

COMPLEX ANALYSIS

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1. HOLOMORPHIC FUNCTIONS

We begin by recalling the basic facts about the field of the complex numbers \mathbf{C} and the power series in the complex plane. Although we recall all the fundamental facts, we assume the reader to be familiar with the complex numbers and the theory of power series, at least in the case of the real line.

1.1. The complex numbers and power series. We begin by reviewing the definition and a few algebraic facts about the complex numbers.

The set of complex numbers is defined as the *field* of numbers z of the form

$$z = x + iy,$$

where $x, y \in \mathbf{R}$ and i is the *imaginary unit* satisfying the identity

$$i^2 = -1.$$

With this definition, and using the algebraic structure of the *real* field \mathbf{R} (a copy of which is contained in \mathbf{C} , as the set of numbers $\{z : z = x + i0\}$), it is easy to see that \mathbf{C} becomes a field with the operations:

- $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$ [sum];
- $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ [product].

Given a complex number $z \in \mathbf{C}$ we write

$$z = x + iy$$

where $x, y \in \mathbf{R}$ are called the *real* and the *imaginary* part of z , resp. We also write

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

Moreover, the *modulus* of z is the non-negative quantity

$$|z| = \sqrt{x^2 + y^2}.$$

For $z, w \in \mathbf{C}$ we have the *triangle inequality*

$$|z + w| \leq |z| + |w|, \tag{1.1}$$

The *complex conjugate* of z is the complex number

$$\bar{z} = x - iy.$$

Notice that $|\bar{z}| = |z|$.

We identify the complex field \mathbf{C} with the plane \mathbf{R}^2 , via the correspondence

$$\mathbf{C} \ni z = x + iy \mapsto (x, y) \in \mathbf{R}^2.$$

This identification carries over also to be an isometry as metric spaces, if we set

$$d_{\mathbf{C}}(z, w) = |z - w|.$$

Consequently, we topologize \mathbf{C} as metric space and with this topology all the properties of \mathbf{R}^2 as topological space carry over to \mathbf{C} .

We now recall a few facts about *connected sets*. A subset E of a metric space is said to be *connected* if it cannot be written as union of two disjoint subsets that are both open and non-empty in the relative topology.

It is well known that an open set in the plane is connected if and only if it is *connected by arcs*, that is, any two points in the set can be joined by a polygon lying in the set. We also remark that this characterization does not hold true if the connected set is not assumed to be open.

A connected open set $\Omega \subseteq \mathbf{C}$ will be called a *domain*.

Throughout these notes, by the notation

$$D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\} \quad (1.2)$$

we will denote the open disk centered at z_0 and of radius $r > 0$, and by $\overline{D(z_0, r)}$ its closure.

If we write the point (x, y) in polar coordinates the real and the imaginary parts of z we obtain the *polar form* of z :

$$z = \rho \cos \theta + i \rho \sin \theta = \rho(\cos \theta + i \sin \theta). \quad (1.3)$$

The non-negative number ρ is the length of the vector (x, y) representing z , that is, $\rho = |z|$. We call the *argument* of the complex number z , and we denote it by $\arg z$ any real value θ for which (1.3) holds.

It is of fundamental importance to notice that the correspondence $z \mapsto \arg z$ is not a single-valued function, since if θ is a value for which (1.3) holds, we have that

$$z = \rho[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]$$

for all $k \in \mathbf{Z}$. Thus, we could say that $\arg z$ is a (doubly infinite) sequence of values and that if θ is one such value, all other values are of the form $\theta + 2k\pi$, for $k \in \mathbf{Z}$. We will return to this important fact in Section 3.2.

Using the expansion in power series in the complex plane for the exponential, sine and cosine that will be fully justified in Subsection 3.1 (in particular after Prop. 3.2), we obtain *Euler's identity*:

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad \theta \in \mathbf{R}. \quad (1.4)$$

In order to prove Euler's identity we need the theory of *power series* in the complex plane.

Let $\{a_n\}$ be a sequence of complex numbers, $n = 0, 1, 2, \dots$, and let $z_0 \in \mathbf{C}$. We call *power series centered at z_0* the series of functions

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n. \quad (1.5)$$

The following results are proven as in the well-known real case.

Proposition 1.1. (i) *Let the power series (1.5) be given. If there exists $\zeta \neq z_0$ such that the series converges, then it converges absolutely on the open disk $D = \{z \in \mathbf{C} : |z - z_0| < |\zeta - z_0|\}$. Moreover, it converges uniformly in all closed disks $\overline{D}_r = \{z \in \mathbf{C} : |z - z_0| \leq r\}$, where $r < |\zeta - z_0|$.*

(ii) *Suppose the series converges for some $\zeta \neq z_0$. There exists $R \in (0, +\infty]$ such that the series converges absolutely in the open disk $D = \{z \in \mathbf{C} : |z - z_0| < R\}$ and if $w \in \mathbf{C}$ is such that $|w - z_0| > R$ the series does not converge in w .*

(iii) (Cauchy-Hadamard criterion) *The value R above is called the radius of convergence of the power series. It can be computed as $1/L$, where*

$$L = \limsup_{n \rightarrow +\infty} |a_n|^{1/n},$$

with the usual convention if $L = 0$ or $L = +\infty$. In the latter case, $R = 0$ and the series converges only for $z = z_0$.

(iv) (Ratio test) If the limit

$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

exists, then also $\lim_{n \rightarrow +\infty} |a_n|^{1/n}$ exists and it equals L .

(v) Let $R > 0$ or $R = +\infty$ be the radius of convergence of the series in (1.5). Then the series converges uniformly in every closed disks $\overline{D}_r = \{z \in \mathbf{C} : |z - z_0| \leq r\}$, where $r < R$.

We now prove Euler's identity (1.4).

We know that the power series defining the exponential function, the cosine and the sine have radius of convergence $+\infty$. Hence the three power series converge for all complex numbers. Observing that $i^{2k} = (-1)^k$, we have

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{+\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{+\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

From Euler's identity we obtain the *exponential form* of a complex number

$$z = \rho e^{i\theta}. \quad (1.6)$$

We recall that one uses the exponential form to obtain the n -th roots of a complex number. If $w = \rho e^{i\theta}$, the n solutions of the equation $w = z^n$ are the complex numbers

$$z_k = \sqrt[n]{\rho} e^{\frac{i}{n}(\theta + 2k\pi)}, \quad \text{for } k = 0, 1, \dots, n-1. \quad (1.7)$$

The numbers $e^{2\pi i \frac{k}{n}}$, $k = 0, 1, \dots, n-1$ are called the n -th roots of the unity.

1.2. Holomorphic functions.

Definition 1.2. Let A be an open set in \mathbf{C} and let $z_0 \in A$. A function $f : A \rightarrow \mathbf{C}$ is said to be (complex) differentiable at z_0 if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists finite.

If f is differentiable at every z_0 in an open subset U of A , we say that f is *holomorphic* on U .

Although the object of our studies are the holomorphic functions, we now see a few properties of the mere complex differentiability.

It is clear that, by the same differentiation rules as in the real case, the sum, the product and the quotient of two functions is complex differentiable at $z_0 \in \mathbf{C}$, as long as the denominator is non-zero, is again complex differentiable at z_0 .

Moreover, a function complex differentiable at $z_0 \in \mathbf{C}$ is necessarily continuous in z_0 . For, the definition of complex differential can be written as

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

where h is a complex number. Then, writing

$$f(z_0 + h) - f(z_0) = h \frac{f(z_0 + h) - f(z_0)}{h},$$

if f is complex differentiable at z_0 we obtain

$$\lim_{h \rightarrow 0} f(z_0 + h) - f(z_0) = \lim_{h \rightarrow 0} h \frac{f(z_0 + h) - f(z_0)}{h} = 0.$$

By the same proofs as in the real case, we obtain the same rules for the complex differentiation of the sum, product, quotient and composition.

Proposition 1.3. (i) Let f, g be functions (complex) differentiable at z_0 , $\alpha, \beta \in \mathbf{C}$. Then $\alpha f + \beta g$, fg and f/g , if $g(z_0) \neq 0$ are differentiable at z_0 and it holds that

$$(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0), \quad (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0),$$

and

$$(f/g)'(z_0) = (f'(z_0)g(z_0) - f(z_0)g'(z_0))/g^2(z_0).$$

(ii) Suppose g is differentiable at z_0 and f is differentiable in $w_0 = g(z_0)$, then $f \circ g$ is differentiable in z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proposition 1.4. Let $A \subseteq \mathbf{C}$ be an open set, $f : A \rightarrow \mathbf{C}$, and suppose that f is differentiable at $z \in A$. Write f in terms of its real and imaginary parts $f = u + iv$.

Then, u and v are differentiable at $(x, y) \equiv z$ in the classical (real) sense and it holds that

$$\begin{cases} \partial_x u(z) = \partial_y v(z) \\ \partial_y u(z) = -\partial_x v(z). \end{cases} \quad (1.8)$$

Proof. Since f is differentiable in z it follows that

$$f(z + h) - f(z) = hf'(z) + o(h)$$

as $h \rightarrow 0$. Writing $f = u + iv$, $h = a + ib$ we obtain that

$$u(z + h) - u(z) = \operatorname{Re}(hf'(z) + o(h)) = L(a, b) + o((a, b)),$$

for a linear function L , that is, u is differentiable (in the classical sense) in z .

If we take h to be real in the definition of the complex differential and leave y fixed, we obtain the partial derivative of f with respect to x ; that is,

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} = \partial_x f(z) = \partial_x u(z) + i\partial_x v(z).$$

Similarly, if we take a purely imaginary values ik for h , we obtain

$$f'(z) = \lim_{k \rightarrow 0} \frac{f(z + ik) - f(z)}{ik} = -i\partial_y f(z) = -i\partial_y u(z) + \partial_y v(z).$$

Therefore, at the point z , $\partial_x f$ and $\partial_y f$ must satisfy the equation

$$\partial_x f(z) = -i\partial_y f(z),$$

which is equivalent to the system of real equations

$$\begin{cases} \partial_x u(z) = \partial_y v(z) \\ \partial_y u(z) = -\partial_x v(z), \end{cases}$$

as we wished to prove. \square

Prop. 1.4 has a natural converse.

Proposition 1.5. *Let $f = u + iv$ be defined in a ngbh of $z \in \mathbf{C}$. Suppose that u and v are differentiable (in the classical sense) in z and that they satisfy equations (1.8) in z . Then f is complex differentiable in z .*

For $h, k \in \mathbf{R}$, using the differentiability of u and v at $z \equiv (x, y)$ we have,

$$\begin{aligned} f(z + h + ik) - f(z) &= u(x + h, y + k) + iv(x + h, y + k) - u(x, y) - iv(x, y) \\ &= \partial_x u(x, y)h + \partial_y u(x, y)k + o(h, k) + i\left(\partial_x v(x, y)h + \partial_y v(x, y)k + o(h, k)\right) \\ &= \partial_x u(x, y)(h + ik) + i\partial_x v(x, y)(h + ik) + o(h, k). \end{aligned}$$

Therefore,

$$\lim_{h+ik \rightarrow 0} \frac{f(z + h + ik) - f(z)}{h + ik} = \partial_x u(z) + i\partial_x v(z)$$

which shows that f is complex differentiable at z . \square

The equations in (1.8) are called the *Cauchy-Riemann equations*. The complex equation from which we have derived them is called *complex version* of the CR-equations, precisely,

$$\partial_x f = -i\partial_y f. \quad (1.9)$$

The following corollary is now obvious.

Corollary 1.6. *Let $A \subseteq \mathbf{C}$ be open, $f : A \rightarrow \mathbf{C}$, and let $f = u + iv$ be its decomposition in real and imaginary parts. Then f is holomorphic in A if and only if u and v are differentiable (in the classical sense) in A and their partial derivatives satisfy the Cauchy-Riemann equations (1.8) in A .*

An important, although immediate, consequence of the above facts is the following proposition.

Proposition 1.7. *A function f that is holomorphic on a connected open set Ω and such that $f'(z) = 0$ for $z \in \Omega$, is constant.*

We now define the complex vector fields

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_x + i\partial_y). \end{aligned}$$

Writing $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ we may think a function f of the complex variable z as depending on x and y , and hence on z and \bar{z} , thinking the latter as *independent* variables. Then, a function $f = f(z, \bar{z})$ turns out to satisfy the CR-equations if $\partial_{\bar{z}} f = 0$. (However, we will not use this notation in the present notes.)

It is worth noticing that the existence of $f'(z)$ implies the existence of the four partial derivatives in (1.8) above. Moreover, using the CR-equations we can rewrite $f'(z)$ in four different ways, for instance,

$$f'(z) = \partial_x u(z) + i\partial_x v(z).$$

Therefore,

$$\begin{aligned} |f'(z)|^2 &= |\partial_x u(z) + i\partial_x v(z)|^2 = |\partial_x u(z)|^2 + |\partial_x v(z)|^2 \\ &= \partial_x u(z)\partial_y v(z) - \partial_y u(z)\partial_x v(z), \end{aligned} \quad (1.10)$$

that is, $|f'(z)|^2$ is the determinant jacobian of the mapping $(x, y) \mapsto (u(x, y), v(x, y))$.

