\nu_\gamma}$ times the product of the weights of a number of simple polygons, defined as follows. We draw with two colors, white and black, both the disconnetted interiors of the polygon and its exterior, call them $A_1, \dots, A_n$ and A_0 respectively. The drawing is done in such a way that A_0 is white and two adjacent sets A_i and A_j, $0 \le i < j \le n$, have different colors (we call A_i and A_j adjacent if their boundaries have a common side). Then we consider the set $\mathcal{P}$ of simple polygons obtained as the boundaries of the black sets, thought as completely disconnetted one from the other. The "disconnection" of the boundaries of the black regions (which originally could touch each other through the corners) is realized by the elementary disconnetion of the intersection elements described in Fig.A2

We claim that the weight of γ is $(-1)^{\nu_\gamma} \prod_{\gamma' \in \mathcal{P}} (-1)^{n_{\gamma'}}$, which is the desired result (recall that $\mathcal{P}$ is the set of polygons obtained as boundaries of the black sets, *after* the disconnection described in Fig.A2). Note that the factor $(-1)^{\nu_\gamma}$ in front of the product of the weights of the disconnected simple polygons is due to the doubling of the centers of the intersections, implied by our definition of disconnection.

In order to prove the claim we explicitely write the contribution from the intersection in both cases (a) and (b) of Fig. A2, and we show that it is equal to the contribution of the two corner elements on the r.h.s. of Fig. A2, unless for a minus sign, to be associated to the new site 0_1.

The contribution of the left hand side of case (a) in Fig. A2 is:

$$\int \mathrm{d}\overline{H}_{\mathbf{x}_0} \mathrm{d}H_{\mathbf{x}_0} \mathrm{d}\overline{V}_{\mathbf{x}_0} \mathrm{d}V_{\mathbf{x}_0} \left[\overline{H}_{\mathbf{x}_1} H_{\mathbf{x}_0} \cdot \overline{H}_{\mathbf{x}_0} H_{\mathbf{x}_3} \cdot \overline{V}_{\mathbf{x}_2} V_{\mathbf{x}_0} \cdot \overline{V}_{\mathbf{x}_0} V_{\mathbf{x}_4} \right] . \quad (4.150)$$

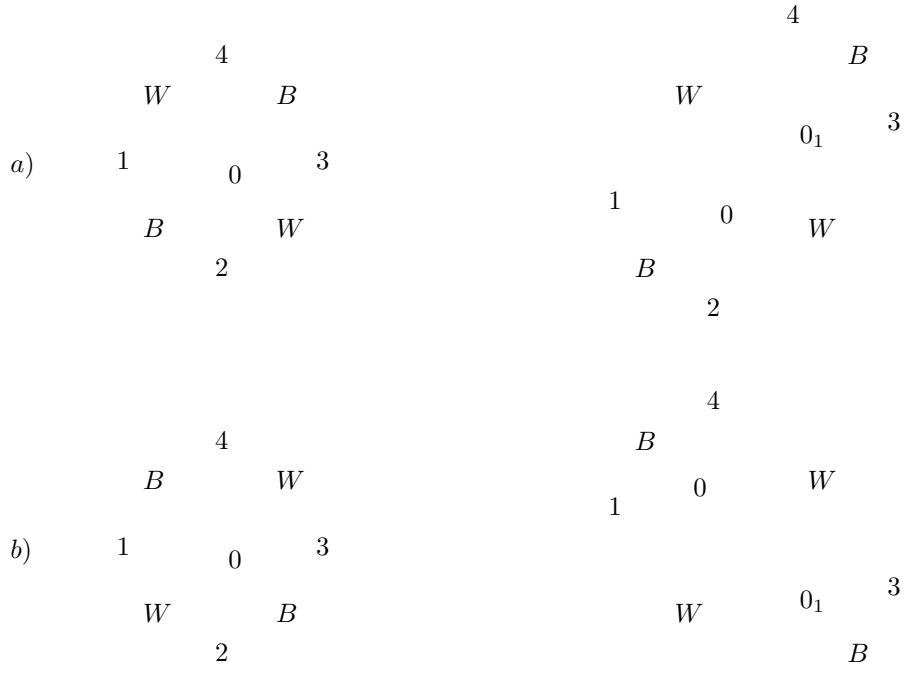

Fig. 4.3 The two elementary operations of disconnecting an intersection. The labels W and B mean that the corresponding regions must be coloured white and black respectively. Note that the operation of disconnecting an intersection involves the doubling of the site 0 at the center of the intersection: in the figure we call 0 and 0_1 its two copies after the disconnection.

