Global Attractor for a Parabolic-Hyperbolic Penrose-Fife Phase Field System

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Abstract

A singular nonlinear parabolic-hyperbolic PDE’s system describing the evolution of a material subject to a phase transition is considered. The goal of the present paper is to analyze the asymptotic behaviour of the associated dynamical system from the point of view of global attractors. The physical variables involved in the process are the absolute temperature θ (whose evolution is governed by a parabolic singular equation coming from the Penrose-Fife theory) and the order parameter χ (whose evolution is ruled by a nonlinear damped hyperbolic relation coming from a hyperbolic relaxation of the Allen-Cahn equation). Dissipativity of the system and the existence of a global attractor are proved. Due to questions of regularity, the one space dimensional case (1D) and the 2D - 3D cases require different sets of hypotheses and have to be settled in slightly different functional spaces.

**Key words:** Penrose-Fife model, parabolic-hyperbolic system, dissipativity, global attractor.

**AMS (MOS) subject classification:** 35B41, 35L70, 80A22.

1 Introduction

In this paper we consider a modification of the thermodynamically consistent model for the description of the kinetics of phase-transitions proposed by O. Penrose and P. Fife in [15, 16]. In particular, we address the question of the existence
of the global attractor for a parabolic-hyperbolic PDE’s system of Penrose-Fife type. The state variables of the process (describing the phase-transition in a smooth bounded subset $\Omega$ of $\mathbb{R}^N$, $1 \leq N \leq 3$) are the absolute temperature $\vartheta > 0$ and the phase parameter $\chi$. The first equation of the system, describing the evolution of $\vartheta$ through the energy balance, takes the form

$$\frac{\partial \vartheta}{\partial t} + b\chi - \Delta \alpha(\vartheta) = m, \quad (1.1)$$

where the subscript $t$ stands for the time derivative, $m$ is a heat source term, and $\vartheta + b\chi$ accounts for the internal energy of the system with $b > 0$ representing the latent heat density of the phase-transition, assumed to be constant. Moreover, $\alpha$ in (1.1) has the form

$$\alpha(r) = -k_2 r^{-1} + k_1 r \quad \text{for all } r > 0. \quad (1.2)$$

The expression above, originally proposed in [3], indicates that the heat flux is singular in proximity of the absolute zero, while the behavior is similar to the linear one in a high temperatures regime.

Recently, (1.1) has been coupled (cf. [4, 5, 8] for the case of PDE’s systems of Caginalp [1] type, [2, 17] for the case of Penrose-Fife models, and references therein) with a hyperbolic relation governing the evolution of the phase variable $\chi$, which can be written as

$$\mu \chi_{tt} + \chi_t - \Delta \chi + g(\chi) + b\vartheta^{-1} = 0, \quad (1.3)$$

where the subscript $tt$ stands for the second time derivative, $g$ usually represents a polynomial function, in general of cubic growth, with positive leading coefficient, and $\mu > 0$ is a small inertial parameter accounting for a delay effect in the response of the system to the generalized force responsible for the phase change (see, e.g., [8] for more details). A common and physically relevant example of $g$ is provided by the derivative of a double-well potential, i.e., $g(\chi) \sim \chi^3 - \chi - b\vartheta_C^{-1}$, with $\vartheta_C > 0$ representing the critical temperature of the system.

Noting that well-posednes for a variational formulation of (1.1)+(1.3), with $\alpha$ as in (1.2), has been recently proved by the first author in [17], in this paper we address the question of existence of the global attractor for the solutions of system (1.1)+(1.3) coupled with the initial conditions, with the (physically meaningful) no-flux conditions for $\chi$, and with third type (Robin) boundary conditions for $\alpha(\vartheta)$, namely expressed by

$$\partial_n \alpha(\vartheta) = \gamma(h - \alpha(\vartheta)), \quad (1.4)$$

where $\gamma > 0$ and $h$ is a boundary source term. For a motivation of this choice of the boundary conditions for equation (1.1), we refer again to [3].

We actually have to notice that so far it seems that very few works have been devoted to the study of Penrose-Fife models from the point of view of long time behaviour. Indeed, the main difficulty of this type of problem comes from the (usual) choice of a more singular heat flux law of the form $\alpha(r) \sim -1/r,$
which gives rise to a lack of coercivity with respect to $\vartheta$ in (1.1). For some results related to this choice of the heat flux law, but in standard (parabolic) Penrose-Fife models, i.e. with

$$\chi_t - \Delta \chi + g(\chi) + b\vartheta^{-1} = 0,$$

(1.5)
in place of our (1.3), we refer to [12, 13, 20].

More precisely, in [20] the authors prove existence of a maximal attractor in the case of no-flux boundary conditions for both unknowns (this choice partly compensates the lack of coercivity through a conservation property) and just in the 1D case. Differently, in [13] the singular character of the flux is corrected by the presence of a 0-order dissipative term $\varepsilon \vartheta$ for small $\varepsilon > 0$ on the left hand side of (1.1). In this setting, the authors can prove the existence of a global attractor with respect to a weak metric (see the Introduction of [18] for a detailed explanation of this point). Moreover, in [12] the existence of an inertial set (cf. [21, § VIII]) is shown in the 1D case.

In the former works [18, 19] we treat the (parabolic) case of $\mu = 0$ and prove the existence of the maximal attractor in three space dimensions both for the standard phase field system (1.1)+(1.5) and for the related conserved model (corresponding to a 4th order dynamics for $\chi$) in the (more coercive) case of the expression (1.2) for $\alpha$. We have to remark that in such a setting we are able to deal with very general, and even singular functions $g$ (cf. also [7]), which seems to be a very hard task in the mixed (parabolic-hyperbolic) case. Moreover, we work in a phase space $\mathcal{X}$ (cf. (2.22) below), which is smaller than that considered, e.g., in [13, 12], and it is correspondingly endowed with a stronger metric whose choice comes from the singular terms characterizing the system and, in fact, from the expression of the free energy. This gives rise to several technical problems since some control of the nonlinear constraints is required as the dissipativity estimates are performed; on the other hand, this method permits us to prove stronger asymptotic properties of the trajectories of the associated dynamical systems.

Here we aim to adapt the approach devised in [18, 19] to the parabolic-hyperbolic case. This gives rise to a series of further (and not merely technical) difficulties. First, we have to notice that it is necessary to restrict the choice of admissible potentials to functions $g$ with a controlled growth, which has to be of subcubic type in the 3D case. Indeed, it seems that, unfortunately, the (physically relevant) case of the cubic growth (which has a critical character in the 3D setting, cf. [11, 9]) cannot be treated, at least with this method, in the framework of the Penrose-Fife system. This is due to the lack of a global coercive Lyapunov functional, which seems hard to be found for the present system, even in the case of zero source data. Such a functional, instead, can be found in the case of the parabolic-hyperbolic Caginalp system; indeed, in [8], it is shown that such a model admits a global attractor in the 3D cubic case, at least for a null heat source.

As a second difficulty, we remark that we have to settle the 1D and 2D-3D problems in slightly different regularity frameworks. Indeed, in the former case we are able to study the dynamical system associated to (1.1)+(1.3) by setting it
in a metric space $\mathcal{X}$ very similar to that used in [18]. Instead, in 2D and in 3D this regularity appears too weak in order to perform the compactness estimates needed to prove existence of the attractor. This problem was not present in the parabolic case studied in [18], since in that case one could take advantage of some regularizing effect of equation (1.5), which, of course, is not provided by the hyperbolic relation (1.3). Hence, in 2D-3D we have to work in a different space $\mathcal{D}$ consisting of more regular functions (cf. (2.28) below) and refine the continuous dependence and dissipativity results by adapting them to the (stronger) topology of $\mathcal{D}$. This argument then allows us to perform the compactness estimates leading to existence of the attractor.

To conclude, we have to remark that, as in [18, 19], the choice of the phase space $\mathcal{X}$ modelled on the structure of the energy functional also presents some inconvenient. Namely, it does not seem possible to address the question of existence of an inertial set (cf. [12]), at least using the metric $d_{\mathcal{X}}$ (or $d_{\mathcal{D}}$, see (2.29) below). Indeed, it appears very difficult to prove any sort of Lipschitz continuity with respect to these metrics “containing” nonlinear functions. It might be possible, anyway, to show the existence of some set which attracts exponentially the $d_{\mathcal{X}}$ (or $d_{\mathcal{D}}$)-bounded set, but with the attraction property holding in some weaker topology.

Let us eventually give the plan of the paper. In the next Section 2, we first state the variational formulation of system (1.1)+(1.3) coupled with the initial and boundary conditions. Moreover, we report (for the sake of clarity) some existence-continuous dependence results proved in [17]. Still in Section 2 we introduce the phase spaces $\mathcal{X}$ and $\mathcal{D}$, where our problem is settled in the 1D and 2D-3D case, respectively, and finally state our main theorems. The remainder of the paper is devoted to the proofs of these results: in Section 3, we deal with the construction of the semigroups associated to our problem and show a dissipativity property in the larger space $\mathcal{X}$. Next, in Section 4 we show the existence of the universal attractor in the 1D case, while in Section 5 we refine the results on dissipativity by working in the smaller space $\mathcal{D}$. This is required for proving the compactness estimates (Section 6) leading to our main result, i.e., the existence of the universal attractor in the 2D and 3D cases. Finally, in Section 7, we make some final comments. In particular, we compare our results with the corresponding ones on the parabolic-hyperbolic Caginalp’s model and formulate some related open questions.

2 Main results

We aim to state here a variational formulation of problem (1.1–1.3) coupled with the proper initial-boundary conditions in a suitable functional framework and to recall related well-posedness results (cf. [17]). First, let us give our basic assumptions on the data:

\begin{align*}
\text{A1} & \quad b, \gamma, k_1, k_2 \quad \text{are positive constants,} \\
\text{A2} & \quad g \in C^1(\mathbb{R}),
\end{align*}

in a metric space $\mathcal{X}$ very similar to that used in [18]. Instead, in 2D and in 3D this regularity appears too weak in order to perform the compactness estimates needed to prove existence of the attractor. This problem was not present in the parabolic case studied in [18], since in that case one could take advantage of some regularizing effect of equation (1.5), which, of course, is not provided by the hyperbolic relation (1.3). Hence, in 2D-3D we have to work in a different space $\mathcal{D}$ consisting of more regular functions (cf. (2.28) below) and refine the continuous dependence and dissipativity results by adapting them to the (stronger) topology of $\mathcal{D}$. This argument then allows us to perform the compactness estimates leading to existence of the attractor.