Another consequence of (1.8) is that if the real and imaginary parts u and v admit mixed partial derivatives (we will see that the real and imaginary parts of a holomorphic function always do), then

$$\Delta u(z) = \partial_x^2 u(z) + \partial_y^2 u(z) = 0$$

$$\Delta v(z) = \partial_x^2 v(z) + \partial_y^2 v(z) = 0.$$

The above equations simply mean that u and v are *harmonic*; Δ being the Laplace operator.

If u and v are harmonic functions and they satisfy the CR-equations (1.8), then we say that v is the *harmonic conjugate* of u (or the conjugate harmonic function).

Often it will be convenient to view holomorphic functions as *mappings* between regions of the complex plane.

Proposition 1.8. *Let f be holomorphic in a neighborhood of a point ζ . Let α_1, α_2 be complex numbers of modulo 1. Let $D_{\alpha_j} f(\zeta)$, $j = 1, 2$, denote the directional derivative of f at ζ in the direction given by α_j ; that is,*

$$D_{\alpha_j} f(\zeta) = \lim_{\mathbf{R} \ni t \rightarrow 0} \frac{f(\zeta + t\alpha_j) - f(\zeta)}{t} \quad j = 1, 2.$$

Then $|D_{\alpha_1} f(\zeta)| = |D_{\alpha_2} f(\zeta)|$. If $f'(\zeta) \neq 0$, the oriented angle from α_1 to α_2 equals the oriented angle from $D_{\alpha_1} f(\zeta)$ to $D_{\alpha_2} f(\zeta)$.

Proof. It suffices to notice that

$$\begin{aligned} D_{\alpha_j} f(\zeta) &= \lim_{\mathbf{R} \ni t \rightarrow 0} \frac{f(\zeta + t\alpha_j) - f(\zeta)}{t\alpha_j} \alpha_j \\ &= f'(\zeta) \alpha_j. \end{aligned}$$

The two assertions now follow. To see the equality between the oriented angles if $f'(\zeta) \neq 0$, notice that

$$\frac{D_{\alpha_1} f(\zeta)}{\alpha_1} = \frac{D_{\alpha_2} f(\zeta)}{\alpha_2},$$

so that $\arg(D_{\alpha_1} f(\zeta)) - \arg(\alpha_1) = \arg(D_{\alpha_2} f(\zeta)) - \arg(\alpha_2)$; that is,

$$\arg(D_{\alpha_2} f(\zeta)) - \arg(D_{\alpha_1} f(\zeta)) = \arg(\alpha_2) - \arg(\alpha_1),$$

as we wished to show. \square

A map that preserves angles between curves and their orientation is called *conformal*.

The first examples of holomorphic functions are polynomials in z , $p(z) = a_0 + a_1z + \cdots + a_nz^n$, but not in \bar{z} . For instance, the function $g(z) = \bar{z}$ is *not* holomorphic, since

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h \rightarrow 0} \frac{\overline{(z+h)} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{h}}{h},\end{aligned}$$

and such limit does not exist.

We recall that a function f is called *rational* if it is the quotient of two polynomials, $f = p/q$. Then f is holomorphic on the set where $q \neq 0$.

1.3. Exercises.

1.1. Represent the following numbers in the complex plane finding their real and imaginary parts, the modulus, the argument and their complex conjugate.

$$\begin{aligned}z_1 &= -8 - 8i, \quad z_2 = 5(\cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi), \quad z_3 = -1 - i\sqrt{3}, \quad z_4 = -i \\ z_5 &= 3, \quad z_6 = 4(\cos \pi + i \sin \pi), \quad z_7 = 1 + i, \quad z_8 = \frac{2}{3}e^{i\frac{7}{6}\pi}\end{aligned}$$

1.2. Compute the value of the following expressions and represent the point in the complex plane:

$$\begin{aligned}\text{(i)} \quad & \frac{-2 + 2i}{1 - i\sqrt{3}} e^{-i\frac{\pi}{2}}, \\ \text{(ii)} \quad & \frac{1+i}{1-i} - (1+2i)(2+2i) + \frac{3-i}{1+i}, \\ \text{(iii)} \quad & \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{11} + \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)^{-6} - \frac{2+i}{2i}, \\ \text{(iv)} \quad & 2i(-1+i) + \frac{3}{(\sqrt{3}+i)} + (1+i)\overline{(1+i)}.\end{aligned}$$

1.3. Find all the complex numbers that satisfy the following conditions:

$$\begin{aligned}\text{(i)} \quad & 2\operatorname{Re}(z(1+i)) + z\bar{z} = 0; \quad \text{(ii)} \quad \begin{cases} \operatorname{Re}(z^2) + \operatorname{Im}(\bar{z}(1+2i)) = 3 \\ \arg z = \pi \end{cases}; \\ \text{(iii)} \quad & \begin{cases} \operatorname{Re} z \geq 0 \\ z\bar{z} = 4 \end{cases}; \quad \text{(iv)} \quad \operatorname{Im}((2-i)z) = 1.\end{aligned}$$

1.4. Let $z, w \in \mathbf{C}$. Show that $|z|, |w| < 1$ implies that

$$\left| \frac{z-w}{1-z\bar{w}} \right| < 1,$$

and that $|z| < 1, |w| = 1$ implies

$$\left| \frac{z-w}{1-z\bar{w}} \right| = 1.$$

Is the equality above true for all z, w on the unit circle, that is, for all z, w with $|z| = |w| = 1$?

1.5. Let $f, g : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be C^1 functions such that their compositions are defined. Prove the following versions of the chain rule:

$$\begin{aligned}\partial_z(f \circ g) &= \partial_z f \cdot \partial_z g + \partial_{\bar{z}} f \cdot \partial_z \bar{g}, \\ \partial_{\bar{z}}(f \circ g) &= \partial_z f \cdot \partial_{\bar{z}} g + \partial_{\bar{z}} f \cdot \partial_{\bar{z}} \bar{g}.\end{aligned}$$

1.6. Prove the *Cauchy-Schwarz inequality*: For $z_j, w_j, j = 1, \dots, n$ in \mathbf{C} we have

$$\left| \sum_{j=1}^n z_j \bar{w}_j \right|^2 \leq \left(\sum_{j=1}^n |z_j|^2 \right) \left(\sum_{j=1}^n |w_j|^2 \right). \quad (1.11)$$

1.7. Show that the CR-equations in polar coordinates take the form

$$\begin{cases} \partial_\rho u = \frac{1}{\rho} \partial_\theta v \\ \frac{1}{\rho} \partial_\theta u = -\partial_\rho v. \end{cases} \quad (1.12)$$

[You will need to show that

$$\begin{cases} \partial_\rho = \cos \theta \partial_x + \sin \theta \partial_y \\ \partial_\theta = -\rho \sin \theta \partial_x + \rho \cos \theta \partial_y, \end{cases}$$

then assume that u and v satisfy the CR equations to obtain (1.12). Viceversa, if (1.12) are satisfied, then the CR equations hold true.]

(This fact will be used in the proof of Prop. 3.5.)

2. COMPLEX INTEGRATION AND CAUCHY'S THEOREM

We begin with some definitions.

If $f = u + iv$ is a continuous complex-valued function defined on an interval $[a, b]$ on the real line, we set

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

Then the mapping $f \mapsto \int_a^b f(t) dt$ is complex linear.

Lemma 2.1. *Let $f \in C([a, b])$. Then*

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt.$$

Proof. The proof is simple. If $\int_a^b f(t) dt = 0$ we have nothing to prove.

Otherwise, let $\alpha \in \mathbf{C}$, $|\alpha| = 1$. Then

$$\begin{aligned} \operatorname{Re} \left(\alpha \int_a^b f(t) dt \right) &= \int_a^b \operatorname{Re} (\alpha f(t)) dt \leq \int_a^b |\operatorname{Re} (\alpha f(t))| dt \\ &\leq \int_a^b |\alpha f(t)| dt = \int_a^b |f(t)| dt. \end{aligned}$$

If α is chosen $\alpha = |\int_a^b f(t) dt| / \int_a^b f(t) dt$, the assertion follows. \square

We recall that a *parametrized curve* is a piecewise C^1 -function $\gamma : [a, b] \rightarrow \mathbf{C}$. Let $\Omega \subseteq \mathbf{C}$ be a domain, $\gamma : [a, b] \rightarrow \mathbf{C}$ a parametrized curve with image contained in Ω and let $f : \Omega \rightarrow \mathbf{C}$ be a continuous function. Then we define the line integral

$$\int_{\gamma} f dz := \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

We call two parametrized curves $\gamma : [a, b] \rightarrow \mathbf{C}$ and $\sigma : [c, d] \rightarrow \mathbf{C}$ *equivalent*, and we write $\gamma \sim \sigma$, if there exists a strictly increasing differentiable function $\varphi : [a, b] \rightarrow [c, d]$ such that $\sigma(\varphi(t)) = \gamma(t)$ for all $t \in [a, b]$. Notice that we require that $\varphi'(t) > 0$ for all $t \in [a, b]$. The one above is easily seen to be an equivalence relation in the family of parametrized curves.

We define a curve γ to be an equivalence class of the parametrized curves.

We recall a few properties of the complex line integrals that can be easily obtained as in the case of line integrals in \mathbf{R}^2 :

- (i) the value of the integral $\int_{\gamma} f dz$ is independent of the parametrization;
- (ii) if σ denotes the curve (to be precise, one of whose representatives is given by the parametrized curve) $\gamma(-t)$, with $\sigma : [-b, -a] \rightarrow \mathbf{C}$, then we write $\sigma = -\gamma$ and we have $\int_{-\gamma} f dz = -\int_{\gamma} f dz$.

Next we set

$$\int_{\gamma} f \overline{dz} := \overline{\int_{\gamma} \overline{f} dz},$$

and the line integrals with respect to dx and dy as

$$\begin{aligned}\int_{\gamma} f dx &:= \frac{1}{2} \left(\int_{\gamma} f dz + \int_{\gamma} f \overline{dz} \right) \\ \int_{\gamma} f dy &:= \frac{1}{2i} \left(\int_{\gamma} f dz - \int_{\gamma} f \overline{dz} \right).\end{aligned}$$

Then, if $f = u + iv$ we have that

$$\begin{aligned}\int_{\gamma} f dz &= \int_{\gamma} (u + iv) dz = \int_{\gamma} (u + iv) (dx + i dy) \\ &= \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx).\end{aligned}$$

Of course we could have started from the definition of the line integral of linear differential forms $pdx + qdy$ and use the above equation to define the complex line integral $\int_{\gamma} f dz$.

We also consider a different line integral, with respect to *arc length*:

$$\int_{\gamma} f ds = \int_{\gamma} f |dz| := \int_a^b f(\gamma(t)) |\gamma'(t)| dt$$

We remark that this one is not an oriented integral, that is, it does not depend on the orientation of the curve γ : $\int_{-\gamma} f |dz| = \int_{\gamma} f |dz|$.

From this definition it follows that if f is a continuous function on a curve γ then

$$\begin{aligned}\left| \int_{\gamma} f dz \right| &= \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \\ &\leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt;\end{aligned}$$

that is,

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| |dz|. \quad (2.1)$$

We also recall the following result from the theory of line integrals in the plane \mathbf{R}^2 .

Proposition 2.2. *Let $\Omega \subseteq \mathbf{C}$ be a domain, $pdx + qdy$ be a linear differential form, with p, q continuous. The following conditions are equivalent:*

- (i) *the line integral $\int_{\gamma} (pdx + qdy)$ depends only on the end points of γ for all curve γ with image in Ω ;*
- (ii) *for all closed curves γ in Ω $\int_{\gamma} (pdx + qdy) = 0$;*
- (iii) *$(pdx + qdy)$ is an exact form; that is, there exists a function $U \in C^1(\Omega)$, such that $\partial_x U = p$ and $\partial_y U = q$ on Ω .*

It follows immediately that, if f is a continuous function, then $f(z)dz$ is an *exact* differential, that is, there exists a holomorphic function F such that $F' = f$ on Ω .

We are then led to consider the following question: When is $f(z)dz = f(z)dx + if(z)dy$ an exact differential? By definition there must exist a function F on Ω have:

$$\begin{aligned}\partial_x F(z) &= f(z) \\ \partial_y F(z) &= if(z).\end{aligned}$$

In this case, F satisfies the CR-equations (that in complex notations can be written as)

$$\partial_x F = -i\partial_y F,$$

and F is differentiable, since f is continuous by assumption.

For sake of clarity, we restate the previous result in its complex version.

Proposition 2.3. *Let $\Omega \subseteq \mathbf{C}$ be a domain, $f : \Omega \rightarrow \mathbf{C}$ continuous on Ω . Then the following conditions are equivalent:*

- (i) *the integral $\int_\gamma f dz$ depends only on the end points of the curve γ ;*
- (ii) *for every closed curve γ in Ω , $\int_\gamma f dz = 0$;*
- (iii) *f is the derivative of a holomorphic function on Ω .*

Proof. The equivalence of (i) and (ii) is obvious.

Suppose (iii) holds, that $f = F'$ with F holomorphic on Ω , then $f(z)dz$ is an exact differential and the result follows from Prop. 2.2.