Multiplying (4.150) by

$$-\int \mathrm{d}\overline{H}_{\mathbf{x}_{0_1}}\,\mathrm{d}H_{\mathbf{x}_{0_1}}\,\mathrm{d}\overline{V}_{\mathbf{x}_{0_1}}\,\mathrm{d}V_{\mathbf{x}_{0_1}}\,[\overline{V}_{\mathbf{x}_{0_1}}\,\overline{H}_{\mathbf{x}_{0_1}}\cdot V_{\mathbf{x}_{0_1}}H_{\mathbf{x}_{0_1}}] = +1\ , \qquad (4.151)$$

we see that it can be equivalently rewritten as

$$-\int \big(\mathrm{d}\overline{H}_{\mathbf{x}_0}\mathrm{d}H_{\mathbf{x}_0}\mathrm{d}\overline{V}_{\mathbf{x}_0}\mathrm{d}V_{\mathbf{x}_0}\big)\big(\mathrm{d}\overline{H}_{\mathbf{x}_{0_1}}\,\mathrm{d}H_{\mathbf{x}_{0_1}}\,\mathrm{d}\overline{V}_{\mathbf{x}_{0_1}}\,\mathrm{d}V_{\mathbf{x}_{0_1}}\big)\cdot \qquad (4.152)$$
$$\cdot\Big[\overline{H}_{\mathbf{x}_1}H_{\mathbf{x}_0}\cdot\overline{V}_{\mathbf{x}_{0_1}}\,\overline{H}_{\mathbf{x}_{0_1}}\cdot\overline{V}_{\mathbf{x}_2}V_{\mathbf{x}_0}\Big]\cdot\Big[\overline{H}_{\mathbf{x}_0}H_{\mathbf{x}_3}\cdot V_{\mathbf{x}_{0_1}}H_{\mathbf{x}_{0_1}}\cdot\overline{V}_{\mathbf{x}_0}V_{\mathbf{x}_4}\Big]$$

Exchanging the names of the fields $\overline{V}_{\mathbf{x}_0}\longleftrightarrow\overline{V}_{\mathbf{x}_{0_1}}$ and $\overline{H}_{\mathbf{x}_0}\longleftrightarrow\overline{H}_{\mathbf{x}_{0_1}}$, we easily recognize that (**??**) is equal to (-1) times the contribution of the r.h.s. of case (a) in Fig. A2. The minus sign compensate the fact that after the doubling the new polygon has a site more than the original one.

The argument can be repeated in case (b), so that the proof of the claim is complete.

This concludes the proof of (4.116) in the case of open boundary conditions (*i.e.* in the case where polygons winding up over the lattice are not allowed).

4.1.10 *The Grassmann representation of the 2d Ising model with periodic boundary conditions.*

In the case periodic boundary conditions are assumed, the representation in terms of multipolygons is the same, except for the fact that also polygons winding up over the lattice are allowed. In order to construct a Grassmann representation for the multipolygon expansion of Ising with p.b.c., let us start with considering the following expression:

$$\int \prod_{\mathbf{x}\in\Lambda_M} \mathrm{d}\overline{H}_{\mathbf{x}}\mathrm{d}H_{\mathbf{x}}\mathrm{d}\overline{V}_{\mathbf{x}}\mathrm{d}V_{\mathbf{x}} e^{S_{\varepsilon,\varepsilon'}(t)} \,, \tag{4.153}$$

where $\varepsilon, \varepsilon' = \pm$ and

$$\begin{aligned} &\overline{H}_{\mathbf{x}+M\widehat{e}_0} = \varepsilon \overline{H}_{\mathbf{x}} \quad , \quad \overline{H}_{\mathbf{x}+M\widehat{e}_1} = \varepsilon' \overline{H}_{\mathbf{x}} \\ &H_{\mathbf{x}+M\widehat{e}_0} = \varepsilon H_{\mathbf{x}} \quad , \quad H_{\mathbf{x}+M\widehat{e}_1} = \varepsilon' H_{\mathbf{x}} \quad , \quad \varepsilon, \varepsilon' = \pm \end{aligned} \tag{4.154}$$

where we recall that M is the side of the lattice Λ_M. Identical definitions are set for the variables $V, \overline{V}$. We shall say that $\overline{H}, H, \overline{V}, V$ satisfy ε–periodic (ε'–periodic) boundary conditions in vertical (horizontal) direction. Note that, unless for a sign and for the replacement $S(t) \to S_{\varepsilon,\varepsilon'}(t)$, (4.153) is the same as the r.h.s. of (4.118).

Clearly, by expanding the exponential in (4.153) and by integrating the Grassmann fields as described in previous section, we get a summation over dimers very similar to the one seen above. In particular the weights assigned to the closed polygons not winding up the lattice are exactly the same as those calculated in previous section. In this case, however, also Grassmann polygons winding up the lattice are allowed. Let us calculate the weight that (4.153) assigns to these polygons (as above we define the weight by descarding the "trivial" factors $t^{|\gamma|}$ and $(-1)^{M^2-n_\gamma}$).