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\begin{align*}
\text{A1} & \quad b, \gamma, k_1, k_2 \quad \text{are positive constants,} \\
\text{A2} & \quad g \in C^1(\mathbb{R}),
\end{align*}
Remark 2.1. Under these assumptions on \( g \), it is easy to see that there exist \( \kappa > 0 \) and a primitive \( \tilde{g} \) of \( g \) such that
\[
\lambda r^2 - \kappa \leq \tilde{g}(r) \leq 2\nu_1(|r|^{p+1} + 1) \quad \forall r \in \mathbb{R}.
\] (2.1)

We also observe that (A3–A4) provide an upper and, respectively, a lower bound on the growth of \( g \) at \( \infty \). Moreover, the coupling of (A4) and (A5) says that \( g \) is “approximately monotone” (increasingly). Indeed, the correction term \( \nu_2(|r|^\iota + 1) \) is controlled at \( \infty \) thanks to \( \iota < \sigma \) and (A4).

Remark 2.2. Assumption (A6) could be generalized in several directions; for instance, less regular data, or even data suitably depending on time, might be considered.

We can now introduce a variational formulation of our problem. To this end, we use the notation \( (\cdot, \cdot) \) both for the scalar product in \( H := L^2(\Omega) \) and in \( (L^2(\Omega))^N \), also denoted by \( H \), and \( |\cdot| \) for the corresponding norm. We also set \( W := H^2(\Omega) \) and denote the scalar product in \( L^2(\Gamma) \) as \( \langle \cdot, \cdot \rangle_\Gamma \). For the sake of convenience, \( V := H^1(\Omega) \) will be endowed with the inner product \( (\langle \cdot, \cdot \rangle) \), defined by
\[
(\langle v_1, v_2 \rangle) := \int_\Omega \nabla v_1 \cdot \nabla v_2 + \gamma(v_1, v_2)_\Gamma \quad \forall v_1, v_2 \in V.
\] (2.2)

This choice is actually motivated by the Robin boundary conditions (1.4). The norm in \( V \) associated to this scalar product will be simply noted as \( \| \cdot \| \).

As usual, we identify \( H \) with a subspace of \( V' \) (topological dual of \( V \)), so that the \( V'–V \) duality \( \langle u, v \rangle \) equals the scalar product \( (u, v) \) for all \( u \in H \) and for all \( v \in V \). Actually, with a small abuse of language, we will sometimes use the symbol \( (u, v) \) in order to denote such a duality even when \( u \) is in \( V' \) but not in \( H \). Then, the Riesz isomorphism \( J : V \to V' \) associated to our choice of the norm and the related scalar product in \( V' \) are given, respectively, by
\[
\langle Jv_1, v_2 \rangle := (v_1, v_2), \quad \forall v_1, v_2 \in V; \quad (w_1, w_2)_s := \langle w_1, J^{-1}w_2 \rangle, \quad \forall w_1, w_2 \in V'.
\] (2.3) (2.4)

Let us also introduce the positive operator on \( H \)
\[
A := 1 - \Delta \quad \text{on} \quad \text{dom}(A) = \{ w \in W : \partial_n w = 0 \text{ on } \Gamma \}.
\] (2.5)
The natural extension of $A$ to the space $V$ is nothing else but the Riesz operator associated to the standard norm $\| \cdot \|_A$ on $V$. This Riesz mapping will be equally indicated as $A$. Obviously, the norm $\| \cdot \|$ associated to the inner product in (2.2) is equivalent to the standard norm $\| \cdot \|_A$. Similar considerations holds also for dual norms in $V'$ and we term as $\| \cdot \|_*$ (resp., $\| \cdot \|_{*,A}$) the norm in $V'$ related to the inner product (2.4) (resp., the standard dual norm).

In the sequel we will also use some spaces of $L^q_{\text{loc}}$-translation bounded functions: as $X$ is a Banach space, $\tau \geq 0$, and $q \in [1, +\infty)$ we set

$$T^q_{\tau}(X) := \left\{ v \in L^q_{\text{loc}}(\tau, +\infty; X) : \sup_{t \geq \tau} \int_t^{t+1} \|v(s)\|_X^q \, ds < +\infty \right\},$$

which is a Banach space with respect to the natural (graph) norm

$$\|v\|_{T^q_{\tau}(X)} := \sup_{t \geq \tau} \int_t^{t+1} \|v(s)\|_X^q \, ds. \quad (2.7)$$

**Remark 2.3.** If $v \in T^q_{\tau}(X)$, setting $\|v\|_{T^q_{\tau}(X)} = c < \infty$, it is easy to check (cf., e.g., [14]) that

$$\int_s^t e^{-\varepsilon(t-r)}\|v(r)\|_X^q \, dr \leq \frac{e^\varepsilon c}{1 - e^{-\varepsilon}} \quad \forall \varepsilon > 0, \; \forall t \geq s \geq \tau.$$

Let us recall the statement of the so-called uniform Gronwall lemma (see, e.g., [21, Lemma III.1.1] for the proof):

**Lemma 2.4.** Let $y, a, \phi \in L^1_{\text{loc}}(0, +\infty)$ be three non negative functions such that

$$\frac{d}{dt} y(t) \leq a(t)y(t) + \phi(t) \quad \text{for a.e. } t \geq 0 \quad (2.8)$$

and let $a_1, a_2, a_3$ be three nonnegative constants such that

$$\|a\|_{T^q_{\tau}(\mathbb{R})} \leq a_1, \quad \|\phi\|_{T^q_{\tau}(\mathbb{R})} \leq a_2, \quad \|y\|_{T^q_{\tau}(\mathbb{R})} \leq a_3. \quad (2.9)$$

Then, we have that

$$y(t + 1) \leq (a_2 + a_3)e^{a_1} \quad \text{for all } t \geq 0. \quad (2.10)$$

Finally, let us define the generalized heat source term as

$$\langle f, v \rangle := (m, v) + \gamma(h, v)_\Gamma \quad \forall v \in V; \quad (2.11)$$

indeed, we remark that (A6) entails $f \in V'$.

Now, we are ready to recall the result [17, Thm. 2.3] related to global existence.
Theorem 2.5. Let us assume (A1–A8) and take $T > 0$. Then, there exists at least one couple $(\vartheta, \chi)$ with the following regularity

\begin{align}
\vartheta & \in H^1(0, T; V') \cap L^\infty(0, T; H) \cap L^2(0, T; V), \\
\vartheta & > 0 \quad \text{a.e. in } Q := \Omega \times (0, T), \\
\vartheta^{-1} & \in L^2(0, T; V), \\
\chi & \in C^1([0, T]; H) \cap C^0([0, T]; V),
\end{align}

satisfying

\begin{align}
(\vartheta + b \chi)_t + J(k_1 \vartheta - k_2 \vartheta^{-1}) &= f \quad \text{in } V', \quad \text{a.e. in } (0, T), \\
\mu \chi_t + \chi + A \chi + g(\chi) &= -b \vartheta^{-1} \quad \text{in } V', \quad \text{a.e. in } (0, T), \\
\vartheta(\cdot, 0) &= \vartheta_0, \quad \chi(\cdot, 0) = \chi_0, \quad \chi_t(\cdot, 0) = \chi_1 \quad \text{a.e. in } \Omega. 
\end{align}

We point out that, due to the expression (2.5) of the operator $A$, the function $g$ considered in (2.17) (as well as in the remainder of the paper) is slightly different from the $g$ appearing in the introduction.

Next, we report the statement of the continuous dependence result proved in [17, Thm. 2.8].

**Theorem 2.6.** Let hypotheses (A1–A5) hold. Then, for $i = 1, 2$, consider two sets of data $\{\vartheta_{0i}, \chi_{0i}, \chi_{1i}, m_i, h_i\}$ satisfying the assumptions (A6–A8). Moreover, denote by $(\vartheta_i, \chi_i)$ a corresponding couple of solutions to Problem (2.16–2.18) in the sense of Theorem 2.5, and note as $M$ a positive constant such that

$$
\max \{\|\chi_1\|_{L^\infty(0,t;V)}, \|\chi_2\|_{L^\infty(0,t;V)}\} \leq M. 
$$

Then, setting $u_i := -\vartheta_i^{-1}$, the following estimate holds

$$
\|\vartheta_1 - \vartheta_2\|_{L^\infty(0,t;V)}^2 + \|\chi_1 - \chi_2\|_{L^\infty(0,t;V)}^2 + \|\chi_1 - \chi_2\|_{L^\infty(0,t;H)}^2 
\leq D \left( \|\vartheta_0 - \vartheta_2\|_2^2 + \|\chi_0 - \chi_2\|_2^2 + \|\chi_1 - \chi_2\|_2^2 + \|f_1 - f_2\|_2^2 \right)
$$

for some positive constant $D = D(M)$ also depending on $T, \Omega, \gamma, b, \mu, \nu_1, \nu_2$ and $p$, where $f_i$ is the datum corresponding to $m_i$ and $h_i$, $(i = 1, 2)$ according to (2.11).

**Remark 2.7.** We point out that our assumptions on the function $g$ are slightly more general than those considered in [17]. However, it seems a trivial matter to extend the proofs of the quoted theorems of [17] to the present setting.

Looking back at the results stated up to this point, we notice that, in order to have existence, we choose rather regular initial data (cf. (A7–A8)); conversely, the continuous dependence theorem holds with respect to much weaker norms. Thus, in order to define the phase space for the asymptotic analysis, we have to make a choice between the two functional settings. We actually work with the stronger norms and take

$$
\mathcal{H} = H \times V \times H.
$$
Then, we put (denoting with $(\cdot)^-$ the negative part function)

$$
\mathcal{X} := \{(\vartheta, \chi, \zeta) \in \mathcal{H} : \vartheta > 0, \log^+ \vartheta \in L^1(\Omega)\}. \tag{2.22}
$$

The following property will allow us to use $\mathcal{X}$ as a phase space for our system, at least in the 1D case.