Conversely, suppose that $\int_\gamma f dz$ depends only on the ends points. Then, the same holds for the integral of the real and imaginary parts of

$$f dz = u dx - v dy + i(v dx + u dy).$$

By Prop. 2.2 again, there exists C^1 real-valued functions U, V such that

$$\begin{aligned}\partial_x U &= u, \quad \partial_y U = -v \\ \partial_x V &= v, \quad \partial_y V = u.\end{aligned}$$

Then, U, V are C^1 functions and they satisfy the CR-equations; that is, $F := U + iV$ is holomorphic and $F' = \partial_x U + i\partial_y V = u + iv = f$. \square

Corollary 2.4. *Let γ be a closed curve. Then, for $n = 0, 1, 2, \dots$, and all $z_0 \in \mathbf{C}$ we have*

$$\int_\gamma (z - z_0)^n dz = 0.$$

Proof. Let $\gamma : [a, b] \rightarrow \mathbf{C}$. Since $(z - z_0)^n$ is the derivative of $F(z) = \frac{1}{n+1}(z - z_0)^{n+1}$, which is then holomorphic, we have

$$\int_\gamma (z - z_0)^n dz = F(\gamma(b)) - F(\gamma(a)) = 0,$$

since $\gamma(b) = \gamma(a)$. \square

On the other hand, a simple calculation shows that if γ is the unit circle, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$,

$$\int_\gamma \frac{1}{z} dz = 2\pi i.$$

2.1. Cauchy's theorem for a rectangle. We now see the simplest version of Cauchy's Theorem, in the case of a rectangle $R = \{z = x + iy : a \leq x \leq b, c \leq y \leq d\}$. We denote by ∂R the boundary of R , oriented counter-clockwise.

Theorem 2.5. *Let Ω be a domain containing R . For any f holomorphic on Ω we have*

$$\int_{\partial R} f dz = 0.$$

Proof. For a rectangle $R' \subseteq \Omega$ we write

$$\eta(R') = \int_{\partial R'} f dz.$$

We divide the rectangle R into 4 rectangles $R^{(1)}, \dots, R^{(4)}$ by bisecting each side into two equal segments.

Since the line integrals over the common sides cancel out, we obtain that

$$\eta(R) = \eta(R^{(1)}) + \dots + \eta(R^{(4)}).$$

At least one rectangle $R^{(k)}$, $k = 1, \dots, 4$ must satisfy

$$|\eta(R^{(k)})| \geq \frac{1}{4} |\eta(R)|.$$

We call this rectangle R_1 . By repeating this construction we obtain a sequence of rectangles R_1, R_2, \dots such that:

- (i) $R \supset R_1 \supset R_2 \supset \dots$;
- (ii) $|\eta(R_n)| \geq \frac{1}{4} |\eta(R_{n-1})|$, so that $|\eta(R_n)| \geq 4^{-n} |\eta(R)|$;
- (iii) if p_n and d_n denote the perimeter and the diameter of R_n , respectively, and p, d the ones of R , then $p_n = 2^{-n} p$ and $d_n = 2^{-n} d$.

By the Bolzano-Weierstrass theorem, $\cap_n R_n$ is non-empty, and since $d_n \rightarrow 0$, $\cap_n R_n$ cannot contain two distinct points. Therefore, there exists $\zeta \in R$ such that $\cap_n R_n = \{\zeta\}$.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $D(\zeta, \delta) \subseteq \Omega$ and, since the function f is holomorphic in Ω , such that

$$|f(z) - f(\zeta) - (z - \zeta)f'(\zeta)| < \varepsilon |z - \zeta|$$

for $z \in D(\zeta, \delta)$.

Recall that, from Cor. 2.4 we know that

$$\int_{\partial R_n} dz = \int_{\partial R_n} (z - \zeta) dz = 0.$$

Now, there exists n_0 such that for $n \geq n_0$ R_n is contained in $D(\zeta, \delta)$, and then, if $z \in \partial R_n$, $|z - \zeta| \leq d_n$. Therefore, by (ii) and (iii) above,

$$\begin{aligned} |\eta(R_n)| &= \left| \int_{\partial R_n} (f(z) - f(\zeta) - (z - \zeta)f'(\zeta)) dz \right| \\ &\leq \varepsilon \int_{\partial R_n} |z - \zeta| |dz| \\ &\leq \varepsilon d_n p_n \\ &\leq \varepsilon 4^{-n} dp. \end{aligned}$$

It then follows that

$$|\eta(R)| \leq 4^n |\eta(R_n)| \leq \varepsilon dp.$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proven. \square

We can weaken the hypotheses in the previous theorem.

Theorem 2.6. *Let Ω and R be as in Thm. 2.5. Let f be holomorphic in the domain Ω' obtained removing from Ω a finite number of points ζ_j , $j = 1, \dots, n$, lying in the interior of R , and assume that*

$$\lim_{z \rightarrow \zeta_j} (z - \zeta_j) f(z) = 0$$

for $j = 1, \dots, n$. Then

$$\int_{\partial R} f dz = 0.$$

Proof. We first argue that it suffices to consider the case of a single exceptional point ζ . In fact, we can divide the rectangle R as finite union of rectangles R_j , each containing a single exceptional point ζ_j , $j = 1, \dots, n$, and observe again that

$$\int_{\partial R} f dz = \sum_{j=1}^n \int_{\partial R_j} f dz.$$

So, let us assume that we have a single exceptional point ζ inside R . We divide R as union of nine rectangles, in such a way that the central one is a square R_0 centered at ζ and has side lengths to be fixed. Then,

$$\begin{aligned} \int_{\partial R} f dz &= \int_{\partial R_0} f dz + \sum_{j=1}^8 \int_{\partial R_j} f dz \\ &= \int_{\partial R_0} f dz \end{aligned}$$

by applying Thm. 2.5 to the integrals $\int_{\partial R_j} f dz$, $j = 1, \dots, 8$.

Given $\varepsilon > 0$ we fix the side lengths of R_0 to be small enough so that

$$|z - \zeta| |f(z)| \leq \varepsilon$$

for $z \in \partial R_0$. We then have

$$\begin{aligned} \left| \int_{\partial R} f dz \right| &= \left| \int_{\partial R_0} f dz \right| \leq \int_{\partial R_0} |f(z)| |dz| \\ &\leq \varepsilon \int_{\partial R_0} \frac{1}{|z - \zeta|} |dz| \\ &\leq 8\varepsilon, \end{aligned}$$

since R_0 is a square, as an elementary argument shows. This proves the theorem. \square

2.2. Cauchy's theorem in a disk. We denote by $D = D(z_0, r)$ the open disk having center z_0 and radius $r > 0$; that is,

$$D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}.$$

Theorem 2.7. *Let f be holomorphic in an open disk D . Then*

$$\int_{\gamma} f(z) dz = 0$$

for all closed curves γ contained in D .

Proof. We are going to use Thm. 2.5. For any $z = x + iy \in D$, let $\sigma = \sigma_z$ be the curve in D consisting of the horizontal segment from (x_0, y_0) to (x, y_0) followed by the vertical segment from (x, y_0) to (x, y) . Define

$$F(z) = \int_{\sigma_z} f dz.$$

Then F is well defined and we can easily compute that

$$\partial_y F(z) = if(z).$$

By Thm. 2.5, since f is holomorphic on D , we have that

$$F(z) = \int_{\sigma_z} f dz = \int_{\tau_z} f dz$$

where τ_z is the curve consisting of the vertical segment from (x_0, y_0) to (x_0, y) followed by the horizontal segment from (x_0, y) to (x, y) . Computing the partial derivatives in x of F we obtain that $\partial_x F(z) = f(z)$. Since the partial derivatives of F are continuous and satisfy the CR-equation, F is holomorphic in D , and its derivative is f .

Therefore, $f(z)dz$ is an exact differential and

$$\int_{\gamma} f(z) dz = 0$$

by Prop. 2.3. \square

The next corollary concerns with the existence of local anti-derivative for a holomorphic function. It follows at once from Prop. 2.3 (or, just from the proof of the previous theorem).

Corollary 2.8. *If f is holomorphic in a disk $D(z_0, r)$ then there exists a function F holomorphic on the same disk such that $F' = f$ on $D(z_0, r)$.*

The conclusion of the previous theorem remains valid if we assume that there exists a finite number of exceptional points for f in D , in the same way as in Thm. 2.6.

Theorem 2.9. *Let f be holomorphic in D' obtained removing from an open disk D a finite number of points ζ_j , $j = 1, \dots, n$, and assume that*

$$\lim_{z \rightarrow \zeta_j} (z - \zeta_j) f(z) = 0$$

for $j = 1, \dots, n$. Then

$$\int_{\gamma} f dz = 0$$

for every closed curve γ contained in D' .

Proof. This proof now follows from the previous arguments. First we can reduce to the case of a single exceptional point ζ . Then we only need to make sure that the curve γ does not pass through ζ . Having fixed $z_0 \in D'$, given $z \in D'$, if the rectangle with opposite vertices in z_0 and z passes through ζ , we can still easily define the indefinite integral F of f on D' . We leave the simple detail to the reader (or see Ahlfors, p. 114). \square

2.3. Cauchy's formula. We begin with the notion of *index of a point* with respect to a curve.

Lemma 2.10. *Let γ be a closed curve and let z_0 be a point not lying on γ . Then the integral*

$$\int_{\gamma} \frac{dz}{z - z_0}$$

is an integral multiple of $2\pi i$.

Proof. Let $\gamma : [a, b] \rightarrow \mathbf{C}$ and define

$$h(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau) - z_0} d\tau.$$

We wish to show that there exists an integer k such that

$$\int_{\gamma} \frac{dz}{z - z_0} = h(b) = 2\pi i k.$$

The function h is defined and continuous on $[a, b]$, $h(a) = 0$ and

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

on the interval $[a, b]$ taken away a finite number of points where $\gamma(t)$ is not differentiable. It follows that

$$\begin{aligned} \frac{d}{dt}(e^{-h(t)}(\gamma(t) - z_0)) &= e^{-h(t)}(-h'(t)(\gamma(t) - z_0) + \gamma'(t)) \\ &= 0 \end{aligned}$$

except at those points t_1, \dots, t_n where $\gamma(t)$ is not differentiable. Therefore, $e^{-h(t)}(\gamma(t) - z_0)$ is constant on each connected component of $[a, b] \setminus \{t_1, \dots, t_n\}$. Since $e^{-h(t)}(\gamma(t) - z_0)$ is also continuous, it follows that it is constant on $[a, b]$; that is,

$$e^{-h(t)}(\gamma(t) - z_0) = c.$$

Since $h(a) = 0$, $c = \gamma(a) - z_0$, so that

$$e^{h(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}.$$

Now, using the fact that $\gamma(b) = \gamma(a)$ we have $e^{h(b)} = 1$, so that

$$h(b) = 2\pi i k$$

for some integer k . \square

Definition 2.11. We call *index of a point* z_0 with respect to a closed curve γ not passing through z_0 , the integer

$$n(\gamma, z_0) \equiv \text{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

The integer $n(\gamma, z_0)$ is also called the *winding number* of γ about z_0 .

Lemma 2.12. *The function*

$$z_0 \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$$

is continuous on $\mathbf{C} \setminus \gamma$. In fact it is constant in every connected component of $\mathbf{C} \setminus \gamma$ and equals 0 in every unbounded connected component.

Proof. This is elementary and it is left as an exercise. (See Exercise 2.1) \square

We now come to the heart of the matter of the local properties of holomorphic functions: The local Cauchy's integral formula.

Theorem 2.13. *Let f be holomorphic in an open disk D and let γ be a closed curve in D . Then, for every point z_0 not on γ we have*

$$n(\gamma, z_0)f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. Let $F(z)$ be defined on $D' = D \setminus \{z_0\}$ as

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

Notice that F is holomorphic on D' and moreover

$$\lim_{z \rightarrow z_0} (z - z_0)F(z) = \lim_{z \rightarrow z_0} f(z) - f(z_0) = 0.$$

Therefore, we can apply Thm. 2.9 to obtain that for every closed curve γ in D'

$$0 = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_\gamma \frac{f(z)}{z - z_0} dz - f(z_0) \int_\gamma \frac{dz}{z - z_0}.$$

Since

$$\int_\gamma \frac{dz}{z - z_0} = n(\gamma, z_0)2\pi i,$$

the conclusion now follows at once. \square

We now wish to think of the point z_0 as the variable. Then we can write

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

If we take γ to be the circle $\partial D(z_0, r)$, z any point inside $D(z_0, r)$, so that $n(\gamma, z) = 1$ (by Lemma 2.12) and we assume f holomorphic on (that is, in a domain containing) $\overline{D(z_0, r)}$, then Cauchy's integral formula takes the form

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (2.2)$$

2.4. Exercises.

2.1. Prove Lemma 2.12.

2.2. Evaluate the following line integrals:

$$(i) \int_{\sigma} e^{\pi z} dz, \quad (ii) \int_{\tau} \frac{\bar{z}}{z^2 + 1} dz,$$

where σ denotes the segment from i to $i/2$, and τ is the portion of the circle $|z| = 2$ lying in $\operatorname{Re} z > 0$ and oriented counter-clockwise. [A: (i) $(1+i)/\pi$.]

2.3. Evaluate the line integrals

$$\int_{\sigma} \cos(z/2) dz,$$

where σ is a curve joining the origin with $\pi + 2i$. [A: $e + 1/e$.]