As an example, let us first calculate the contribution from the simplest polygon γ winding up the lattice, the horizontal straight line winding once

in horizontal direction. Its weight is given by:

$$\int \overline{V}_{\mathbf{0}} V_{\mathbf{0}} \cdot \overline{H}_{\mathbf{0}} H_{\widehat{e}_1} \cdot \overline{V}_{\widehat{e}_1} V_{\widehat{e}_1} \cdot \overline{H}_{\widehat{e}_1} H_{2\widehat{e}_1} \cdots \overline{H}_{(M-1)\widehat{e}_1} H_{M\widehat{e}_1} \ . \tag{4.155}$$

Now, we can rewrite $\overline{H}_{M\widehat{e}_1}$ as $\varepsilon' H_{\mathbf{0}}$. Also, permutating the field $H_{\mathbf{0}}$ from the last position to the third one, we see that (4.155) is equal to:

$$\begin{aligned}(-\varepsilon') \int \overline{V}_{\mathbf{0}} V_{\mathbf{0}} H_{\mathbf{0}} \overline{H}_{\mathbf{0}} \cdot \overline{V}_{\widehat{e}_1} V_{\widehat{e}_1} H_{\widehat{e}_1} \overline{H}_{\widehat{e}_1} \cdots \overline{V}_{(M-1)\widehat{e}_1} V_{(M-1)\widehat{e}_1} H_{(M-1)\widehat{e}_1} \overline{H}_{(M-1)\widehat{e}_1} = \\ = (-\varepsilon')(-1)^M = (-\varepsilon')(-1)^{n_\gamma}\end{aligned} \tag{4.156}$$

where, in the last identity, we used that the length of the straight polygon γ is exactly M. Repeating the lengthy construction of previous section, it can be (straightforwardly) proven that a generic polygon γ winding up once in horizontal direction has a weight (as assigned by (4.153)) equal to $(-\varepsilon')(-1)^{n_\gamma}$. Analogously a polygon γ winding up once in horizontal direction has a weight (as assigned by (4.153)) equal to $(-\varepsilon)(-1)^{n_\gamma}$.

Let us now consider the simplest polygon γ winding up h times in horizontal direction and v times in vertical direction, that is the union of h distinct horizontal lines and v distinct vertical lines each of them winding once over the lattice in horizontal or vertical direction, respectively. Repeating the same simple calculation of (4.155)–(**??**), we easily see that the weight assigned by (4.153) to γ is $(-\varepsilon')^h(-\varepsilon)^v(-1)^{M(h+v)}$. Note that γ has $(-1)^{h\cdot v}$ self intersections, so that $n_\gamma = M(h+v) - h \cdot v$ and the weight can be rewritten as $(-\varepsilon')^h(-\varepsilon)^v(-1)^{h\cdot v}(-1)^{n_\gamma}$. Again, repeating the lengthy construction of previous section, it can be (straightforwardly) proven that a generic polygon γ winding up h times in horizontal direction and v times in vertical direction has a weight (as assigned by (4.153)) equal to $(-\varepsilon')^h(-\varepsilon)^v(-1)^{h\cdot v}(-1)^{n_\gamma}$.

Since the weight assigned to a generic polygon is the one just computed, which is in general different from $(-1)^{n_\gamma}$, it is clear that there exists *no choice* of $\varepsilon, \varepsilon' = \pm 1$ such that (4.153) is equal to $(-1)^{M^2}(2\cosh^2 \beta J)^{-M^2}$ times Ξ_I, where now Ξ_I is the Ising model partition function in the volume Λ_M with periodic boundary conditions. However it is easy to realize that $(-1)^{M^2}\Xi_I(2\cosh^2 \beta J)^{-M^2}$ is equal to a suitable linear combination of the expressions in (4.153), with different choices of $\varepsilon, \varepsilon' = \pm 1$: it holds that

$$(-1)^{M^2} \frac{Z}{(2\cosh^2 \beta J)^{M^2}} = \frac{1}{2} \sum_{\varepsilon,\varepsilon'=\pm 1} \int \prod_{\mathbf{x}\in\Lambda_M} \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}} (-1)^{\delta_{(\varepsilon,\varepsilon')}} e^{S_{\varepsilon,\varepsilon'}(t)} \ , \tag{4.157}$$

where $\delta_{+,-} = \delta_{-,+} = \delta_{-,-} = 0$ and $\delta_{+,+}. In order to verify the last identity it is sufficient to verify that the weight assigned to each polygon \gamma$ is exactly $(-1)^{n_\gamma}$. If γ winds up the lattice h times in horizontal direction and v times in vertical direction, from the calculation above it follows that the weight is:

$$\frac{1}{2}\sum_{\varepsilon,\varepsilon'=\pm 1}(-1)^{\delta_{(\varepsilon,\varepsilon')}}(-\varepsilon')^h(-\varepsilon)^v(-1)^{h\cdot v}(-1)^{n_\gamma} = \tag{4.158}$$

$$= \frac{1}{2}(-1)^{n_\gamma}\Big[(-1)^{h+v+hv+\delta_{+,+}} + (-1)^{v+hv+\delta_{+,-}} + (-1)^{h+hv+\delta_{-,+}} + (-1)^{hv+\delta_{-,-}}\Big]$$

The expression between square brackets on the last line is equal to $(-1)^{hv}[-(-1)^{h+v}+(-1)^v+(-1)^h+1]$. Now, if h and v are both even, this is equal to $(+1)[-1+1+1+1]=2$; if h is even and v is odd (or viceversa), it is equal to $(+1)[+1-1+1+1]=2$; if they are both odd, it is equal to $(-1)[-1-1-1+1]=2$. That is, (4.159) is identically equal to $(-1)^{n_\gamma}$, as wanted, and (4.157) is proven.

Chapter 5