**Lemma 2.8.** The set $\mathcal{X}$ is a complete metric space with respect to the distance

$$
d_{\mathcal{X}}((\vartheta_1, \chi_1, \zeta_1), (\vartheta_2, \chi_2, \zeta_2)) := |\vartheta_1 - \vartheta_2| + \int_{\Omega} |\log^\vartheta_1 - \log^\vartheta_2| + \|\chi_1 - \chi_2\| + |\zeta_1 - \zeta_2|, \tag{2.23}
$$

**Proof.** Let $(\vartheta_n, \chi_n, \zeta_n)$ be a Cauchy sequence in $\mathcal{X}$. Then it is clear that there exists $(\vartheta, \chi, \zeta) \in \mathcal{H}$ such that $(\vartheta_n, \chi_n, \zeta_n) \to (\vartheta, \chi, \zeta)$ in $\mathcal{H}$. Furthermore, we can assume that, at least for a subsequence such a convergence holds also pointwise in $\Omega$. Then, by the Fatou Lemma,

$$
\int_{\Omega} \log^+ \vartheta \leq \liminf_{n \to +\infty} \int_{\Omega} \log^+ \vartheta_n < +\infty, \tag{2.24}
$$

which shows that $\vartheta > 0$ almost everywhere and $\log \vartheta \in L^1(\Omega)$. Thus $(\vartheta, \chi, \zeta) \in \mathcal{X}$. Finally, since $\log^+ \vartheta_n$ is a Cauchy sequence in $L^1(\Omega)$, we easily conclude that (the whole sequence) $\log^+ \vartheta_n$ converges to $\log^+ \vartheta$ in $L^1(\Omega)$, which concludes the proof of the Lemma. \(\Box\)

The next step consists in proving well-posedness and continuity of the map

$$
\mathcal{X} \to \mathcal{X}, \quad (\vartheta_0, \chi_0, \zeta_1) \to (\vartheta(t), \chi(t), \chi(t)) \quad \forall \ t \geq 0. \tag{2.25}
$$

This is stated in the following theorem which will be proved in the next Section 3.

**Theorem 2.9.** Assume (A1–A8). Then, the map in (2.25) is well posed and defines a strongly continuous semigroup $S_{\mathcal{X}}(t)$ on $\mathcal{X}$ associated to the Problem (2.16–2.18).

We are now ready to state our first dissipativity result, whose proof will be given in the next Subsection 3.2.

**Theorem 2.10.** Let $\mu_0 > 0$ be fixed and let $0 < \mu \leq \mu_0$. Assume that (A1–A8) hold. Then, the semigroup $S_{\mathcal{X}}(t)$ given by Theorem 2.9 possesses an absorbing set $\mathcal{B}_\mathcal{X} = \mathcal{B}_\mathcal{X}(\mu)$ which is bounded with respect to the metric $d_\mathcal{X}$.

Furthermore, we have the following Theorem, whose proof will be detailed in Section 4.

**Theorem 2.11.** Let (A1–A8) hold. If $\Omega \subset \mathbb{R}$, then the semigroup $S_{\mathcal{X}}$ defined in Theorem 2.9 possesses a global attractor $\mathcal{A}_\mathcal{X} = \mathcal{A}_\mathcal{X}(\mu)$ which is a compact set of the phase space $\mathcal{X}$.
We now aim to extend the preceding results to the 2D-3D cases. However, to construct the attractor in this setting we have to work with slightly more regular initial data. The additional regularity will depend on the exponent $p$ in (A3) in the 3D setting. Then, let us start by introducing some auxiliary spaces. Given $r \in (0, 1/8]$, let us first set
\[ V_r := \{ (\vartheta, \chi, \zeta) \in X : \vartheta, \vartheta^{-1} \in V, \chi \in H^{1+2r}(\Omega), \zeta \in H^{2r}(\Omega) \}. \tag{2.26} \]
This space $V_r$ turns out to have a (complete) metric structure with respect to the natural distance (also depending on the choice of $r$)
\[ d_V((\vartheta_1, \chi_1, \zeta_1), (\vartheta_2, \chi_2, \zeta_2)) := \|\vartheta_1 - \vartheta_2\| + \|\vartheta_1^{-1} - \vartheta_2^{-1}\| + \|\chi_1 - \chi_2\|_{H^{1+2r}(\Omega)} + \|\zeta_1 - \zeta_2\|_{H^{2r}(\Omega)}. \tag{2.27} \]
Moreover, we have the following property (whose proof is similar to that of [18, Prop. 3.15] and is therefore omitted).

**Proposition 2.12.** For any $r \in (0, 1/8]$, the following compact immersion holds true:
\[ V_r \subset X. \]
Namely, if $(\vartheta_n, \chi_n, \zeta_n)$ is a $d_V$-bounded sequence in $V_r$, then there exist $(\vartheta, \chi, \zeta) \in V_r$ and a subsequence of $(\vartheta_n, \chi_n, \zeta_n)$ converging to $(\vartheta, \chi, \zeta)$ in $d_X$.

While the space $V_r$ corresponds, roughly speaking, to the regularity of the attractor, we also have to introduce a slightly larger space $D_r$ for the initial data and construct a related new semigroup $S_D$. Thus, still for $r \in (0, 1/8]$, let us set
\[ D_r := \{ (\vartheta, \chi, \zeta) \in X : \chi \in H^{1+r}(\Omega), \zeta \in H^r(\Omega) \}. \tag{2.28} \]
Next, as it can be proved by following the lines of [18, Prop. 3.5], $D_r$ is a complete metric space with respect to the distance (depending, of course, on $r$)
\[ d_D((\vartheta_1, \chi_1, \zeta_1), (\vartheta_2, \chi_2, \zeta_2)) := |\vartheta_1 - \vartheta_2| + \int_{\Omega} |\log^{-} \vartheta_1 - \log^{-} \vartheta_2| \nonumber + \|\chi_1 - \chi_2\|_{H^{1+r}(\Omega)} + \|\zeta_1 - \zeta_2\|_{H^r(\Omega)}. \tag{2.29} \]
The next step consists in constructing the new semigroup working on the more regular solutions:

**Theorem 2.13.** Assume (A1–A7). Assume $r$ be an arbitrary number in $(0, 1/8]$ if $N = 2$ and take, instead,
\[ r := \min \{1/8, (3 - p)/2\} \tag{2.30} \]
if $N = 3$, where $p$ is given in (A3). Next, suppose that
\[ \chi_0 \in H^{1+r}(\Omega) \quad \text{and} \quad \chi_1 \in H^r(\Omega). \tag{A9} \]
Then, the restriction to $D_r$ of the map in (2.25) defines a strongly continuous semigroup $S_D(t)$ on $D_r$. 
Remark 2.14. In the 2D case, the choice of the maximal admissible \( r = 1/8 \) is just given for uniformity in the procedure; larger \( r \) might actually be considered. Instead, the constraints in (2.30) seem to be necessary in the 3D setting, although both the values 1/8 and \((3 - p)/2\) are probably not optimal.

Theorem 2.15. Assume (A1–A7) and (A9), with \( r \) as in (2.30) in the 3D case. Then, there exists an absorbing set \( \mathcal{B}_D \), bounded in the metric \( d_D \), for the semigroup \( S_D(t) \).

It is not difficult to prove (cf., again, [18, Prop. 3.15]) that

Proposition 2.16. For any \( r \in (0, 1/8] \), the following compact immersion holds true:

\[ \mathcal{V}_r \subset D_r. \]

We are now ready to state our existence result for the universal attractor in the 2D and 3D cases. Its proof is given in Section 6.

Theorem 2.17. Take the assumptions of Theorem 2.15 and suppose additionally that

\[ h \in L^\infty(\Gamma). \quad \text{(A10)} \]

Then \( S_D \) admits a global attractor which is a compact set in the phase space \( D_r \).

3 Continuity and Dissipativity in \( \mathcal{X} \)

3.1 Proof of Theorem 2.9

Let us start by showing that, as \( (\vartheta_0, \chi_0, \chi_1) \in \mathcal{X} \), it follows that the corresponding triplet \( (\vartheta(t), \chi(t), \chi_t(t)) = S_\mathcal{X}(t)(\vartheta_0, \chi_0, \chi_1) \) belongs to \( \mathcal{X} \) for every \( t > 0 \).

Within proving this fact we proceed similarly as in [18, Section 4]; thus, we just give here the highlights of the procedure. First of all, we define

\[ \hat{\alpha}(r) := \int_1^r \alpha(s) \, ds + \kappa_\alpha \quad \text{for} \quad r \in (0, +\infty), \]

where \( \alpha(r) = k_1 r - k_2/r \) and \( \kappa_\alpha > 0 \) is a constant chosen so that \( \hat{\alpha} \) is nonnegative; more precisely, we can find \( \lambda_\alpha > 0 \) and a convex and nonnegative function \( \hat{\alpha}_{\text{rest}} : (0, +\infty) \to \mathbb{R} \) such that

\[ \hat{\alpha}(r) = \lambda_\alpha \log r + \lambda_\alpha r^2 + \hat{\alpha}_{\text{rest}}(r) \quad \text{for all} \quad r \in (0, +\infty). \]

Using the fact that \( \vartheta > 0 \) a.e. in \( Q \) (cf. (2.13)) together with [18, Lemma 4.1] (applied with the choices of \( u = \vartheta \), \( \eta = \alpha(\vartheta) \), and \( \mathcal{J} \) given by the convex functional induced on \( H \) by \( \hat{\alpha} \)), and relying on the convexity and lower semicontinuity of the summands in the decomposition (3.2) we obtain that the functions

\[ t \mapsto \int_\Omega \hat{\alpha}(\vartheta(t)), \quad t \mapsto \int_\Omega \log^-(\vartheta(t)), \quad t \mapsto |\vartheta(t)|^2 \]

(3.3)
are absolutely continuous in \([0, T]\). This implies that \((\vartheta(t), \chi(t), \chi_i(t))\) belongs to \(\mathcal{X}\) for any \(t \geq 0\), i.e., the map \(S_\mathcal{X}(t)\) is well defined.

Next, let us prove the continuity of \(S_\mathcal{X}\) in the metric \(d_\mathcal{X}\). Assume that \(\{(\vartheta_0, \chi_0, \chi_1)\} \subset \mathcal{X}\) is a sequence of initial data converging to \((\vartheta_0, \chi_0, \chi_1) \in \mathcal{X}\) in \(d_\mathcal{X}\). Moreover, let us fix \(t > 0\) and set \((\vartheta_n, \chi_n, \partial_t \chi_n)(t) := S_\mathcal{X}(t)(\vartheta_0, \chi_0, \chi_1)\) (resp., \((\vartheta, \chi, \partial_t \chi)(t) := S_\mathcal{X}(t)(\vartheta_0, \chi_0, \chi_1)\)). Then, let us notice that we are in the position of applying Theorems 2.5, 2.6. This means in particular that we can perform energy-type estimates, uniform with respect to \(n\), entailing that the solutions satisfy, independently of \(n\), the bounds corresponding to the regularity properties (2.12–2.15). This gives rise, as \(n \nrightarrow \infty\), to the corresponding weak, or weak-* convergence properties, holding for the whole sequences thanks to uniqueness of the limit. Let us note that, in particular, the convergence
\[
\alpha(\vartheta_n) \rightarrow \alpha(\vartheta) \quad \text{weakly in } L^2(0, T; V)
\]  
(3.4)
is a byproduct of this procedure.