2.4. Using Cauchy's formula evaluate the following integrals:

$$(i) \int_{|z|=1} \frac{e^z}{z} dz, \quad (ii) \int_{|z|=2} \frac{1}{z^2 + 1} dz.$$

2.5. Let $\gamma = \{|z| = 3\}$. Evaluate the function

$$g(z) = \int_{\gamma} \frac{2w^2 - w - 2}{w - z} dw$$

when $|z| \neq 3$.

2.6. (The holomorphic implicit function theorem.) Let A, B be domains in \mathbf{C} , $F : A \times B \rightarrow \mathbf{C}$. Suppose that $F \in C^1(A \times B)$ and that F is holomorphic in both variables separately; that is, the function $z \mapsto F(z, w)$ is holomorphic in A for any $w \in B$ fixed, and the function $w \mapsto F(z, w)$ is holomorphic in B for any fixed $z \in A$. Suppose that

- (i) $(z_0, w_0) \in A \times B$ is such that $F(z_0, w_0) = 0$;
- (ii) $\partial_w F(z_0, w_0) \neq 0$.

Then there exist nbh's of U and V of z_0 and w_0 resp., and a unique holomorphic function $f : U \rightarrow V$ such that $f(z_0) = w_0$ and $F(z, f(z)) = 0$ for all $z \in U$. Moreover, for $z \in U$,

$$f'(z) = -\frac{\partial_z F(z, f(z))}{\partial_w F(z, f(z))}.$$

[Hint: Identify F with a map $F : \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$. Use formula (1.10) and Dini's theorem. Then use the Cauchy-Riemann equations to obtain the holomorphicity of f .]

3. EXAMPLES OF HOLOMORPHIC FUNCTIONS

We begin by reviewing a few facts about power series in the complex plane.

3.1. Power series. We now prove that the sum of a convergent power series is a holomorphic function in the disk of convergence.

Proposition 3.1. *Let the power series $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$ be given and suppose that it has radius of convergence $R > 0$. Let f be the function sum of the power series, that is, $f(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n$.*

(i) Define the derived series the power series

$$\sum_{n=0}^{+\infty} (n+1)a_{n+1}(z - z_0)^n.$$

Then the derived series has the same radius of convergence R .

(ii) Let ζ be a point in $D(z_0, R)$. Let $r > 0$ be such that the closed disk $\overline{D(\zeta, r)}$ is contained in $D(z_0, R)$. Then f admits power series expansion at ζ , that is, there exist complex coefficients b_k such that on the disk $D(\zeta, r)$

$$f(z) = \sum_{k=0}^{+\infty} b_k(z - \zeta)^k.$$

(iii) The function f is holomorphic in $D(z_0, R) = \{z \in \mathbf{C} : |z - z_0| < R\}$ and it holds that

$$f'(z) = \sum_{n=0}^{+\infty} (n+1)a_{n+1}(z - z_0)^n.$$

(iv) Every derivative of f is holomorphic in $D(z_0, R) = \{z \in \mathbf{C} : |z - z_0| < R\}$ and, for $n = 0, 1, \dots$, it holds that

$$f^{(n)}(z_0) = n!a_n.$$

Notice in particular that this means that f is sum of its Taylor series centered in z_0 .

Proof. Statement (i) follows at once from the Cauchy-Hadamard criterion. Since the derived series has the same radius of convergence, it converges uniformly in the closed disks contained in $D(z_0, R)$.

(ii) For simplicity, we may assume that $z_0 = 0$. Let $r = R - |\zeta| > 0$ and write $z = \zeta + (z - \zeta)$, so that

$$z^n = (\zeta + (z - \zeta))^n = \sum_{k=0}^n \binom{n}{k} \zeta^{n-k} (z - \zeta)^k,$$

and

$$f(z) = \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^n \binom{n}{k} \zeta^{n-k} (z - \zeta)^k \right). \quad (3.1)$$

Now, for $z \in D(\zeta, r)$, let $r' = |z - \zeta| < r$, so that $|\zeta| + r' = R' < R$. Then

$$\begin{aligned} \left| a_n \left(\sum_{k=0}^n \binom{n}{k} \zeta^{n-k} (z - \zeta)^k \right) \right| &\leq |a_n| \sum_{k=0}^n \binom{n}{k} |\zeta|^{n-k} |z - \zeta|^k \\ &\leq |a_n| \sum_{k=0}^n \binom{n}{k} |\zeta|^{n-k} r'^k \\ &= |a_n| (|\zeta| + r')^n \\ &= |a_n| R'^n. \end{aligned}$$

Since $R' < R$, the series $\sum_{n=0}^{+\infty} |a_n| R'^n$ converges, so does the series

$$\sum_{n=0}^{+\infty} \left| a_n \left(\sum_{k=0}^n \binom{n}{k} \zeta^{n-k} (z - \zeta)^k \right) \right|.$$

Thus, we may interchange the summation order in (3.1) and obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^n \binom{n}{k} \zeta^{n-k} (z - \zeta)^k \right) \\ &= \sum_{k=0}^{+\infty} \left(\sum_{n=k}^{+\infty} a_n \binom{n}{k} \zeta^{n-k} \right) (z - \zeta)^k \\ &=: \sum_{k=0}^{+\infty} b_k (z - \zeta)^k, \end{aligned}$$

that converges absolutely. This proves (ii).

(iii) For simplicity, we assume again that $z_0 = 0$. Now we show that f is holomorphic in $D(0, R)$. For $\zeta \in D(0, R)$ we use the power expansion at ζ as before, we compute

$$\begin{aligned} \lim_{\mathbf{C} \ni h \rightarrow 0} \frac{f(\zeta + h) - f(\zeta)}{h} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\sum_{k=0}^{+\infty} b_k (\zeta + h - \zeta)^k - \sum_{k=0}^{+\infty} b_k (\zeta - \zeta)^k \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\sum_{k=0}^{+\infty} b_k h^k - b_0 \right] \\ &= \lim_{h \rightarrow 0} \sum_{k=1}^{+\infty} b_k h^{k-1} \\ &= b_1. \end{aligned}$$

Hence, $f'(\zeta) = b_1$ and f is complex differentiable at each $\zeta \in D(0, R)$; that is, f is holomorphic in $D(0, R)$.

Finally, (iv) is now obvious. \square

We now recall the Taylor expansion of a few noticeable functions. The items (ii)-(v) serve in particular to define the functions on the right hand sides of the equal sign.

Proposition 3.2. *The following power series have sum and radius of convergence, resp.,*

- (i) $\sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}$, $r = 1$;
- (ii) $\sum_{n=0}^{+\infty} \frac{z^n}{n!} = e^z$, $r = +\infty$;
- (iii) $\sum_{n=0}^{+\infty} \frac{z^{2n}}{(2n)!} = \cosh z$, $r = +\infty$;
- (iv) $\sum_{n=0}^{+\infty} \frac{z^{2n+1}}{(2n+1)!} = \sinh z$, $r = +\infty$;
- (v) $\sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z = \frac{e^{iz} + e^{-iz}}{2}$, $r = +\infty$;
- (vi) $\sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $r = +\infty$.

We conclude this part showing a few properties of the exponential function e^z .

From the power series expansion, it is immediate to see that

$$\begin{aligned} \frac{d}{dz} e^z &= \sum_{n=1}^{+\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{+\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \sum_{n=0}^{+\infty} \frac{z^n}{n!} = e^z. \end{aligned}$$

Setting $g(z) = e^z e^{a-z}$ for some fixed $a \in \mathbf{C}$, we see that

$$g'(z) = e^z e^{a-z} + e^z (-e^{a-z}) = 0.$$

Hence, $g(z) = c$, for some constant $c \in \mathbf{C}$. Since $e^0 = 1$, $g(0) = e^a$, so that $g(z) = e^a$ for all $z \in \mathbf{C}$.

Therefore, $e^z e^{a-z} = e^a$ for all $a, z \in \mathbf{C}$, or, in other words

$$e^{a+b} = e^a e^b$$

for all $a, b \in \mathbf{C}$.

This also implies that

$$1 = e^z e^{-z}$$

so that $e^z \neq 0$ for all $z \in \mathbf{C}$, and $\frac{1}{e^z} = e^{-z}$.

We remark that, from Euler's identity it now follows that the exponential function e^z is *periodic of period $2\pi i$* .¹

$$e^{z+2\pi i} = e^{x+i(y+2\pi)} = e^x e^{i(y+2\pi)} = e^x (\cos(y+2\pi) + i \sin(y+2\pi)) = e^x (\cos y + i \sin y) = e^z.$$

3.2. The complex logarithm.

We now define the complex logarithm. We wish to define the inverse function of the exponential, that is to solve in α the equation

$$e^\alpha = z,$$

and then set $\alpha = \log z$.

Since the exponential is never 0, z must be $\neq 0$. If $z = \varrho e^{i\theta}$ and we write $\alpha = a + ib$ in real and imaginary part, then we must have $e^{a+ib} = \varrho e^{i \arg z}$, that is,

$$\begin{cases} a = \log \varrho \\ b = \arg z. \end{cases}$$

¹A function $f(z)$ is said to be periodic of period T if $f(z+T) = f(z)$ for all z .

Recall that $\arg z$ is not a single-valued function. Thus, for $z \neq 0$ we define $\log z$ as a multiple-valued function as

$$\log z = \operatorname{Log} |z| + i \arg z. \quad (3.2)$$

Here we denote by $\operatorname{Log} r$ the classical logarithm of the positive number r . This notation is consistent with Definition 3.4 below.

Notice that, for $z \neq 0$, if β is one value of $\log z$, all other values of $\log z$ are of the form $\beta + 2k\pi i$, for $k \in \mathbf{Z}$. Moreover, $e^{\log z} = z$ for all such values, while instead is *not* true that $\log e^z = z$, for $z \neq 0$, since $\log z$ is not a single-valued function.

In order to define the logarithm as a single-valued function, we introduce the notion of *branch*; also called *determination*.

Definition 3.3. If F is a multiple-valued function, we call *branch* of F a single-valued continuous function f defined on a domain \mathcal{D} such that $F|_{\mathcal{D}} = f$.

Definition 3.4. Let S be the domain

$$S = \mathbf{C} \setminus \{(-\infty, 0]\} = \{z = \varrho e^{i\theta} : \varrho > 0 \quad -\pi < \theta < \pi\}.$$

We call *principal branch* of $\arg z$, and we denote it by $\operatorname{Arg} z$, the restriction of $\arg z$ to the domain S such that for $z = x > 0$, $\operatorname{Arg} z = 0$.

We define the *principal branch* of $\log z$, and we denote it by $\operatorname{Log} z$, the single-valued function on S defined as

$$\operatorname{Log} z = \operatorname{Log} |z| + i \operatorname{Arg} z. \quad (3.3)$$

Notice that, in order to define $\arg z$ and $\log z$ as a single-valued functions we need to restrict z in a domain that does not contain any circle around the origin. We have chosen to remove the half-line $(-\infty, 0]$, called the *branch cut*, from the complex plane, but this clearly is not the only choice, as we will soon see.

Proposition 3.5. Let S and $\log z$ be defined as above. Then $\operatorname{Log} z$ is holomorphic on S . Moreover, the derivative of $\operatorname{Log} z$ is the function $1/z$. Thus, in particular $\operatorname{Log} z$ is the anti-derivative of the holomorphic function $1/z$ on S .

Proof. It suffices to see that the CR-equations in polar coordinates take the form (see Exercise I.7)

$$\begin{cases} \partial_{\rho} u = \frac{1}{\rho} \partial_{\theta} v \\ \frac{1}{\rho} \partial_{\theta} u = -\partial_{\rho} v. \end{cases} \quad (3.4)$$

Clearly, the function $\operatorname{Log} z$ has real and imaginary parts that are $C^1(S)$ and satisfy such equations. The last part of the statement follows from a direct computation (using again Exercise I.7, and the fact that $f'(z) = \partial_x u(z) + i \partial_x v(z)$). \square

Remark 3.6. If α_1, α_2 are non-zero complex numbers and $\mathcal{L}z$ denotes a given branch of the logarithm, then it is not in general true that $\mathcal{L}(\alpha_1 \alpha_2) = \mathcal{L}\alpha_1 + \mathcal{L}\alpha_2$. On the other hand, it is true that, as multiple-valued functions,

$$\log(\alpha_1 \alpha_2) = \log \alpha_1 + \log \alpha_2.$$

However, there exist two (and in fact infinite pairs of) branches $\mathcal{L}_{(1)}, \mathcal{L}_{(2)}$ such that

$$\mathcal{L}(\alpha_1 \alpha_2) = \mathcal{L}_{(1)} \alpha_1 + \mathcal{L}_{(2)} \alpha_2.$$

The function defined in (3.3) is called the principal branch of the logarithm. We can also define other branches of the logarithm on the same domain S by setting

$$\operatorname{Log}_{k_0} z = \operatorname{Log} |z| + i(\operatorname{Arg} z + 2k_0\pi).$$

Notice that the function $z \mapsto \operatorname{Arg} z + 2k_0\pi$ is the restriction to S of $z \mapsto \arg z$ that takes the value $2k_0\pi$ when $z = x$ and $x > 0$; and notice also that $\operatorname{Log}_0 z$ is again the principal branch.

We now define other determinations of the $\log z$ using different branch cuts.

Define S_{θ_0} to be the plane taken away the half-line $\ell := \{z = re^{i\theta_0} : r \geq 0\}$, that is, $S_{\theta_0} = \mathbf{C} \setminus \ell$, and define a branch of $\arg z$ on S_{θ_0} by setting

$$\arg_{S_{\theta_0}} z = \arg z, \theta_0 < \arg z < \theta_0 + 2\pi.$$

Notice that $S_{-\pi} = S$ and $\arg_{-\pi} z = \operatorname{Arg} z$.

Now, the same definition as in (3.3) gives a holomorphic function in S_{θ_0} :

$$\mathcal{L}_{\theta_0} z = \operatorname{Log} |z| + i(\arg_{S_{\theta_0}} z). \quad (3.5)$$

Remark 3.7. Notice that the function $\operatorname{Log}(1+z)$ is well defined and holomorphic for $|z| < 1$, since then the argument of the logarithm lies in the right half plane, i.e. the principal argument of the complex number $1+z$ is between $-\pi/2$ and $\pi/2$ (recall that here $|z| < 1$). Moreover, the derivative $\operatorname{Log}(1+z)$ is the function $1/(1+z)$ that has power series expansion about 0

$$\sum_{n=0}^{+\infty} (-1)^n z^n.$$

Integrating term by term we obtain the expansion

$$\operatorname{Log}(1+z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{z^n}{n}, \quad (3.6)$$

valid for $|z| < 1$.

Using the logarithmic function we can now define the multiple-valued function z^α , for $\alpha \in \mathbf{C}$. In fact we set

$$z^\alpha = e^{\alpha \log z}, \quad (3.7)$$

whenever $\log z$ is defined; e.g. when $z \in S$.

When we choose a branch of the logarithm, so that we have a well-defined single-valued function, we obtain a branch of z^α ; hence a single-valued holomorphic function.

We remark, as it is easy to check, that the function z^α is single-valued if α is an integer, it is finitely many-valued if α is rational, and countably infinite-valued if α is irrational.

Remark 3.8. We conclude this part with an important warning on the notation. In order to avoid awkward notation, in the remaining of these notes, we will *always* use the logarithm function as a single-valued function, and we will write $\log z$ to indicate *any given branch* of the logarithm—hence with a change of notation w.r.t. the previous treatment.

The use of this notation should however cause no confusion.

3.3. The binomial series.

Proposition 3.9. *Set*

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad \text{for } n = 1, 2, \dots, \quad \text{and} \quad \binom{\alpha}{0} = 1,$$

and define the binomial series

$$\sum_{n=0}^{+\infty} \binom{\alpha}{n} z^n = (1+z)^\alpha. \quad (3.8)$$

Then, the binomial series has radius of convergence $r = 1$ if α is not a non-negative integer, $r = +\infty$ if $\alpha \in \mathbf{N}$.

Proof. Notice that the right hand side in (3.8) is a holomorphic function at least in the region $\arg(1+z) \in (-\pi/2, \pi/2)$, that is, $\operatorname{Re}(1+z) > 0$.

If $\alpha = N \in \mathbf{N}$, the binomial coefficient $\binom{\alpha}{n} = 0$ when $n > N$, and the series reduces to Newton's binomial expansion.

If α is not a non-negative integer, then all the coefficients $a_n = \binom{\alpha}{n}$ are non-zero. By Prop. 1.1 (iv),

$$\lim_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow +\infty} \left| 1 - \frac{\alpha}{n} \right| = 1,$$

as we wished to show.

Finally, notice that the two functions in the equality (3.8) are holomorphic in $|z| < 1$ and they coincide on the set $\{|z| < 1, \operatorname{Im} z = 0\}$. Using the identity principle (see Subsection 4.3) it follows that they coincide in $|z| < 1$. \square

3.4. Exercises.

3.1. (i) Given the power series

$$\sum_{n=2}^{\infty} \frac{a^{n+1}}{n(n-1)} z^n$$

where $a > 0$. Find the radius of convergence and compute the sum of the series when $a = 2$.

(ii) Find the radius of convergence and compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n} (z-2)^n.$$

3.2. Expand in power series the following functions about the assigned points:

$$f(z) = \frac{2z-8}{z^2-8z+12} \text{ at } z_0 = 0, \quad g(z) = \frac{1}{1+z^2} \text{ at } z_0 = 2i.$$

Determine their radii of convergence and compute $f^{(n)}(0)$ and $g^{(n)}(2i)$ for all n .

3.3. Determine the radius of convergence of the following series

$$\sum_{n=1}^{+\infty} a_n z^n,$$

where

$$(i)a_n = (\log n)^2, \quad (ii)a_n = n!, \quad (iii)a_n = \frac{n^2}{4^n + 3n}, \quad (iv)a_n = \frac{(n!)^3}{(3n)!}.$$

3.4. Determine the radius of convergence and compute the sum of the following series

$$\sum_{n=3}^{+\infty} n z^n \qquad \sum_{n=2}^{+\infty} \frac{n}{n^2 - 1} z^n \qquad .$$

3.5. Find a branch of:

- (a) $\log(z + i)$ that is analytic in $\mathbf{C} \setminus \{x - i : x \geq 0\}$;
- (b) $\log(z + i)$ that is analytic in $\mathbf{C} \setminus \{x - i : x \leq 0\}$;
- (c) $\log(z^2 + i)$ that is analytic at $z = 1 - i$ and takes the value $i\frac{3}{2}\pi$ there.

3.6. Describe the two branches of \sqrt{z} .

4. CONSEQUENCES OF CAUCHY'S INTEGRAL FORMULA

4.1. Expansion of a holomorphic function in Taylor series. We recall that if f is holomorphic in a domain Ω , $z_0 \in \Omega$ and $\overline{D(z_0, r)} \subseteq \Omega$, $\gamma = \partial D(z_0, r)$, then formula (2.2) holds:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Theorem 4.1. *Let f, Ω, z_0 be as above. Then f admits power series expansion about z_0 . The series converges in the largest disk $D(z_0, R) \subseteq \Omega$.*

Proof. Let $0 < r < R$ be such that $\overline{D(z_0, r)} \subseteq \Omega$ and let $\gamma = \partial D(z_0, r)$ be the circle centered at z_0 and radius r . For $z \in D(z_0, r)$ we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0 - (z - z_0)} d\zeta \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} d\zeta. \end{aligned}$$

Since $|z - z_0| < r = |\zeta - z_0|$, we have

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1,$$

so that

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{+\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n,$$

and the series converges uniformly for $\zeta \in \gamma$.

Therefore, we can interchange integration and summation order to obtain that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} \cdot \sum_{n=0}^{+\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{+\infty} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n. \end{aligned}$$

Hence,

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

This concludes the proof. \square

It is worth observing that in the course of the proof we only have used the continuity of the function f on the curve γ . That is, we have the following.

Proposition 4.2. *Let γ be a curve in the complex plane, g a continuous function on γ . Let*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

Then f is holomorphic on $\mathbf{C} \setminus \gamma$.

Proof. Let $z_0 \in \mathbf{C} \setminus \gamma$ and let $r > 0$ be given by $r = \text{dist}(z_0, \gamma)$. For $0 < r' < r$, for $z \in \overline{D(z_0, r')}$, as in the proof of Thm. 4.1 we can write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z_0} \cdot \sum_{n=0}^{+\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{+\infty} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n. \end{aligned}$$

This shows that f admits power series expansion about z_0 and converging in every disk $D(z_0, r')$, with $r' < \text{dist}(z_0, \gamma)$. The conclusion now follows. \square

An immediate consequence of Thm. 4.1 is the following.

Corollary 4.3. *Let Ω be a domain and let f be holomorphic in Ω . Then f is indefinitely complex differentiable in Ω , that is, $f^{(n)}$ exists on Ω for all n , and hence $f^{(n)}$ is holomorphic in Ω for all n .*

Proof. It suffices to prove the statement for f any disk $D(z_0, r) \subseteq \Omega$. This follows at once from the previous Thm. 4.1 \square

4.2. Further consequences of Cauchy's integral formula. A first, classical, consequence of Cauchy's integral formula is the following.

Theorem 4.4. (Morera) *Let f be continuous on a domain Ω . Suppose that*

$$\int_{\gamma} f(\zeta) d\zeta = 0$$

for all closed curves γ in Ω . Then, f is holomorphic in Ω .

Proof. By Prop. 2.3 we know that f admits an anti-derivative F , which of course is holomorphic on Ω . By Cor. 4.3 all of the derivatives of F , so f in particular, are holomorphic on Ω . \square

Recall that for a power series expansion $f(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n$ we have that $f^{(n)}(z_0) = n!a_n$. Then, we just have obtained the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

valid when $\gamma = \partial D(z_0, r) \subseteq \Omega$. More generally we have

Theorem 4.5. (Cauchy's formula for the derivatives) *Let f be holomorphic in a domain Ω , $z_0 \in \Omega$, $\zeta \in D(z_0, r)$ and $\overline{D(z_0, r)} \subseteq \Omega$, $\gamma = \partial D(z_0, r)$. Then, for every $n = 0, 1, 2, \dots$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \quad (4.1)$$

Proof. By passing the differentiation under the integral sign we have

$$\frac{d}{dz}f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

Assuming the statement true for $n - 1$ we see that

$$\begin{aligned} \frac{d}{dz}f^{(n-1)}(z) &= \frac{d}{dz} \left(\frac{(n-1)!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^n} d\zeta \right) \\ &= \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \end{aligned}$$

Thus, we only need to justify that we can pass the differentiation under the integral sign. One way to see this, we notice that

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{h} \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) d\zeta \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta. \end{aligned}$$

The convergence of $((\zeta - z - h)(\zeta - z))^{-1}$ to $(\zeta - z)^{-2}$ is uniform for $z \in \overline{D(z_0, r')}$, with $0 < r' < r$ fixed, and $\zeta \in \partial D(z_0, r)$. Then the conclusion follows for the first derivative. The argument is analogous for the higher derivatives and we are done. \square

Corollary 4.6. (Cauchy's estimates) *Let f be holomorphic in a domain Ω , $z_0 \in \Omega$ and $\overline{D(z_0, r)} \subseteq \Omega$, $\gamma = \partial D(z_0, r)$. Then, for every $n = 0, 1, 2, \dots$ we have*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{|\zeta - z_0| = r} |f(\zeta)|. \quad (4.2)$$

Proof. From (4.1) we have

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \\ &\leq \frac{n!}{2\pi} \int_{\gamma} \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |d\zeta| \\ &\leq \frac{n!}{2\pi} \sup_{|\zeta - z_0| = r} |f(\zeta)| \frac{1}{r^{n+1}} \int_{\gamma} |d\zeta| \\ &= \frac{n!}{r^n} \sup_{|\zeta - z_0| = r} |f(\zeta)|. \end{aligned}$$

This proves the corollary. \square

Definition 4.7. A function f that is holomorphic on the whole complex plane \mathbf{C} is called *entire*.

Corollary 4.8. (Liouville's theorem) *Let f be a bounded entire function. Then f is constant*

Proof. This follows at once from Cor. 4.6. If f is bounded, there exists $C > 0$ such that for all $r > 0$, $z_0 \in \mathbf{C}$,

$$\sup_{|\zeta - z_0| = r} |f(\zeta)| \leq C.$$

By (4.2) we have then

$$|f^{(n)}(z_0)| \leq C \cdot \frac{n!}{r^n}$$

for all $r > 0$. Letting $r \rightarrow +\infty$ we obtain $f^{(n)}(z_0) = 0$ for $n = 1, 2, \dots$ and all $z_0 \in \mathbf{C}$. Hence, f is constant (by Prop. 1.7). \square

Corollary 4.9. (Fundamental theorem of algebra) *Let $p(z)$ be a non-constant polynomial of degree n . Then $p(z)$ has exactly n -roots in \mathbf{C} , counting multiplicity.*

Proof. It suffices to prove that a non-constant polynomial has at least one root. Let $p(z) = a_0 + a_1z + \dots + a_nz^n$, with $a_n \neq 0$ and $n \geq 1$, since p is assumed to be non-constant. If p did not have any root in \mathbf{C} the function $1/p$ would be entire and bounded; therefore constant, by Cor. 4.8. It would follow that $1/p = c$; that is, $p = 1/c$ would be constant, a contradiction. \square

We conclude this part with a very important result about the holomorphicity of a limit of sequence of holomorphic functions.

Theorem 4.10. (Weierstrass convergence theorem) *Let Ω be a domain and $\{f_n\}$ be a sequence of holomorphic functions on Ω . Suppose that the sequence $\{f_n\}$ converges uniformly on compact subsets of Ω to a function f .*

Then f is holomorphic on Ω and the sequence $\{f'_n\}$ converges uniformly on compact subsets to f' .