By standard compactness results and linearity and continuity of the mapping \(\delta_t : v \mapsto v(t)\) from \(C^0([0, T]; H)\) to \(H\) and from \(C^0([0, T]; V)\) to \(V\), it then follows that \(\vartheta_n(t) \rightarrow \vartheta(t)\) weakly in \(H\), that \(\chi_n(t) \rightarrow \chi(t)\) weakly in \(V\), and that \(\partial_t \chi_n(t) \rightarrow \partial_t \chi(t)\) weakly in \(H\); all this, for all \(t \geq 0\). We have now to prove that these convergences are strong and, in order to do that, it suffices to show that
\[
\limsup_{n \nrightarrow \infty} |\vartheta_n(t)| \leq |\vartheta(t)| \quad \text{and}
\limsup_{n \nrightarrow \infty} |\nabla \chi_n(t)| \leq |\nabla \chi(t)|, \quad \limsup_{n \nrightarrow \infty} |\partial_t \chi_n(t)| \leq |\partial_t \chi(t)|.
\]  
(3.5)  
(3.6)
This is done by a semicontinuity argument. Actually, we test (2.16), written for \((\vartheta_n, \chi_n, \partial_t \chi_n)\) by \(\alpha(\vartheta_n)\) and integrate in \((0, t)\). Then, we test (2.17) by \(k_2 \partial_t \chi_n\) and integrate over \((0, t)\). Finally, we sum the two results together. Noting that two terms cancel and observing that the term with \(g\) can be integrated by parts thanks to the regularity properties (A2), (A3) and (2.15), we get
\[
\int_\Omega \tilde{\alpha}(\vartheta_n(t)) + \|\alpha(\vartheta_n)\|^2_{L^2(0, t; V)} + \frac{\mu k_2}{2} |\partial_t \chi_n(t)|^2
\]
\[
+ \frac{k_2}{2} |\nabla \chi_n(t)|^2 + k_2 \|\partial_t \chi_n\|^2_{L^2(0, t; H)} + k_2 \int_\Omega \tilde{g}(\chi_n(t))
\]
\[
= \int_\Omega \tilde{\alpha}(\vartheta_0) + \int_0^t \langle f, \alpha(\vartheta_n) \rangle + \frac{\mu k_2}{2} |\chi_1n|^2 + \frac{k_2}{2} |\nabla \chi_0n|^2
\]
\[
+ k_2 \int_\Omega \tilde{g}(\chi_0n) - bk_1 \int_0^t \int_\Omega \partial_n \partial_t \chi_n
\]
(3.7)
holding for all \(t > 0\). Then, let us note that from the energy estimates and well known compactness results it follows
\[
\vartheta_n \rightarrow \vartheta \quad \text{strongly in } L^2(0, T; H),
\]  
(3.8)
so that, by the first of (2.15), as we take the lim sup of the above relation for \( n \to \infty \), the last term on the right hand side above passes to the limit. The same is true for the term with \( \widehat{g} \). Indeed, for any \( \sigma > 0 \) we have that

\[
\chi_n \to \chi \quad \text{strongly in } C^0([0,T]; H^{1-\sigma} (\Omega)),
\]

(3.9)

so that we can use (A2), (2.1), and the Lebesgue Theorem (observe that here it is essential that \( p < 3 \) in the 3D case). Noting that the term with \( f \) can be treated thanks to (3.4), we finally deduce that the lim sup of the left hand side is not greater than the quantity obtained by rewriting the right hand side without the index \( n \). Comparing with the limit system tested by the same type of test functions used before and owing again to the decomposition (3.2) of \( \widehat{\alpha} \), we then get the inequalities in (3.5–3.6), which actually concludes the proof of Theorem 2.9.

\[\Box\]

### 3.2 Proof of Theorem 2.10

Let us (only in this Section) assume, just for simplicity of notation, that \( k_1 = k_2 = 1 \). In the sequel, the symbol \( c \) will indicate a generic positive constant only depending on the data of the problem, but neither on \( \mu \) nor on \( t \), and possibly varying even within a single line. Let us introduce the functional

\[
\mathcal{G}(v) := \int_{\Omega} \widehat{g}(v(x)) \, dx, \quad \forall v \in V.
\]

(3.12)

Notice that \( \mathcal{G}(v) \) is finite thanks to (A3). More precisely, using (A2–A5) and relying on the “approximated monotonicity” property stated in (A5), it is not difficult to prove that

\[
\mathcal{G}(v) \geq -c_1
\]

(3.10)

\[
(g(v), v) \geq \frac{1}{2} \mathcal{G}(v) + \frac{\lambda}{2} |v|^2_H - c_2
\]

(3.11)

for any \( v \in V \), where the positive constants \( c_1, c_2 \) depend on all the parameters appearing in (A2–A5). Moreover, let us introduce the nonnegative functional

\[
L(v) := \int_{\Omega} (v - \log v)(x) \, dx, \quad \forall v \in V.
\]

(3.13)

Consider now the auxiliary variables \( \xi := \chi_t + \varepsilon \chi \) and \( u := \vartheta^{-1} \) for a fixed \( \varepsilon \in (0, 1/4) \) and test (2.17) by \( \xi \). This gives

\[
\frac{1}{2} \frac{d}{dt} \left( \| \chi \|^2_H + \mu |\xi|^2 + 2 \mathcal{G}(\chi) \right) + \varepsilon \| \chi \|^2_H + (1 - \varepsilon \mu) |\xi|^2 + \varepsilon (g(\chi), \chi)
\]

\[= \varepsilon (1 - \varepsilon \mu)(\chi, \xi) - b(u, \chi_t) - \varepsilon b(u, \chi).
\]

(3.13)
Testing now (2.16) by $\delta \vartheta - u + 1$, where $\delta$ is a positive constant to be chosen later, we have
\[
\frac{1}{2} \frac{d}{dt} \left( 2L(\vartheta) + \delta |\vartheta|^2 \right) + \|u\|^2 + \delta \|\vartheta\|^2 + (1 + \delta)|\nabla \log \vartheta|^2 + \gamma \int_\Gamma \vartheta \\
\leq \langle f, \delta \vartheta - u + 1 \rangle - \delta b(\chi_t, \vartheta) - b(\chi_t, -u + 1) \\
+ \gamma \int_\Gamma (u + \gamma(1 + \delta)|\Gamma|).
\]  
(3.14)

Now, taking (3.10) and (3.12) into account, we define the (nonnegative) functional
\[
E(t) := \delta |\vartheta(t)|^2 + 2L(\vartheta(t)) + \|\chi(t)\|^2_A + \mu |\xi(t)|^2 + 2G(\chi(t)) + 2c_1.
\]  
(3.15)

We add now the term $2\varepsilon L(\vartheta)$ to both sides in (3.13) and estimate it on the right hand side as follows
\[
2\varepsilon L(\vartheta) \leq \frac{\delta}{4} \|\vartheta\|^2 + c + \varepsilon \|u\|^2,
\]  
(3.16)

where $c$ depends also on $\delta$, of course. Now, summing up (3.13) and (3.14), using (3.16), splitting some terms by the Young inequality, and observing that a pair of terms cancel, we obtain
\[
\frac{1}{2} \frac{d}{dt} E + \frac{\delta}{2} \|\vartheta\|^2 + \gamma \int_\Gamma \vartheta + (1 + \delta)|\nabla \log(\vartheta)|^2 + \varepsilon \|\chi\|^2_A + (1 - \varepsilon \mu) |\xi|^2 \\
+ \varepsilon (g(\chi), \chi) + \frac{1}{2} \|u\|^2 + 2\varepsilon L(\vartheta) \\
\leq c + c\|f\|^2 + \varepsilon (1 - \varepsilon \mu)(\chi, \xi) - \varepsilon b(u, \chi) - \delta b(\chi_t, \vartheta) - b(\chi_t, 1),
\]  
(3.17)

and it remains to estimate the terms on the right hand side. Actually, if $\varepsilon$ is sufficiently small (say $\varepsilon \leq 1/2\mu_0$), we clearly have
\[
\varepsilon (1 - \varepsilon \mu)(\chi, \xi) \leq \frac{\varepsilon}{4} |\chi|^2 + c\varepsilon |\xi|^2.
\]  
(3.18)

Moreover,
\[
-\varepsilon b(u, \chi) \leq \frac{\varepsilon}{4} |\chi|^2 + c\varepsilon |u|^2.
\]  
(3.19)

Finally,
\[
\delta b(\chi_t, \vartheta) \leq \frac{1}{4} |\xi|^2 + 2\delta^2 b^2 |\vartheta|^2 + c\varepsilon^2 |\chi|^2 \quad \text{and}
\]
\[
b(\chi_t, 1) \leq \frac{1}{4} |\xi|^2 + c\varepsilon^2 |\chi|^2 + c.
\]  
(3.20)

Now, collecting estimates (3.17–3.21), using (A3) and (3.11), and choosing $\varepsilon$ and $\delta$ sufficiently small, we obtain that there exist positive constants $c_3, \varepsilon_0$ depending on data, on $\varepsilon$ and $\delta$, but not on $\mu$, and such that
\[
\frac{d}{dt} E + \varepsilon_0 E + \varepsilon_0 |\nabla \vartheta|^2 + \frac{1}{2} \|u\|^2 + |\nabla \log \vartheta|^2 \leq c_3 (1 + \|f\|^2).
\]  
(3.22)
Hence, by the Gronwall lemma in the differential form we easily get that

\[ E(t) \leq E(0)e^{-\varepsilon_0 t} + \frac{c_3(1 + \|f\|_*^2)}{\varepsilon_0} \]  \hspace{1cm} (3.23)

for all \( t \in (0, +\infty) \). It is also clear that, for \( \varepsilon \) small enough and for some \( c_4, c_5 > 0 \) still depending on \( \varepsilon, \delta \), but not on \( \mu \),

\[ E(t) \geq c_4(\|\vartheta(t)\|^2 + \|\log^+ \vartheta(t)\|_{L^1(\Omega)} + \|\chi(t)\|^2_A + \mu|\chi(t)|^2) \]  \hspace{1cm} (3.24)

and

\[ E(0) \leq c_5(1 + |\vartheta_0|^2 + \|\log^+ \vartheta_0\|_{L^1(\Omega)} + \|\chi_0\|_{L^{p+1}(\Omega)} + \mu|\chi_1|^2) \]  \hspace{1cm} (3.25)

(note that we used here (2.1)). The thesis is now a consequence of (3.23–3.25). \( \Box \)

Integrating now (3.22) over \((t, t + 1)\) for a generic \( t \geq 0 \), we have the following

**Corollary 3.1.** Assume the same hypotheses of Theorem 2.10. Then, for any \( d_X \)-bounded set \( \mathcal{M} \subset X \), there exists a constant \( \mathcal{R}_\mathcal{M} \) such that

\[ \|\vartheta^{-1}\|_{T_*^2(V)}^2 + \|\vartheta\|_{T_*^2(V)}^2 \leq \mathcal{R}_\mathcal{M} \]  \hspace{1cm} (3.26)

for all trajectories \((\vartheta(t), \chi(t), \chi_t(t)) = S_X(t)(\vartheta_0, \chi_0, \chi_1)\) with \((\vartheta_0, \chi_0, \chi_1) \in \mathcal{M}\).