Proof. Let $z_0 \in \Omega$, $r > 0$ such that $\overline{D(z_0, r)} \subseteq \Omega$, $\gamma = \partial D(z_0, r)$ and $0 < r' < r$. For $z \in \overline{D(z_0, r')}$, which of course is compact in Ω , by Cauchy's formula (2.2), for all $n = 1, 2, \dots$ we have

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(\zeta)}{\zeta - z} d\zeta.$$

Since $|\zeta - z| \geq r - r' = \delta_0 > 0$, for all $z \in \overline{D(z_0, r')}$, the sequence

$$\frac{f_n(\zeta)}{\zeta - z} \rightarrow \frac{f(\zeta)}{\zeta - z}$$

uniformly in ζ , for $\zeta \in \gamma$, which is a compact set in Ω . Then we can pass to the limit under the integral sign and obtain that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

for $z \in \overline{D(z_0, r')}$. Since f is continuous on γ , by Prop. 4.2 it follows that f is holomorphic on $z \in \overline{D(z_0, r')}$, for every r' with $0 < r' < r$. Hence f is holomorphic on $D(z_0, r)$; hence on Ω .

Next, by Cauchy's formula for the derivatives, it follows that

$$f'_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta,$$

for $z \in \overline{D(z_0, r')}$. Arguing as before we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} f'_n(z) &= \lim_{n \rightarrow +\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \\ &= f'(z), \end{aligned}$$

so that $f'_n(z) \rightarrow f'(z)$, for $z \in \Omega$. The convergence is also uniform on compact subsets since, for $z \in \overline{D}(z_0, r')$,

$$\begin{aligned} |f'_n(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} d\zeta \right| \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \gamma} |f_n(\zeta) - f(\zeta)| \int_{\gamma} \frac{1}{|\zeta - z|^2} |d\zeta| \\ &\leq \frac{r}{(r - r')^2} \sup_{\zeta \in \gamma} |f_n(\zeta) - f(\zeta)|. \end{aligned}$$

The conclusion now follows. \square

4.3. The identity principle. We now discuss some local properties of holomorphic functions.

Proposition 4.11. *Let f be holomorphic on a domain Ω and let $z_0 \in \Omega$. Suppose that*

$$f^{(k)}(z_0) = 0 \quad \text{for } k = 0, 1, 2, \dots$$

Then $f \equiv 0$ in Ω .

Proof. Let $D(z_0, r) \subseteq \Omega$. Then, for $z \in D(z_0, r)$

$$f(z) = \sum_{k=0}^{+\infty} a_k (z - z_0)^k = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k = 0.$$

Then f is identically 0 in $D(z_0, r)$. Now let Ω_1 be the set of the points in Ω such that f and all its derivatives calculated at that point are 0, and let $\Omega_2 = \Omega \setminus \Omega_1$.

The argument above shows that Ω_1 is open, while it is closed as intersection of closed sets. Moreover it is non-empty by assumption. Since Ω is connected, $\Omega_1 = \Omega$ and therefore $f \equiv 0$ in Ω . \square

Definition 4.12. Let Ω be a domain and $z_0 \in \Omega$. We say that a (holomorphic) function f has a zero of order k at $z = z_0$ if

$$f(z_0) = f'(z_0) = \dots = f^{(k-1)}(z_0) = 0 \quad \text{and} \quad f^{(k)}(z_0) \neq 0.$$

If f has a zero of order k at z_0 , by Taylor expansion, we can write

$$f(z) = (z - z_0)^k f_k(z) \tag{4.3}$$

with f_k holomorphic in Ω and $f_k(z_0) \neq 0$. (The fact that f_k is holomorphic in Ω requires just a moment's thought. The function f_k is certainly holomorphic in the disk $D(z_0, r) \subseteq \Omega$ where the expansion of f converges. So, $f_k = f/(z - z_0)^k$ is holomorphic in a ngbh of z_0 . On the other hand, $f/(z - z_0)^k$ is clearly holomorphic in $\Omega \setminus D(z_0, r)$.)

Since f_k is in particular continuous, there exists $\delta > 0$ such that $f_k \neq 0$ in $D(z_0, \delta)$. Therefore, by (4.3) we have that

$$f(z) = (z - z_0)^k f_k(z)$$

has a zero of order k at z_0 and does not vanish at any other point of $D(z_0, \delta)$.

Therefore, we have the following.

Proposition 4.13. *A holomorphic function f has only isolated zeros, unless $f \equiv 0$.*

If two functions f and g holomorphic on a domain Ω coincide in infinite points having an accumulation point in Ω , then $f \equiv g$ on Ω .

Proof. Let z_0 be a zero for f , and assume f is not identically zero. Then, f cannot have all the derivatives vanishing at z_0 , so it must have a zero of finite order k . Hence,

$$f(z) = (z - z_0)^k f_k(z)$$

has a zero of order k at z_0 and does not vanish at any other point in a suitable ngbh of z_0 ; that is, z_0 is an isolated zero.

For the second part of the statement it suffices to apply the first part to the holomorphic function $f - g$. \square

4.4. The open mapping theorem and the principle of maximum modulus.

Theorem 4.14. *Let f be holomorphic in a ngbh of a point z_0 and let $f'(z_0) \neq 0$. Then, there exist a ngbh U of z_0 and a ngbh V of $f(z_0)$, a function g holomorphic on V such that*

$$g(f(z)) = z$$

for all $z \in U$.

Proof. It is possible to give a proof using the Taylor expansion of f at z_0 . By applying two translations and then a multiplication by a non-zero constant, we may assume that $z_0 = 0 = f(z_0)$, and that $f'(z_0) = 1$. Then, in a nbgh of the origin, f can be written as

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k.$$

If such a g exists, it must have expansion

$$g(w) = \sum_{n=1}^{+\infty} b_n w^n,$$

and it must hold that $f(g(w)) = w$, for w in $V' \subseteq V$. Therefore,

$$g(w) + \sum_{k=2}^{+\infty} a_k g(w)^k = w.$$

From this equation one can obtain the coefficients b_n iteratively from the a_k 's, and show that the power series having the b_n 's as coefficients has positive radius of convergence. We leave the details as an exercise. We will provide a different proof using Rouché's theorem, Thm. 5.26. \square

Theorem 4.15. (The open mapping theorem) *Let f be holomorphic and non-constant on a domain Ω . Then f is an open mapping, that is, the image of open sets through f are open.*

Proof. It suffices to show that for every $z_0 \in \Omega$, there exists a ngbh V of $f(z_0)$ such that $f(\Omega)$ contains V .²

²This condition would show that $f(\Omega)$ is open. Since the argument is independent of the open set on which f is defined, it can be repeated for any open set $A \subseteq \Omega$.

By translation, we may assume $z_0 = 0$ and $f(0) = 0$. In a suitably small ngbh of 0, as in (4.3) we may write

$$f(z) = z^m f_m(z) = a_m z^m (1 + h(z))$$

where, since $f_m(0) \neq 0$, $a_m \neq 0$ and $h(0) = 0$. Therefore, in a suitably small ngbh of 0, for a holomorphic function h_1 defined in such a ngbh,

$$f(z) = \left(az(1 + h_1(z)) \right)^m$$

Setting $f_1(z) = az(1 + h_1(z))$, by Thm. 4.14 is locally invertible, with a holomorphic inverse. Therefore, f_1 is an open mapping,³ and in particular there exist an (open) nbgh U of 0 and $\delta > 0$ such that

$$f_1(U) \supseteq D(0, \delta),$$

and $f(z) = f_1(z)^m$,

$$f(U) = (f_1(U))^m \supseteq D(0, \delta^m).$$

Hence, f is an open mapping. \square

Corollary 4.16. (Maximum modulus principle) *Let Ω be a domain, f holomorphic in Ω . Suppose there exist $z_0 \in \Omega$ and $r > 0$ such that $D(z_0, r) \subseteq \Omega$ and*

$$|f(z_0)| \geq |f(z)| \quad \text{for all } z \in D(z_0, r).$$

Then f is constant in Ω .

Proof. Seeking a contradiction, suppose that f is not constant in Ω . Then f is non-constant in $D(z_0, r)$ (since Ω is connected) and by the open mapping theorem

$$f(D(z_0, r)) \supseteq D(f(z_0), \delta)$$

for some $\delta > 0$. But this contradicts the hypothesis $|f(z_0)| \geq |f(z)|$ for all $z \in D(z_0, r)$. Hence, f must be constant. \square

4.5. The general form of Cauchy's theorem. In Cauchy's Thm. 2.7 we were concerned with functions holomorphic on an open disk. We now want to present a version of that theorem and a corresponding Cauchy's formula for more general open sets, more precisely for a multi-connected domain. In this section, we are not going to provide all the proofs, since the required techniques fall aside the scope and main course of these lectures, and for lack of adequate time. Full proofs can be found in [A] and [L], for instance.

We begin by extending the notion of line integral. Given curves $\gamma_1, \dots, \gamma_n$ we will call chain the union C of the curves, that is, formally we set $\mathcal{C} = \gamma_1 \cup \dots \cup \gamma_n$, or we can also use the more appealing notation

$$\mathcal{C} = \gamma_1 + \dots + \gamma_n.$$

Moreover, recalling that, for a given curve γ , the curve $-\gamma$ has the opposite orientation (see the beginning of Sec. 2), we give the following definition.

³For, if $A \subseteq \Omega$ is open and g is the inverse of f , then $f(A)$ is the inverse image of the open set A through g , that is, $f(A) = g^{-1}(A)$.

Definition 4.17. We call *chain* the formal expression

$$\mathcal{C} = k_1\gamma_1 + \cdots + k_n\gamma_n,$$

where k_j is an integer and $\gamma_1, \dots, \gamma_n$ are curves. We call the chain a *cycle* if $\gamma_1, \dots, \gamma_n$ are closed curves.

Given a function f continuous on the chain \mathcal{C} defined as above we set

$$\int_{\mathcal{C}} f(z) dz := \sum_{j=1}^n k_j \int_{\gamma_j} f(z) dz.$$

Definition 4.18. Let Ω be a domain, γ_0, γ_1 be curves in Ω such that $\gamma_j : [a, b] \rightarrow \Omega$, $j = 0, 1$. We say that γ_0 and γ_1 are *homotopic* if there exists a continuous function

$$\Gamma : [a, b] \times [0, 1] \rightarrow \Omega$$

such that $\Gamma(\cdot, s)$ is piecewise $C^1([a, b])$ for all $s \in [0, 1]$ and

- (i) $\Gamma(\cdot, 0) = \gamma_0$;
- (ii) $\Gamma(\cdot, 1) = \gamma_1$;
- (iii) $\Gamma(a, s) = \gamma_0(a) = \gamma_1(a)$ for all $s \in [0, 1]$;
- (iv) $\Gamma(b, s) = \gamma_0(b) = \gamma_1(b)$ for all $s \in [0, 1]$.

Notice in particular that we require γ_0 and γ_1 to have the same end points and the same orientation, and that $\Gamma(\cdot, s)$ are curves, and that they also have the same end points and orientation.

There exists also a definition of homotopy in the class of continuous functions $\sigma : [a, b] \rightarrow \mathbf{C}$. We call such functions *paths*. Then, two paths σ_1, σ_2 contained in an open set A are said to be *homotopic in A* if there exists a continuous function Γ satisfying properties (i)-(iv) above (with γ_1, γ_2 replaced by σ_1, σ_2 resp.)

Recall that a domain Ω is *simply connected* if every closed path σ in Ω is homotopic to a point $z_0 \in A$.

Recall that in Def. 2.11 we defined the index of a point z_0 with respect to closed curve γ not passing through z_0 , the integer

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}.$$

Definition 4.19. Let Ω be a domain and let γ_1, γ_2 be curves in Ω . We say that γ_1, γ_2 are *homologous in Ω* if

$$n(\gamma_1, z_0) = n(\gamma_2, z_0), \quad \text{for all } z_0 \in \mathbf{C} \setminus \Omega.$$

In this case we write $\gamma_1 \sim \gamma_2 \pmod{\Omega}$.

We say that a curve γ is *homologous to 0 in Ω* if

$$n(\gamma, z_0) = 0, \quad \text{for all } z_0 \in \mathbf{C} \setminus \Omega.$$

These definitions can be extended to chains in an obvious manner, since $n(\gamma_1 + \gamma_2, z_0) = n(\gamma_1, z_0) + n(\gamma_2, z_0)$. Notice that the relation \sim of homologous curves is an equivalence relation.

We now compare the definition of homotopy and homology between curves.

Lemma 4.20. *Let Ω be a domain, γ a closed curve in Ω . If γ is homotopic to a point $z_0 \in \Omega$, then γ is homologous to 0 in Ω .*

Proof. Let $\Gamma(t, s)$ be the homotopy of γ to z_0 , so that $\Gamma(\cdot, 0) = \gamma$ and $\Gamma(\cdot, 1) = z_0$. Write $\Gamma(\cdot, s) = \gamma_s$. Let $\zeta_0 \in \mathbf{C} \setminus \Omega$. Now we claim that the function

$$G : [0, 1] \ni s \mapsto n(\gamma_s, \zeta_0)$$

is continuous. For, we observe that the function $1/(\zeta - \zeta_0)$ is holomorphic in Ω since $\zeta_0 \notin \Omega$. Next, if $\delta > 0$ is chosen small enough, if $|s_1 - s_2| < \delta$ there exist finitely many disks D_1, \dots, D_n whose union contains γ_{s_1} and γ_{s_2} and it is contained in Ω . Then the holomorphic function $1/(z - \zeta_0)$ admits anti-derivative in $D_1 \cup \dots \cup D_n$, by virtue of Cor. 2.8. This easily implies that $\int_{\gamma_{s_1}} 1/(\zeta - \zeta_0) d\zeta = \int_{\gamma_{s_2}} 1/(\zeta - \zeta_0) d\zeta$.

Then, G is a continuous function taking integer values, so it must be constant on $[0, 1]$. Therefore, $G(0) = G(1)$, that is,

$$n(\gamma, \zeta_0) = \frac{1}{2\pi i} \int_{\{z_0\}} \frac{1}{\zeta - \zeta_0} d\zeta = 0,$$

for all $\zeta_0 \in \mathbf{C} \setminus \Omega$. \square

The converse of the statement in the lemma is not true. It is easy to provide a counterexample. Consider the domain Ω given by the plane taken away two points, say the origin and $z_0 = 2$. Let γ be the close curve consisting of the circle $\partial D(0, 1)$ starting and $z = 1$ followed by the circle $\partial D(1, 1)$ starting at $z = 1$, and then and the circle $\partial D(0, 1)$ followed by the circle $\partial D(1, 1)$ this time each covered in the opposite direction with respect to the first time. This curve is homologous to 0 in Ω , but cannot be continuously deformed to a point without leaving Ω , i.e. it is not homotopic to a point in Ω .

The previous statement can also be rephrased as if two closed curves γ_1, γ_2 are homotopic in Ω , then they are homologous— it suffices to consider the closed curve $\gamma = \gamma_1 \cup \{-\gamma_2\}$.

We have the following characterization of simply connectivity.

Theorem 4.21. *A domain Ω is simply connected if and only if $n(\mathcal{C}, z_0) = 0$ for all cycles \mathcal{C} in Ω and all $z_0 \in \mathbf{C} \setminus \Omega$.*

Proof. The necessity follows from Lemma 4.20, applied to each closed curve γ part of the given cycle \mathcal{C} .

For the sufficiency we refer to [A], Thm. 14 in Section 4.2. \square

The proof of the sufficiency in the previous theorem also uses the following classical result that, although very intuitive, requires a long and delicate argument.

We call *Jordan curve* a homeomorphic image of the unit circle ∂D . (Hence, a *simple* curve is a Jordan curve.)⁴

Theorem 4.22. (Jordan curve Thm.) *Given a Jordan curve γ , the open set $\mathbf{C} \setminus \gamma$ is union of two disjoint connected open sets U_b, U_u , one being bounded and the other one unbounded, both having as boundary the curve γ . We call the interior of γ the bounded open connected component U_b of $\mathbf{C} \setminus \gamma$.*

⁴A proof of this theorem, in its full generality, can be found in an algebraic topology textbook, such as *Algebraic Topology*, by E. H. Spanier, Springer Ed., p. 198.

The significance of the notion of homologous curves resides in the general form of Cauchy's theorem.

Theorem 4.23. (General form of Cauchy's Thm.) *Let Ω be a domain. Then for all functions f holomorphic in Ω and cycles γ homologous to 0 in Ω we have*

$$\int_{\gamma} f(z) dz = 0.$$

Proof. We present an argument due to Beardon. The reader can find the complete details in [A], Subsection 4.5 in Ch. 4.

It is easy to see that we may assume that Ω is bounded. Let \mathcal{Q}_k be the collection of dyadic squares in \mathbf{R}^2 (that we identify with \mathbf{C} as usual) of side length 2^{-k} , $k \in \mathbf{N}$. Let \mathcal{Q} be the collection of all squares in \mathcal{Q}_k that are contained in Ω and let

$$\Omega_k = \bigcup_{Q \in \mathcal{Q}} Q.$$

Then $\Omega_k \subseteq \Omega$ and $\text{dist}(\Omega_k, \partial\Omega) < \sqrt{2}2^{-k}$.

Let γ a cycle homologous to 0 in Ω be given. We choose k large enough so that γ is contained in Ω_k . The boundary of Ω_k is a finite union of closed polygonal curves, and we denote it by $\partial\Omega_k$. Notice that

$$\partial\Omega_k = \sum_{Q \in \mathcal{Q}} \partial Q.$$

Let $\zeta \in \Omega \setminus \Omega_k$. It follows that $n(\partial\Omega_k, \zeta) = 0$. For, there exists $\tilde{Q} \in \mathcal{Q}_k \setminus \mathcal{Q}$ such that $\zeta \in \tilde{Q}$. Now, \tilde{Q} contains points in ${}^c\Omega$, say ζ_0 . The points ζ and ζ_0 can be joined by a line segment that does not meet γ , since γ is contained in Ω_k . Since $n(\gamma, \zeta_0) = 0$, it follows that $n(\gamma, \zeta) = 0$, for all $\zeta \in \Omega \setminus \Omega_k$. In particular, $n(\gamma, \zeta) = 0$ for all $\zeta \in \partial\Omega_k$.

Next, let f be holomorphic in Ω . Let $z \in \Omega_k$. If $z \in Q_0$, for some $Q_0 \in \mathcal{Q}$ (that is, lies in the interior of one square Q_0), then

$$\frac{1}{2\pi i} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } Q = Q_0 \\ 0 & \text{if } Q \neq Q_0, \end{cases}$$

and hence

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega_k} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Since both sides of the identity above are continuous function of z , it follows that the identity holds true for all $z \in \Omega_k$.

Therefore,

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} \left(\frac{1}{2\pi i} \int_{\partial Q} \frac{f(\zeta)}{\zeta - z} d\zeta \right) dz \\ &= \int_{\partial\Omega_k} f(\zeta) \left(\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - z} dz \right) d\zeta \\ &= 0, \end{aligned}$$

where we have switched the integration order since the integrands are continuous functions of z and ζ , and since the last inner integral on the right hand side is $-n(\gamma, \zeta) = 0$.

This proves the theorem. \square

In some special cases, the idea of the proof of the previous theorem can be illustrated by the following lemma that reduces the integration along a general curve to the integration over a number of circles.

Lemma 4.24. *Let $\Omega = \tilde{\Omega} \setminus \{z_1, \dots, z_n\}$, where $\tilde{\Omega}$ is a domain and $z_1, \dots, z_n \in \tilde{\Omega}$. Let γ be curve in Ω , homologous to 0 in $\tilde{\Omega}$.*

Then, there exist $\delta_j > 0$, $j = 1, \dots, n$ such that the closed disks $\overline{D(z_j, r_j)}$ are all contained in $\tilde{\Omega}$ and are all disjoint, and if $\gamma_j = \partial D(z_j, r_j)$, $j = 1, \dots, n$, then γ is homologous to cycle $\sigma := \sum_{j=1}^n n(\gamma, z_j) \gamma_j$ in Ω .

An obvious consequence of the Thm. 4.23 is the following.

Corollary 4.25. *Let Ω and f be as above and let γ be a curve homotopic to a point in Ω . Then*

$$\int_{\gamma} f(z) dz = 0.$$

We apply this corollary to define the logarithm of a non-zero holomorphic function on a simply connected domain.

Corollary 4.26. *Let Ω be a simply connected domain and let f be holomorphic on Ω . Suppose that $f(z) \neq 0$ for all $z \in \Omega$. Then $\log f(z)$ is a well defined function on Ω and it is holomorphic.*

Proof. The function $f'(z)/f(z)$ is holomorphic on Ω . Since Ω is simply connected,

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$

for all closed curves γ in Ω . Hence, by Prop. 2.3, there exists a holomorphic function g on Ω such that

$$g'(z) = \frac{f'(z)}{f(z)}.$$

We may choose g so that $g(z_0) = \log f(z_0)$ (since $f(z) \neq 0$ for all $z \in \Omega$ and since g was defined only up to an additive constant).

We wish to show that $g(z) = \log f(z)$, that is, $e^{g(z)} = f(z)$. We consider the function $f(z)e^{-g(z)}$ and compute its derivative:

$$\begin{aligned} \frac{d}{dz}(f(z)e^{-g(z)}) &= (f'(z) - f(z)g'(z))e^{-g(z)} \\ &= 0. \end{aligned}$$

Hence $f(z)e^{-g(z)} = C$ is constant on Ω (since in particular it is connected), and $C \neq 0$. By our previous choice,

$$C = f(z_0)e^{-g(z_0)} = 1.$$

Therefore,

$$f(z) = e^{g(z)}$$

and we are done. \square

4.6. Exercises.

4.1. State and prove a holomorphic version of de l'Hôpital's rule.

4.2. Let $|z_0| < r < |w_0|$ and let γ be the circle of radius r centered at the origin, oriented counter-clockwise. Show that

$$\int_{\gamma} \frac{1}{(z - z_0)(z - w_0)} dz = \frac{2\pi i}{z_0 - w_0}.$$

4.3. Let $a > 0$. Show that each of the following series of functions represents a holomorphic function:

- (i) $\sum_{n=1}^{+\infty} e^{-an^2 z}$ for $\operatorname{Re} z > 0$;
- (ii) $\sum_{n=1}^{+\infty} \frac{e^{-anz}}{(a+n)^2}$ for $\operatorname{Re} z > 0$;
- (iii) $\sum_{n=1}^{+\infty} (a+n)^{-z}$ for $\operatorname{Re} z > 1$.

4.4. Determine the set

$$\Omega = \{z \in \mathbf{C} : -\pi < \arg(e^z + 1) < \pi\}$$

(that is, $e^z + 1 \notin (-\infty, 0]$). Show that the function $\sqrt{e^z + 1}$ is well defined and holomorphic on Ω . (Cfr. Cor. 4.25.)

5. ISOLATED SINGULARITIES OF HOLOMORPHIC FUNCTIONS

Definition 5.1. We call a *Laurent series* a doubly infinite series

$$\sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n.$$

We say that such a series converges (absolutely, uniformly, resp.) if each of the two series

$$\sum_{n=0}^{+\infty} a_n(z - z_0)^n \quad \text{and} \quad \sum_{n=-1}^{-\infty} a_n(z - z_0)^n$$

converge (absolutely, uniformly, resp.).

We denote by $A_{r,R}(z_0) = \{z : r < |z - z_0| < R\}$ the annulus centered at z_0 , with the obvious conventions if $r, R = 0, +\infty$.

Theorem 5.2. *Let f be holomorphic in $A = A_{r,R}(z_0)$. Then f admits Laurent expansion about z_0*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n. \quad (5.1)$$

The double series converges absolutely and uniformly on compact subsets of $A_{r,R}(z_0)$; hence in particular on all closed annuli $A_{s,S}(z_0)$ where $r < s < S < R$. Moreover,

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

for $n \in \mathbf{Z}$ and $\gamma_\rho = \partial D(z_0, \rho)$, $r < \rho < R$.

The expansion (5.1) is unique.

Proof. We may assume $z_0 = 0$.

Let s, S be such that $r < s < S < R$ and consider the annulus $A_{s,S} \subseteq A$. Since $-\gamma_s \cup \gamma_S$ is homologous to 0 in A , where $\gamma_\rho = \partial D(0, \rho)$ is oriented counter-clockwise, and $n(\gamma, z) = 1$ for $z \in A_{s,S}$, it follows that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in A_{s,S}$. The proof now proceeds as in the proof of Thm. 4.1. For $\zeta \in \gamma_s$, that is, for $|\zeta| = s < |z|$ we write

$$\frac{1}{\zeta - z} = -\frac{1}{z} \cdot \frac{1}{1 - \zeta/z} = -\frac{1}{z} \sum_{n=0}^{+\infty} \frac{\zeta^n}{z^n}$$

while for $\zeta \in \gamma_S$, i.e. $|\zeta| = S > |z|$, we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1 - z/\zeta} = \sum_{n=0}^{+\infty} \frac{z^n}{\zeta^{n+1}}.$$

The uniform convergence of the series shows that

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \int_{\gamma_S} f(\zeta) \frac{z^n}{\zeta^{n+1}} d\zeta + \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \int_{\gamma_s} f(\zeta) \frac{\zeta^n}{z^{n+1}} d\zeta \\ &= \sum_{n=0}^{+\infty} \left(\frac{1}{2\pi i} \int_{\gamma_S} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right) z^n + \sum_{j=-1}^{+\infty} \left(\frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{\zeta^{j+1}} d\zeta \right) z^j, \end{aligned}$$

for $z \in A_{s,S}$.

Hence, we have obtained that for $z \in A_{s,S}(z_0)$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n,$$

where

$$a_n = \begin{cases} \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta & \text{if } n < 0 \\ \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta & \text{if } n \geq 0. \end{cases}$$

The proof will be complete if we show that the value of

$$\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

is independent of ρ , for $r < \rho < R$. But this is immediate, since if we set $\gamma = \gamma_{\rho_1} \cup \{-\gamma_{\rho_2}\}$, with $r < \rho_1, \rho_2 < R$, then γ is a curve (more precisely, a cycle) homologous to 0 in A so that

$$\int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = 0$$

and we are done. \square

Theorem 5.3. (Riemann removable singularity Thm.) *Let Ω be a domain, $z_0 \in \Omega$, $\Omega' = \Omega \setminus \{z_0\}$. If f is holomorphic in Ω' , then there exists F holomorphic in Ω such that $F|_{\Omega'} = f$ if and only if f is bounded in a ngbh of z_0 .*

Proof. It is clear that if the extension F exists, this is unique (since $F(z_0)$ must be equal to $\lim_{z \rightarrow z_0} F(z) = \lim_{z \rightarrow z_0} f(z)$).