## 4 Global attractor in the 1D case

The aim of this section is the proof of Theorem 2.11; hence hereafter we will suppose that \( \Omega \) is a subset of \( \mathbb{R} \), i.e., we consider the 1D case. We assume, for simplicity and without loss of generality, \( k_1 = k_2 = b = 1 \) and consider the absorbing set \( \mathcal{B}_X \) given by Theorem 2.10. Taking

\[ (\vartheta_0, \chi_0, \chi_1) \in \mathcal{B}_X, \]  \hspace{1cm} (4.1)

we decompose the pair \((\vartheta, \chi)\) solving (2.12–2.18) as

\[ (\vartheta, \chi) = (\vartheta_d, \chi_d) + (\vartheta_c, \chi_c) \]  \hspace{1cm} (4.2)

with

\[ \vartheta_d \equiv 0, \]  \hspace{1cm} (4.3)

\[ \mu \partial_t \chi_d + \partial_t \chi_d + A \chi_d = 0, \]  \hspace{1cm} (4.4)

\[ \chi_d(0) = \chi_0, \quad \partial_t \chi_d(0) = \chi_1, \]  \hspace{1cm} (4.5)

and

\[ \partial_t \vartheta_c + J(\vartheta_c - \vartheta_c^{-1}) = f - \partial_t \chi, \]  \hspace{1cm} (4.6)

\[ \mu \partial_t \chi_c + \partial_t \chi_c + A \chi_c + g(\chi) = -u, \]  \hspace{1cm} (4.7)

\[ \vartheta_c(0) = \vartheta_0, \quad \chi_c(0) = 0, \quad \partial_t \chi_c(0) = 0, \]  \hspace{1cm} (4.8)
where \( u := \vartheta^{-1} \). Our aim is now to show that \((\chi_d(t), \partial_t \chi_d(t))\) is vanishing in \( V \times H \) as \( t \nearrow +\infty \) and \((\partial_c(t), \chi_c(t), \partial_t \chi_c(t))\) lies in a compact set \( K_\mu(t) \) of \( X \) (depending on \( t \) and \( \mu \)) for all fixed \( t \geq 1 \). Actually, \( K_\mu(t) \) will be given by a bounded set in \( V_r \) with \( r = 1/8 \) (cf. (2.26)).

In order to accomplish the first purpose, we recall here a Lemma whose proof can be obtained by suitably modifying the simple argument given in [8, Lemma 5.1]. Actually, the situation here is even simpler due to the 1D setting.

**Lemma 4.1.** Let \( \mu_0 > 0 \) be fixed and \( \mu \in (0, \mu_0] \). Assume that the hypotheses of Theorem 2.10 hold. Then there exist two constants \( M \geq 0 \) and \( \kappa > 0 \), independent of \( \mu \), such that the following bound holds

\[
\|\chi_d(t)\|^2 + \mu |\partial_t \chi_d(t)|^2 \leq M e^{-\kappa t} \quad \forall t \geq 0.
\]  

(4.9)

Regarding the compact part, we are able to prove the following:

**Lemma 4.2.** Let \( \mu_0 > 0 \) be fixed and \( \mu \in (0, \mu_0] \). Assume that the hypotheses of Theorem 2.10 hold and let \( \Omega \subset \mathbb{R} \). Then, for every \( t \geq 1 \) there exists a compact set \( K_\mu = K_\mu(t) \subset X \) such that the solution \((\partial_c, \chi_c)\) of (4.6–4.8) satisfies

\[
(\partial_c(t), \chi_c(t), \partial_t \chi_c(t)) \in K_\mu(t) \quad \forall t \geq 1.
\]  

(4.10)

**Proof.** In this proof, let us denote by \( C \) a positive constant depending (increasingly) on the size of \( B_X \), \( t \), \( 1/\mu \) and on the structural quantities of the problem. Moreover, referring e.g. to [21, p. 54], let us consider (for all \( s \in \mathbb{R} \)) the family of Hilbert spaces

\[
H^s := \text{dom}(A^{s/2}) \quad \text{with inner product } \langle \cdot, \cdot \rangle_{H^s} = \langle A^{s/2} \cdot, A^{s/2} \cdot \rangle.
\]

First of all, let us note that thanks to Theorem 2.10 and Lemma 4.1, we have the useful bound

\[
\|\chi(t)\| + |\partial_t \chi(t)| + \|\chi_d(t)\| + |\partial_t \chi_d(t)| + \|\chi_c(t)\| + |\partial_t \chi_c(t)| \leq C.
\]  

(4.11)

Then, following the idea devised in [8, Lemma 5.2], we test (4.7) by \( A^{1/4} \partial_t \chi_c \) and integrate over \((0, t)\). Integrating by parts and using (4.8), we get the inequality

\[
\frac{\mu}{2} |A^{1/8} \partial_t \chi_c(t)|^2 + \frac{1}{2} |A^{5/8} \chi_c(t)|^2 \leq -\langle g(\chi(t)), A^{1/4} \chi_c(t) \rangle + \int_0^t \langle g'(\chi) \partial_t \chi, A^{1/4} \chi_c \rangle - \int_0^t \langle u, A^{1/4} \partial_t \chi_c \rangle.
\]  

(4.12)

Next, thanks to (A3) and (4.11), it is easy to show that the first two terms on the right hand side are bounded. As for the third integral, we may note that

\[
-\int_0^t \langle u, A^{1/4} \partial_t \chi_c \rangle \leq \frac{1}{2} \int_0^t (\|u\|^2 + \|A^{1/4} \partial_t \chi_c\|^2).
\]  

(4.13)

Then, letting \( v \in V \) with \( \|v\| \leq 1 \), thanks to the continuous embedding of \( V \) into \( H^{1/2}(\Omega) \) we have

\[
|A^{1/4}v| \leq C.
\]  

(4.14)
and consequently, thanks also to (4.11), it follows
\[ \|A^{1/4}\partial_t\chi_c\|_\ast = \sup_{\|v\|\leq 1} \left| \langle \partial_t\chi_c, A^{1/4}v \rangle \right| \leq C. \]

Hence, thanks to the last inequality and Corollary 3.1 we deduce from (4.13) that
\[ -\int_0^t \langle u, A^{1/4}\partial_t\chi_c \rangle \leq C. \]

Thus, we readily obtain the bound
\[ \mu|A^{1/8}\partial_t\chi_c(t)|^2 + |A^{5/8}\chi_c(t)|^2 \leq C. \]  \hspace{1cm} (4.15)

Next, we have to find an estimate for \( \partial_c \) and \( \partial_c^{-1} \). Hence, let us test (4.6) by \( \partial_t\partial_c + \partial_t(-\partial_c^{-1}) \) (observe that \( \partial_c = \vartheta \) so that we can omit here the subscript “c” and note, as before, \( u := \vartheta^{-1} \)), obtaining
\[
\int_\Omega \left( \partial_t^2 + \frac{\partial_t^2}{\vartheta^2} \right)(t) + \frac{1}{2}\frac{d}{dt}\|\vartheta - u(t)\|^2 \\
= \left\langle f, \frac{d}{dt}(\vartheta - u)(t) \right\rangle - \int_\Omega \partial_t\chi(t) \left( \frac{\partial_t}{\vartheta^2} + \vartheta \right)(t). \]  \hspace{1cm} (4.16)

Then, let us employ the Hölder inequality and the continuous embedding of \( V \) into \( L^\infty(\Omega) \) (holding since \( N = 1 \)) in order to get
\[
-\int_\Omega \partial_t\chi(t) \left( \frac{\partial_t}{\vartheta^2} + \vartheta \right)(t) \leq |\partial_t\chi(t)||\frac{\partial_t}{\vartheta}(t)||u(t)||_{L^\infty(\Omega)} + |\partial_t\chi(t)||\vartheta(t)| \\
\leq \delta \left( \left| \frac{\partial_t}{\vartheta}(t) \right|^2 + |\vartheta(t)|^2 \right) + C_\delta |\partial_t\chi(t)|^2 (1 + \|u(t)\|^2) \]  \hspace{1cm} (4.17)

for all positive \( \delta \) and some positive constant \( C_\delta \). Now, using (4.11) and (4.17) and choosing \( \delta \) sufficiently small, (4.16) becomes
\[
\int_\Omega \left( \partial_t^2(t) + \frac{\partial_t^2}{\vartheta^2}(t) \right) + \frac{d}{dt}\left( \|\vartheta - u(t)\|^2 - \langle f, \vartheta(t) - u(t) \rangle + \tilde{C} \right) \leq C(1 + \|u(t)\|^2). \]  \hspace{1cm} (4.18)

Hence, we obtained exactly an inequality of the same type as (2.8) with
\[
a(t) = 0, \quad \phi(t) = C(1 + \|u(t)\|^2), \]

and
\[
y(t) = \|\vartheta(t)\|^2 + \|u(t)\|^2 + 2|\nabla \log \vartheta(t)|^2 - 2\gamma|\Gamma| \\
- \langle f, \vartheta(t) - u(t) \rangle + \tilde{C}, \]  \hspace{1cm} (4.19)

where \( \tilde{C} \) is chosen in such a way that \( y(t) \geq 0 \) for all \( t \in (0,T) \). Indeed we have
\[
- \langle f, \vartheta(t) - u(t) \rangle \geq -\zeta\left( \|\vartheta(t)\|^2 + \|u(t)\|^2 \right) - C_\zeta \left( \|h\|^2_{L_2(\Gamma)} + |m|^2 \right) \]
for all positive $\zeta$ and some $C_{\zeta}$. Hence, choosing $\zeta \leq 1/2$ and $\tilde{C}$ in such a way that

\[ \tilde{C} - C_{\zeta}(\|h\|_{L^2(I)}^2 + |m|^2) - 2\gamma|\Gamma| \geq 0, \]

we have the positivity of $y(t)$ for a.e. $t \in (0, T)$. Moreover, thanks to Corollary 3.1, hypotheses (2.9) are satisfied. Thus, applying Lemma 2.4 we get

\[ y(t + 1) \leq C \quad \forall t \geq 0. \quad (4.20) \]

Finally, collecting (4.15) and (4.19–4.20) we get the bound

\[ \|\chi_c(t)\|_{H^{5/4}} + \|\partial_t\chi_c(t)\|_{H^{1/4}} + \|\vartheta_c(t)\| + \|\vartheta_c^{-1}(t)\| \leq C \quad \forall t \geq 1, \quad (4.21) \]

so that we conclude that, at least for $t \geq 1$, the solution $(\vartheta_c(t), \chi_c(t))$ to (4.6–4.8) lies in a bounded set $K_{\mu}$, depending on $\mu$ and $t$, of the space $\mathcal{V}_r$ defined in (2.26) and with $r = 1/8$. Hence, the thesis follows from the compactness of the embedding of $\mathcal{V}_r$ into $\mathcal{X}$ (cf. Prop. 2.12). \( \square \)

As a direct consequence of Lemmas 4.1, 4.2, we have that

\[ \lim_{t \to +\infty} \alpha_{\mathcal{X}}(S_{\mathcal{X}}(t)B_{\mathcal{X}}) = 0, \quad (4.22) \]

where $\alpha_{\mathcal{X}}$ is the Kuratowski measure of noncompactness (cf. [10, p. 17]) defined as

\[ \alpha_{\mathcal{X}}(B) = \inf \{d > 0 : B \text{ has a finite cover of balls of } \mathcal{X} \text{ of diameter } \leq d\}. \]

At this point, standard arguments in the theory of dynamical systems (cf., e.g., [10, 21]) permit us to conclude the proof of Theorem 2.11, i.e., to get the existence of the global (compact) attractor $A_{\mathcal{X}} = A_{\mathcal{X}}(\mu) \subset \mathcal{X}$ for the process $S_{\mathcal{X}}$. \( \square \)

5 Dissipativity in $\mathcal{D}_r$

In order to construct the universal attractor for the 2-3D system we first need to refine our well-posedness and dissipativity results by taking care of the more regular initial data.

5.1 Proof of Theorem 2.13

It is structured very similarly to the proof of Theorem 2.9. Namely, we first have to perform an energy estimate corresponding to the stronger regularity setting and then a semicontinuity argument. We just give the highlights of this procedure. Let us assume, similarly with before, that \{$(\vartheta_{0n}, \chi_{0n}, \chi_{1n})$\} $\subset \mathcal{D}_r$ is a sequence of initial data converging to $(\vartheta_0, \chi_0, \chi_1) \in \mathcal{D}_r$ in $d_\mathcal{D}$. Moreover, let us fix $t > 0$ and set $(\vartheta_n, \chi_n, \partial_t\chi_n)(t) := S_\mathcal{D}(t)(\vartheta_{0n}, \chi_{0n}, \chi_{1n})$ (resp., $(\vartheta, \chi, \chi_t)(t) := S_\mathcal{D}(t)(\vartheta_0, \chi_0, \chi_1)$). Of course, the regularity of data depends on the choice of $r$ which is given in (2.30). Then, the energy estimate can be given simply testing (2.17) by $A^\ast\partial_t\chi_n$
and performing standard computations. For the control of the term with $g$, which is the only difficult point, we can proceed as in (5.13) below. Note that this gives bounds, uniform in $n$, of
\[
\| \partial_t \chi_n \|_{L^\infty(0,T;H^r(\Omega))} + \| \chi_n \|_{L^\infty(0,T;H^{1+r}(\Omega))},
\]
(5.1)

together with the related weak-* convergences.

Then, the semicontinuity argument is operated by multiplying again (2.17) by $A^r \partial_t \chi_n$ and integrating over $\Omega \times (0,t)$, with $t \geq 0$ arbitrary. We get
\[
\frac{\mu}{2} |A^{r/2} \partial_t \chi_n(t)|^2 + \int_0^t |A^{r/2} \partial_t \chi_n|^2 + \frac{1}{2} |A^{(1+r)/2} \chi_n(t)|^2
\]
\[\leq \frac{\mu}{2} |A^{r/2} \partial_t \chi_n|^2 + \frac{1}{2} |A^{(1+r)/2} \chi_n|^2
\]
\[\quad - b \int_0^t (u_n, A^r \partial_t \chi_n) - \int_0^t (g(\chi_n), A^r \partial_t \chi_n)
\]
(5.2)

and we have to show that, as we take the limsup, the latter two terms on the right hand side tend to the right limits. Indeed, the integral with $u_n$ is easily treated thanks to the boundedness of $u_n$ in $L^2(0,T;V)$. As for the last term, we need to integrate it by parts in time, getting
\[
- \int_0^t (g(\chi_n), A^r \partial_t \chi_n) = \int_0^t (g'(\chi_n) \partial_t \chi_n, A^r \chi_n)
\]
\[\quad - (g(\chi_n(t)), A^r \chi_n(t)) + (g(\chi_n), A^r \chi_n).
\]
(5.3)

Let us just show that the first term on the right hand side passes to the limit. Indeed, the other ones are easier to deal with. Thus, noting that (5.1) yields in particular
\[
A^r \chi_n \to A^r \chi \quad \text{strongly in } L^\infty(0,T;H^{1-2r}(\Omega))
\]
(5.4)

and owing to the continuous embedding of $H^{1-2r}(\Omega)$ into $L^{6/(1+4r)}(\Omega)$, it is sufficient to show that
\[
g'(\chi_n) \partial_t \chi_n \to g'(\chi) \partial_t \chi \quad \text{weakly in } L^1(0,T;L^\sigma(\Omega)), \quad \sigma := 6/(5-4r).
\]
(5.5)

Actually, it is clear that, thanks to (5.1), (A2–A3), and the Lebesgue Theorem,
\[
g'(\chi_n) \to g'(\chi) \quad \text{strongly in } L^2(Q), \quad \partial_t \chi_n \to \partial_t \chi \quad \text{weakly in } L^2(Q).
\]
(5.6)

Thus, to get (5.5) it is enough to show boundedness in the space $L^1(0,T;L^\sigma(\Omega))$.

Using the Hölder and Young inequalities, we obtain
\[
\int_\Omega |g'(\chi_n) \partial_t \chi_n|^\sigma \leq \|g'(\chi_n)|^\sigma\|_{L^{(5-4r)/(2-2r)}} \|\partial_t \chi_n|^\sigma\|_{L^{(5-4r)/(3-2r)}}
\]
\[\leq c \left( 1 + \|\chi_n\|_{L^{3/(3-4r)}} \right) \|\partial_t \chi_n\|_{L^{6/(3-2r)}}.
\]
(5.7)

Next, let us take the $1/\sigma$ power, integrate in time, and note that, due to (5.1) and to the continuous embedding of $H^r(\Omega)$ into $L^{6/(3-2r)}(\Omega)$, the term with $\partial_t \chi_n$ is bounded uniformly in time. Since the term with $\chi_n$ is easily controlled by using condition (2.30) on $r$, the assert follows. □
5.2 Proof of Theorem 2.15

We have to show that there exists a bounded set $B_D$ such that for any bounded set $B \subset D_r$, $S_D(t)B \subset B_D$ for $t$ large enough. Actually, since $B$ is also bounded in $\mathcal{X}$, we can take advantage of the dissipativity estimates already proved in Theorem 2.10 and assume for simplicity that $B \subset B_\mathcal{X}$ (of course the inclusion is actually true just for $S_D(t)B$ as $t$ is large enough).

Let us then consider the equation
\begin{equation}
\mu \dot{\chi} + \chi_t + A\chi + g(\chi) = -bu
\end{equation}
where $u = \vartheta^{-1}$. We deal directly with the 3D case, which is the most difficult one. Then, let us take the auxiliary variable $\xi := \chi_t + \epsilon\chi$, with $\epsilon > 0$ to be chosen later, and test (5.8) by $A^r\xi$ getting the following equality:
\begin{align}
\frac{\mu}{2} \frac{d}{dt} |A^{r/2}\xi|^2 + (1 - \mu\epsilon)|A^{r/2}\xi|^2 + \frac{1}{2} \frac{d}{dt} |A^{(r+1)/2}\chi|^2 + \epsilon |A^{(r+1)/2}\chi|^2 \\
= \epsilon (1 - \mu\epsilon)(A^{r/2}\chi, A^{r/2}\xi) - (A^{r/2}g(\chi), A^{r/2}\xi) - b(u, A^r\xi).
\end{align}

We recall that the symbol $c$ denotes some positive constant depending on the data of the problem but neither on $\mu$ nor on $t$. Moreover, we indicate by $c_\eta$ a constant which, in addition, might depend increasingly on the positive number $\eta$.

Let us now bound the right hand side of (5.9). First, we notice that
\begin{equation}
\epsilon (1 - \mu\epsilon)(A^{r/2}\chi, A^{r/2}\xi) \leq \frac{\epsilon}{2} |A^{r/2}\xi|^2 + \frac{\epsilon}{2} |A^{r/2}\chi|^2.
\end{equation}
Then, we bound the last term simply noting that
\begin{align}
-b(u, A^r\xi) & \leq c\|u(t)\|^2 + c\|A\chi(t)\|_s^2 + c\|A^r\chi(t)\|_s^2 \\
& \leq c\|u(t)\|^2 + c,
\end{align}
where the last $c$ can be thought as being independent of time thanks to Theorem 2.10.

Let us now come to the estimate of the second term on the right hand side of (5.9). First of all, we notice that, for $N = 3$, and $s := 6/(5 - 2r)$, $W^{1,s}(\Omega)$ is continuously embedded into $D(A^{r/2})$. Thus, being
\begin{equation}
-(A^{r/2}g(\chi), A^{r/2}\xi) \leq \frac{1}{4} |A^{r/2}\xi|^2 + \|g(\chi)\|^2_{D(A^{r/2})},
\end{equation}
we just have to compute
\begin{align}
\|g(\chi)\|_{D(A^{r/2})}^s & \leq c \int_{\Omega} |g(\chi)|^s + c \int_{\Omega} |g'(\chi)|^s |\nabla\chi|^s \\
& \leq c + c \int_{\Omega} |\chi|^{6p/(5-2r)} + c\|g'(\chi)^{6/(5-2r)}\|_{L^{(5-2r)/(2-2r)}} \|\nabla\chi|^{6/(5-2r)} \|_{L^{(5-2r)/3}} \\
& \leq c + c \int_{\Omega} |\chi|^{6p/(5-2r)} + c \left(1 + \|\chi|^{p-1}\|^{6/(5-2r)}_{L^{(1-r)}} \right) |\nabla\chi|^{6/(5-2r)}_H \\
& \leq c + c \int_{\Omega} |\chi|^{6p/(5-2r)} + c \left(1 + \|\chi|^{(6p-6)/(5-2r)}_{L^{(3p-3)/(1-r)}} \right) |\nabla\chi|^{6/(5-2r)}_H,
\end{align}
\begin{equation}
\Box
\end{equation}
where we have repeatedly used assumption (A3) and the Hölder and Young inequalities.

Thus, it is now easy to see that the choice \((2.30)\) guarantees that all the terms on the above right hand side are bounded uniformly in time. Finally, collecting all the estimates \((5.9–5.13)\) and choosing \(\varepsilon\) sufficiently small (let us say \(\varepsilon < 1/2\mu_0\)), we get (for \(\varepsilon_0 = \varepsilon\))

\[
\frac{d}{dt}\Phi(t) + \varepsilon_0 \Phi(t) \leq c + c\|u(t)\|^2 \quad \text{for } t \geq 0
\]

with

\[
\Phi(t) := \mu\|\chi_t(t)\|^2_{H^r(\Omega)} + \|\chi(t)\|_{H^{1+r}(\Omega)}^2.
\]

Now, using a generalized version of the Gronwall lemma (cf., e.g., [6, Lemma 2.5]) on \([0, t]\) \((t \in (0, +\infty))\) together with (A9), the bound for \(u = \vartheta^{-1}\) of Corollary 3.1, and Remark 2.3 (applied with \(\tau = s = 0\)), we obtain

\[
\Phi(t) \leq 2\Phi(0)e^{-\varepsilon_0 t} + \frac{c}{\varepsilon_0} + 2\int_0^t e^{-\varepsilon_0(t-y)}\|u(y)\|^2\,dy
\]

\[
\leq 2\Phi(0)e^{-\varepsilon_0 t} + \frac{c}{\varepsilon_0} + \frac{e^{\varepsilon_0 c}}{1 - e^{-\varepsilon_0}} \quad \text{for all } t \in (0, +\infty).
\]

This concludes the proof of Theorem 2.15. \(\Box\)

6 Global attractor in the 3D case

The aim of this section is the proof of Theorem 2.17, that is the existence of the attractor for the semigroup \(S_D(t)\) defined in Theorem 2.13 and related to the 2-3D cases. Still, we will directly assume the 3D setting, which is more complicated. The 2D case can be addressed with minor modifications. Hence, let us consider the absorbing set \(B_D\) provided by Theorem 2.15 and redefine it by setting

\[
C_D := \bigcup_{t \geq 0} S_D(t)B_D.
\]

It is a standard matter to show that \(C_D\) is still a bounded absorbing set for \(S_D(t)\). Furthermore, by construction \(C_D\) is positively invariant, i.e. \(S_D(t)C_D \subset C_D\) for all \(t \geq 0\). Of course, \(C_D = C_D(r)\) depends on the choice of \(r\), i.e., actually, on \(p\) in (A3).

Let us then assign an initial datum

\[
(\vartheta_0, \chi_0, \chi_1) \in C_D
\]

and assume that the system evolves from this datum. Let us also decompose the system as in Section 4 (with the same notation) and start by proving some further regularity for the temperature.
First step. We now perform an estimate of Moser type (cf. also [17, Section 4]) on \( u = \vartheta^{-1} \). With this aim, we take a coefficient \( q \geq 3 \) and test (2.16) by \( -u^q \), obtaining

\[
\frac{1}{(q-1)} \frac{d}{dt} \left( \int_{\Omega} u^{q-1} \right) + \frac{4qk_1}{(q-1)^2} \int_{\Omega} |\nabla u^{q-1}|^2 + \frac{4qk_2}{(q+1)^2} \int_{\Omega} |\nabla u^{q+1}|^2 \\
+ \gamma k_2 \int_{\Gamma} u^{q+1} = \gamma k_1 \int_{\Gamma} u^{q-1} - \int_{\Omega} mu^q - \gamma \int_{\Gamma} hu^q + b \int_{\Omega} X_t u^q. \quad (6.3)
\]

Now, let us multiply (6.3) by \((q - 1)\) and estimate the terms on the right hand side. Since it is sufficient to perform a finite number of steps, we do not need to take care of the dependence on \( q \) of the various constants; therefore, the generic positive constant \( c \) will now be allowed to depend also on \( q \).

Let us start with the last term: using the that \(|\chi_i(t)|\) is bounded (cf. (4.11)) together with the Hölder and Young inequalities, we get

\[
(q - 1)b \int_{\Omega} X_t u^q \leq c \left( \int_{\Omega} u^{2q} \right)^{1/2} \leq c \|u^{2q}\|_{L^{2q+2}(\Omega)}^{1/2} \cdot |\Omega|^{1/2(q+1)} \\
\leq c \|u\|_{L^{2q+2}(\Omega)}^q \leq c \|u^{q+1}\|_{L^2(\Omega)}^{2(q+1)} \\
\leq \frac{1}{4} \|u^{q+1}\|_{L^2(\Omega)}^2 + c. \quad (6.4)
\]

Let us note now that we can analogously treat the term with \( m \) since \( m \in H \) (cf. (A6)). Finally, regarding the boundary integrals, using assumption (A10) we have

\[
(q - 1)\gamma \left( k_1 \int_{\Gamma} u^{q-1} - \int_{\Gamma} hu^q \right) \leq (q - 1) \frac{\gamma k_2}{2} \int_{\Gamma} u^{q+1} + c. \quad (6.5)
\]

Now, collecting (6.3–6.5) and using the continuous embedding of \( V \) into \( L^6(\Omega) \) we get (for all \( t > 0 \))

\[
\frac{d}{dt} \left( \int_{\Omega} u^{q-1}(t) \right) + \frac{4qk_1}{(q-1)} \int_{\Omega} |\nabla u^{q-1}(t)|^2 + c^* \left( \int_{\Omega} u^{3(q+1)}(t) \right)^{1/3} \leq c, \quad (6.6)
\]

where \( c^* \) is a positive constant suitably depending on \( q \), \( \gamma \) and \( k_2 \). Now, let us start the bootstrap procedure by choosing \( q = q_0 := 3 \). Indeed, with such a choice, the assumptions of the uniform Gronwall Lemma 2.4 are fulfilled. Indeed, Corollary 3.1 provides a bound for \( u \) in \( T^6_q(H) \). Thus, we obtain that

\[
\|u(t)\|_{L^{q+1}_q(\Omega)}^{q+1-i} \leq c \quad \forall t \geq i + 1 \quad (6.7)
\]

holds for \( i = 0 \). Next, integrating (6.6) between \( t \) and \( t+1 \) with arbitrary \( t \geq i+1 \) we get that, for the present just as \( i = 0 \),

\[
u(t) \text{ is bounded in } T^{q+1}_{i+1}(L^{3(q+1)}(\Omega)). \quad (6.8)
\]

This allows us to perform an induction argument. Actually, assuming (6.7–6.8) at the level \( i \), we see that the assumptions of the uniform Gronwall Lemma are fulfilled as we take \( q_{i+1} := q_i + 2 \) (i.e. \( q_{i+1} - 1 = q_i + 1 \), which is required to control the first term in (6.6)). This means, in particular, that the bound (6.8) holds for any \( i \in \mathbb{N} \). Let us notice, furthermore, that \( q_i = q_0 + 2i = 2i + 3 \), so \( q_i \not\to \infty \) for \( i \not\to \infty \). However, it is clear that also the explicit constants involved in the bounds above might possibly explode as \( i \to \infty \).
Second step. Let us now multiply \((2.16)\) by \(\partial_t \alpha(v) = k_1 \partial_t + k_2 \partial_t / \vartheta^2\), so that, for all \(t \geq 0\), we get

\[
k_1 |\vartheta_t(t)|^2 + k_2 |\partial_t (\log \vartheta(t))|^2 + \frac{1}{2} \frac{d}{dt} \|k_1 \vartheta - k_2 \vartheta^{-1}\|^2(t)
\]

\[
\leq \frac{d}{dt} \langle f, k_1 \vartheta - k_2 \vartheta^{-1} \rangle(t) - bk_1 \int_\Omega \chi_t(t) \vartheta_t(t) - bk_2 \int_\Omega \chi_t(t) \frac{\vartheta_t(t)}{\vartheta(t)} \frac{1}{\vartheta(t)}
\]

(6.9)

and again we have to control the right hand side. First, owing to (A6),

\[
-bk_1 \int_\Omega \vartheta_t(t) \chi_t(t) \leq \frac{k_1}{4} |\vartheta_t(t)|^2 + c |\chi_t(t)|^2.
\]

(6.10)

Moreover, we have to treat the last term in (6.9). Using the Hölder inequality with exponents \(6/(3 - 2r)\), \(2\) and \(3/r\) we get

\[
-bk_2 \int_\Omega \chi_t(t) \frac{\vartheta_t(t)}{\vartheta(t)} \frac{1}{\vartheta(t)} \leq \frac{k_2}{4} |\partial_t (\log \vartheta(t))|^2 + c \|\chi_t(t)\|_{L^2(3-2r)(\Omega)}^2 \|\vartheta^{-1}(t)\|_{L^{3/r}(\Omega)}^2.
\]

Next, we define the functional

\[
\Psi(\vartheta) := \|k_1 \vartheta - k_2 \vartheta^{-1}\|^2 - \langle f, \alpha(\vartheta) \rangle + C(f),
\]

(6.12)

where we have set \(C(f) := \|f\|^2/2\) so that

\[
\Psi(\vartheta) \geq \frac{1}{2} \|k_1 \vartheta - k_2 \vartheta^{-1}\|^2 \geq 0.
\]

(6.13)

Thus, using that the \(L^{6/(3-2r)}\) norm of \(\chi_t\) is uniformly bounded thanks to (6.2), the invariance of \(C_D\), and the continuous embedding of \(D(A^{\gamma/2})\) into \(L^{6/(3-2r)}(\Omega)\), relation (6.9) can be clearly rewritten as

\[
\frac{d}{dt} \Psi(\vartheta) + c \left( |\vartheta_t|^2 + |\partial_t (\log \vartheta)|^2 \right) \leq c \left( 1 + \|\vartheta^{-1}\|^2_{L^{3/r}(\Omega)} \right)
\]

(6.14)

for all \(t \in (0, +\infty)\) and for some \(c > 0\) also depending on \(f\). Then, the uniform Gronwall Lemma 2.4, applied with a starting time \(\tau > 0\), yields

\[
\Psi(\vartheta(t), \chi(t)) \leq c \left( 1 + \|\vartheta^{-1}\|^2_{L^2(3/r)(\Omega)} \right) + \|\Psi(\vartheta, \chi)\|_{\mathcal{T}_2^1(\mathbb{R})} \quad \forall \ t \geq \tau + 1
\]

(6.15)

and the right hand side is a bounded quantity thanks to (6.8) provided we choose \(i\) so large that \((g_i + 1) = 2i + 4 \geq 1/r\) (and correspondingly we take \(\tau \geq i + 1\)). Actually, the contribution of \(\Psi\) on the right hand side gives no trouble since

\[
\|\Psi(\vartheta, \chi)\|_{\mathcal{T}_2^1(\mathbb{R})} \leq c \|k_1 \vartheta - k_2 \vartheta^{-1}\|^2_{\mathcal{T}_2(V)} + \|f\|^2 \leq c + c(C_D) + \|f\|^2
\]

(6.16)

by Corollary 3.1, (A6), and (2.11). Finally, using again (6.13), (6.15) yields

\[
\|\vartheta(t)\|^2 + \|\vartheta^{-1}(t)\|^2 \leq c(C_D) \quad \text{for all } t \geq \tau + 1.
\]

(6.17)
Third step. We take again (for simplicity of notation and without any loss of
generality) the constants \( b, k_1 \) and \( k_2 \) equal to 1. Then, let us recall the decom-
position of (2.17) introduced in Section 4. Namely, we have:

\[
\begin{align*}
\mu \partial_{tt} \chi_d + \partial_t \chi_d + A \chi_d &= 0, \\
\chi_d(0) &= \chi_0, \\
\partial_t \chi_d(0) &= \chi_1
\end{align*}
\]  

and

\[
\begin{align*}
\mu \partial_u \chi_c + \partial_t \chi_c + A \chi_c + g(\chi) &= -u, \\
\chi_c(0) &= 0, \\
\partial_t \chi_c(0) &= 0.
\end{align*}
\]

As before, we start by handling the dissipative part, for which we have the fol-
lowing extension of Lemma 4.1:

**Lemma 6.1.** There exist \( \kappa > 0 \) and \( M \geq 0 \) such that

\[
\| \chi_d(t) \|_{H^{1+r}(\Omega)}^2 + \mu \| \partial_t \chi_d(t) \|_{H^r(\Omega)}^2 \leq M e^{-\kappa t} \text{ for all } t \geq 0.
\]  

**Proof.** It suffices to test (6.18) with \( A^r(\partial_t \chi_d) + \varepsilon A^r(\chi_d) \) and perform standard computations. \( \square \)

The compactness property is given by the following

**Lemma 6.2.** Suppose that \( \mu_0 > 0 \) is fixed and let \( \mu \in (0, \mu_0] \). Then, let \( \mathcal{V}_r \) be
defined as in (2.26), with \( r \) specified by (2.30). Then, for any sufficiently large \( t \) there exists a set \( K_{\mu} = K_{\mu}(r, t) \), compactly embedded into \( \mathcal{D}_r \) and such that

\[
(\vartheta(t), \chi_c(t), \partial_t \chi_c(t)) \subset K_{\mu}.
\]

**Proof.** Let us test (6.20) by \( A^{2r} \partial_t \chi_c \). Then, we can proceed similarly as in the proof of Lemma 4.2, getting

\[
\begin{align*}
\frac{\mu}{2} |A^r \partial_t \chi_c(t)|^2 + \frac{1}{2} |A^{(2r+1)/2} \chi_c(t)|^2 + \int_0^t |A^r \partial_t \chi_c|^2 \\
&\leq -\int_0^t \langle A^r g(\chi), A^r \partial_t \chi_c(t) \rangle - \int_0^t \langle u, A^{2r} \partial_t \chi_c \rangle,
\end{align*}
\]

and again the term with \( g \) needs to be carefully handled. Actually, noting that, for \( N = 3 \) and \( s := 6/(5 - 4r) \), \( W^{1,s}(\Omega) \) is continuously embedded into \( D(A^r) \), we have

\[
\int_0^t \langle A^r g(\chi), A^r \partial_t \chi_c \rangle \leq \frac{1}{4} \int_0^t |A^r \partial_t \chi_c|^2 + \int_0^t \| g(\chi) \|^2_{D(A^r)}.
\]
Thus,

\[-\|g(\chi)\|_{D(A')}^* \leq c + c \int_{\Omega} |\chi|^{6p/(5-4r)} + c \int_{\Omega} |\chi|^{6p/(5-4r)}|\nabla \chi|^{6/(5-4r)} \int_{\Omega} |\chi|^{6p/(5-4r)}|\nabla \chi|^{6/(5-4r)} \leq c + c \int_{\Omega} |\chi|^{6p/(5-4r)} + c \int_{\Omega} |\chi|^{6p/(5-4r)} + c \left(1 + \|\chi\|^{6p/(5-4r)} \right) |\nabla \chi|^{6/(5-4r)} \int_{\Omega} |\chi|^{6p/(5-4r)}|\nabla \chi|^{6/(5-4r)} \int_{\Omega} |\chi|^{6p/(5-4r)}|\nabla \chi|^{6/(5-4r)} \right],

(6.26)

where we have repeatedly used assumption (A3) and the Hölder and Young inequalities. Let us now observe that, thanks to (6.2) and the invariance of \( C_D \), \( \forall t \geq 0 \) \( \nabla \chi(t) \) lies in \( D(\lambda r/2) \), which is continuously embedded into \( L^{6/(3-2r)}(\Omega) \). Moreover, it is easy to check that the remaining terms on the right hand side are bounded uniformly in time. Thus, (6.24) gives \( \forall t \geq 0 \) a boundedness of \( \chi \) in the norm of \( D(A^{(2r+1)/2}) \) and of \( \partial_t \chi \) in the norm of \( D(A') \). Thus, the proof follows now by coupling the compactness estimate (6.17) with the above procedure. □

The proof of Theorem 2.17 is now easily concluded by taking advantage of the Lemmas proved above and reasoning exactly as in the end of Section 4. □

7 Final observations

In this final section we aim to present some further remarks on our approach and make a comparison of our results with those obtained in [8, 9] for the so-called parabolic-hyperbolic Caginalp model (actually the quoted works deal with a slightly more general setting and the system below is a particular case):

\[(\vartheta + b\chi)_t - \Delta \vartheta = f, \quad (7.1)\]
\[\mu \chi_{tt} + \chi_t - \Delta \chi + g(\chi) = b \vartheta. \quad (7.2)\]

For such a system, where the dependence on \( \vartheta \) is linear, the authors of [8, 9] are able to show existence of a family \( \{Z_\mu\} \) of global attractors in the phase space \( H \times V \times H \). Moreover, they consider the parabolic problem formally obtained putting \( \mu = 0 \) in (7.2) and construct a global attractor \( \tilde{Z} \subset H \times V \) for the associated dynamical process (note that here the phase space no longer accounts for \( \partial_t \chi \)). Finally, in the case of \( f = 0 \), they succeed in proving the upper semicontinuity as \( \mu \searrow 0 \) of the lifting \( Z_0 \) of \( \tilde{Z} \) to the three-component phase space \( H \times V \times H \) with respect to the family \( \{Z_\mu\} \). This means that

\[\lim_{\mu \to 0} \text{dist}_{H \times V \times H} (Z_\mu, Z_0) = 0,\]
where dist(·, ·) denotes the unilateral Hausdorff distance with respect to the metric of $H \times V \times H$. This result holds also for the 3D system in the critical case of $p = 3$ (cf. (A3)).

Extending this type of results to the case of the Penrose-Fife system is in our opinion a very hard task. We actually notice that the existence of a compact attractor $\tilde{A}$ for the parabolic Penrose-Fife model (1.1)+(1.5) has been shown in [18] for a wide class of admissible potentials $g$. However, analyzing the relation between $\tilde{A}$ and the family $\{A_\mu\}$ constructed in this paper seems to present at least the following two difficulties.

– First, we have to consider the nonlinear dependence on $\vartheta$ of both equations (1.1) and (1.3). This limits the number of a priori bounds which are available for the coupled system. For instance, it seems not possible to work on the equations obtained through a formal differentiation in time of (1.1) and (1.3). This kind of procedure is more or less an obliged step (cf., e.g., [11, Thm. 2.5]) for proving the upper semicontinuity of the limit attractor. Indeed, in our dissipativity estimates (cf. the functional $E$ defined in (3.15)), the norm of $\partial_t \chi_\mu$ is weighted by $\mu$; instead, an estimate on $\partial_t \chi_\mu$ independent of $\mu$ (cf. [11, (2.29)]) would be needed for this kind of analysis. We also have to remark that the nonlinear dependence on $\vartheta$ is also the reason why, in order to construct the attractor (cf. Theorem 2.17), it seems necessary to work in Sobolev spaces of fractional order (cf. (2.28)) and make accurate use of some technical arguments (e.g., the Moser type estimate on $\vartheta^{-1}$, cf. Section 6).

– The second additional difficulty characterizing the parabolic-hyperbolic Penrose-Fife system is the lack of a global Lyapunov functional in the case of third-type boundary conditions on $\alpha(\vartheta)$, even if $f$ is assumed to be 0. Actually, as one performs the “energy” estimate for the system (1.1)+(1.3) (take $k_1 = k_2 = b = 1$), i.e., tests the first relation by $\vartheta - \vartheta^{-1}$, the second by $\chi_t$, and sums together, the result is

$$\frac{d}{dt} E(t) + \|\vartheta - \vartheta^{-1}\|^2 + |\partial_t \chi|^2 = -\int_\Omega \vartheta \chi_t$$ (7.3)

where the “energy” $E$ is given by

$$E(t) = \int_\Omega \left( -\log \vartheta(t) + \frac{1}{2} \vartheta^2(t) + \frac{\mu}{2} \chi_t^2 + \frac{1}{2} |\nabla \chi(t)|^2 + \hat{g}(\chi(t)) \right).$$ (7.4)

Then, it is easy to see that the “remainder” term on the right hand side of (7.3) can be estimated just by “paying” a positive constant $c$, which makes impossible to integrate (7.3) in time on the whole half line $(0, +\infty)$. We note that the resulting estimate would be a necessary ingredient for proving, e.g., further regularity properties of the attractor, such as a uniform (in time) bound in a “smaller” space, an information which is also needed in the convergence analysis (cf. [9, Sec. 5]). We also point out that the occurrence of the remainder term is linked to the choice of a “special” heat flux law (cf. [3]), i.e. to the presence of the term $k_1 \vartheta$ as a summand in $\alpha(\vartheta)$ (cf. (1.2)). However, even neglecting that term in the formulation of (1.1) would not help. Indeed, without that contribution, the system loses coercivity in $\vartheta$ and its long time analysis seems to be even more difficult.
(cf. the Introduction of [18] for further remarks on this point and the paper [20] for partial results in the 1D case).

References