Since z_0 is an isolated singularity for f ,

$$f(z) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n$$

and we wish to show that $a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = 0$, for $n \leq -1$.

Using the fact that $f(\zeta)$ is bounded for $|\zeta - z_0| \rightarrow 0$, we have

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + \rho e^{i\theta})|}{\rho^{n+1}} \rho d\theta \leq \frac{C}{\rho^n}.$$

Since ρ can be chosen arbitrarily small, $a_n = 0$ for $n \leq -1$.

The converse is obvious. \square

Definition 5.4. Let $\Omega = \{z : 0 < |z - z_0| < r\}$ and let f be holomorphic in Ω . If

$$\lim_{z \rightarrow z_0} |f(z)| = +\infty$$

we say that z_0 is a *pole* for f .

Proposition 5.5. Let Ω be as above and f holomorphic in Ω . Then, f has a pole in z_0 if and only if the Laurent expansion of f about z_0 is given by

$$f(z) = \sum_{n=-m}^{+\infty} a_n(z - z_0)^n$$

with $m \geq 1$ and $a_{-m} \neq 0$. The integer m is called the *order* of the pole.

Proof. If f has the expansion above, that it is immediate to see that $z = z_0$ is a pole.

Conversely, since f is holomorphic in Ω , assuming that $f \not\equiv 0$, there exists $r' > 0$ such that $f(z) \neq 0$ for z in $\{0 < |z - z_0| < r'\}$. Then set $g(z) = 1/f(z)$ is holomorphic in $\{0 < |z - z_0| < r'\}$ and it is bounded as $z \rightarrow z_0$. Then g has a removable singularity in $z = z_0$. Calling g the holomorphic extension, it holds that $g(z_0) = 0$. Let m be the order of vanishing of g in z_0 , that is

$$g(z) = (z - z_0)^m g_m(z),$$

with g_m holomorphic and $g_m(z_0) \neq 0$. Then,

$$f(z) = \frac{1}{(z - z_0)^m} \cdot \frac{1}{g_m(z_0)} = \frac{1}{(z - z_0)^m} \cdot \tilde{f}(z),$$

where \tilde{f} is holomorphic in $\{|z - z_0| < r''\}$ and $\tilde{f}(z_0) \neq 0$. \square

Definition 5.6. A function f that is holomorphic in a domain $\Omega \setminus S$, where S is union of isolated points in Ω in which f has poles, is called a *meromorphic* function on Ω .

Definition 5.7. Let $\Omega = \{z : 0 < |z - z_0| < r\}$, f holomorphic in Ω . We say that f has an *essential singularity* in z_0 if $\lim_{z \rightarrow z_0} f(z)$ does not exist.

Theorem 5.8. (Casorati-Weierstrass Thm.) Let f be holomorphic in $\{0 < |z - z_0| < r\}$ and let f have an essential singularity in $z = z_0$. Then for each $\varepsilon > 0$, the set $f(\{0 < |z - z_0| < \varepsilon\})$ is dense in \mathbf{C} .

Proof. Seeking a contradiction suppose that there exist $w_0 \in \mathbf{C}$, $\delta, \varepsilon > 0$ such that

$$|f(z) - w_0| > \delta$$

for all $z \in A = \{0 < |z - z_0| < \varepsilon\}$. Then, the function

$$g(z) = \frac{1}{f(z) - w_0}$$

is holomorphic in A and is bounded there. Hence, g has a removable singularity in z_0 and $1/g$ has at most a pole in $z = z_0$. Then,

$$f = \frac{1}{g} + w_0$$

has at most a pole in $z = z_0$, a contradiction. \square

We mention that a much more precise result holds true. However its proof requires techniques that are not currently available to us. Consequently, we state what is called *the Great Picard Theorem*, but we defer its proof to [C, Thm. 4.2 Ch. XII].

Theorem 5.9. (The Great Picard Thm.) *Let f be a holomorphic function having an essential singularity at z_0 . Then, in each ngbh of z_0 (the point z_0 excluded), f assumes every complex value, with at most one exception, infinitely many times.*

We now want to study the behaviour of a holomorphic function at the “point at ∞ ”, that is, for $|z| \rightarrow +\infty$.

Let Ω be a domain such that for some $R > 0$, $\Omega \supseteq \{z : |z| > R\}$. Consider now the open set

$$\Omega' = \{z \in \mathbf{C} : 1/z \in \Omega\},$$

which is the image of $\Omega \setminus \{0\}$ through the mapping $z \mapsto \frac{1}{z}$. Then $\Omega' \supseteq \{z : 0 < |z| < 1/R\}$.

Definition 5.10. Let Ω, Ω' be as above. Let f be holomorphic in Ω and set $g(z) = f(1/z)$ for $z \in \Omega'$. We say that

- (i) f has a removable singularity at ∞ ,
- (ii) f has a pole at ∞ ,
- (iii) f has an essential singularity at ∞ ,

resp., if g has a removable singularity, a pole, or an essential singularity at $z = 0$, resp.

Proposition 5.11. *Let f be an entire function. Then $\lim_{|z| \rightarrow +\infty} |f(z)| = +\infty$ if and only if f has a pole at ∞ , if and only if f is a polynomial. The function f has a removable singularity at ∞ if and only if f is constant.*

Proof. Since f is entire, $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $z \in \mathbf{C}$. Hence,

$$g(z) \equiv f(1/z) = \sum_{n=0}^{+\infty} a_n z^{-n} = \sum_{n=0}^{-\infty} a_{-n} z^n,$$

for all $z \in \mathbf{C} \setminus \{0\}$. By the uniqueness of the Laurent expansion, the conclusions now follow. \square

5.1. The residue theorem. Recall Cauchy’s formula: If Ω is a domain, f holomorphic in Ω and γ is a curve homologous to 0 in Ω , then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega.$$

Now we consider the case in which f is holomorphic in Ω except at most in finitely many points z_1, \dots, z_n in which f has isolated singularities. Recall that f admits Laurent expansion at each of the z_j ’s:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n, \quad z \in \{0 < |z - z_j| < r_j\},$$

for suitable r_j , $j = 1, \dots, n$.

Definition 5.12. With the notation above, we call the coefficient a_{-1} the *residue* of f at z_j and we write

$$a_{-1} = \text{Res}_f(z_j).$$

Theorem 5.13. (The Residue Thm.) *Let Ω be a domain, f holomorphic in Ω except at finitely many points $z_1, \dots, z_n \in \Omega$. Let γ be a curve homologous to 0 in Ω , $z_j \notin \text{im}(\gamma)$ for $j = 1, \dots, n$. Then*

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i \sum_{j=1}^n n(\gamma, z_j) \text{Res}_f(z_j). \quad (5.2)$$

Proof. The proof relies on Lemma 4.24 (that we did not prove). By the lemma,

$$\int_{\gamma} f(\zeta) d\zeta = \sum_{j=1}^n n(\gamma, z_j) \int_{\gamma_j} f(\zeta) d\zeta.$$

Next, if we expand f in Laurent series about z_j we have

$$\begin{aligned} \int_{\gamma_j} \sum_{k=-\infty}^{+\infty} a_k (\zeta - z_j)^k d\zeta &= \sum_{k=-\infty}^{+\infty} a_k \int_{\gamma_j} (\zeta - z_j)^k d\zeta \\ &= 2\pi i a_{-1}. \end{aligned}$$

This proves the theorem. \square

Remark 5.14. If f has a pole of order m in $z = z_0$, then

$$\text{Res}_f(z_0) = \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right) (z_0). \quad (5.3)$$

For, if f has expansion $f(z) = \sum_{n=-m}^{+\infty} a_n (z - z_0)^n$, then

$$\begin{aligned} \left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right) (z_0) &= \left(\frac{d}{dz} \right)^{m-1} \left(\sum_{n=-m}^{+\infty} a_n (z - z_0)^{n+m} \right) (z_0) \\ &= \left(\frac{d}{dz} \right)^{m-1} \left(\sum_{k=0}^{+\infty} a_{k-m} (z - z_0)^k \right) (z_0) \\ &= (m-1)! a_{-1}, \end{aligned}$$

as we wished to show.

Example 5.15. Let

$$f(z) = \frac{z^2}{(z+1)(z-1)^2}.$$

Then $\text{Res}_f(1) = 3/4$.

This can be computed by expanding $\frac{z^2}{z+1}$ about $z = 1$, expansion that is

$$\frac{1}{2} + \frac{3}{4}(z-1) + \dots$$

Hence, by the previous Remark,

$$\text{Res}_f(1) = \frac{d}{dz} \left(\frac{z^2}{z+1} \right) (1) = \frac{3}{4}.$$

5.2. The Riemann sphere. We now wish to describe the behaviour of a function f that is holomorphic in a domain $\{|z| > R\}$, as $|z| \rightarrow \infty$. To this end we introduce the so-called *Riemann sphere*, that is, the one-point compactification of \mathbf{C} .

We set

$$\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$$

defining a neighborhood system for $\{\infty\}$ the sets

$$V_R = \{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}.$$

Definition 5.16. If we let S^2 to be the sphere in \mathbf{R}^3 , centered in $(0, 0, 1/2)$ and of radius $1/2$. We embed \mathbf{C} in \mathbf{R}^3 by identifying it with the x, y -plane $z = 0$. Consider the segment starting at the north pole $N = (0, 0, 1)$, intersecting the sphere at the point $s \in S^2$, $s \neq N$ and the plane $z = 0$ at a point z_s , and defining $z_N = \infty$.

We call *stereographic projection* the mapping

$$P : S^2 \ni s \mapsto s_z \in \mathbf{C}_\infty.$$

It is easy to the explicit formula for the stereographic projection:

$$z = P(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3},$$

and its inverse

$$P^{-1}(z) = \left(\frac{z + \bar{z}}{1 + |z|^2}, \frac{z - \bar{z}}{i(1 + |z|^2)}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Moreover, the *chordal* distance on the sphere induces a distance in \mathbf{C}_∞ (that is bounded, since \mathbf{C}_∞ is compact) given by, for $z, w \in \mathbf{C}$.

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}},$$

and

$$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

We leave the elementary, straightforward details to the reader.

We go back to the analysis of isolated singularities at ∞ .

Definition 5.17. Let f be holomorphic in the set $\{|z| > R\}$ for some $R > 0$. We say that f has a removable singularity, a pole or an essential singularity at ∞ , resp. if the function $g(z) = f(1/z)$ has a removable singularity, a pole or an essential singularity at $z = 0$, resp.

We define the residue of f at ∞ the residue of the function $h(z) := -\frac{1}{z^2}f(1/z)$ at $z = 0$:

$$\text{Res}_f(\infty) = \text{Res}_h(0).$$

5.3. Evaluation of definite integrals. We will make use of the residue theorem to evaluate definite real integrals.

Theorem 5.18. *Let $f : \mathbf{R} \rightarrow \mathbf{C}$ be a continuous function that is the restriction of a function, still denoted by f , meromorphic on the upper half plane*

$$\mathcal{U} = \{z = x + iy : y > 0\}$$

and having a finite number of poles in \mathcal{U} . Suppose there exist $C, \varepsilon, R > 0$ such that

$$|f(z)| \leq \frac{C}{|z|^{1+\varepsilon}} \quad \text{for } |z| \geq R.$$

Then

$$\int_{-\infty}^{+\infty} f(x) dx = 2\pi i \sum_{\substack{z_j \text{ pole for } f \\ z_j \in \mathcal{U}}} \text{Res}_f(z_j).$$

Proof. (See [L] Thm 2.1 page 192). \square

As an example we compute the following integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx.$$

The function f is the restriction to the real line of the meromorphic function $f(z) = 1/(1+z^4)$, that has simple poles in $z_{1,\dots,4} = \pm e^{i\pi/4}, \pm e^{i3\pi/4}$. The residues at the points with positive imaginary parts are easily computed as

$$\text{Res}_f(e^{i\pi/4}) = \lim_{z \rightarrow e^{i\pi/4}} (z - e^{i\pi/4})f(z) = \frac{1}{4}e^{-i3\pi/4}$$

and

$$\text{Res}_f(e^{i3\pi/4}) = \lim_{z \rightarrow e^{i3\pi/4}} (z - e^{i3\pi/4})f(z) = -\frac{1}{4}e^{-i5\pi/4}.$$

Therefore,

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}.$$

Theorem 5.19. *Let $R = R(x, y)$ be a rational function in two variables (x, y) , continuous on the unit circle $\{x^2 + y^2 = 1\}$. Let*

$$I = \int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta$$

Then,

$$I = 2\pi i \sum_{\substack{z_j \text{ pole for } f \\ z_j \in D(0,1)}} \text{Res}_f(z_j),$$

where

$$f(z) = \frac{R((z + 1/z)/2, (z - 1/z)/(2i))}{iz}.$$

Proof. This integral can be easily transformed into a complex line integral of a meromorphic function, as follows.

Notice that, for $z = e^{i\theta} \in \partial D(0, 1)$

$$\begin{aligned}\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}, \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}.\end{aligned}$$

Therefore, the integral I above equals

$$\begin{aligned}I &= \int_0^{2\pi} \frac{R(\cos \theta, \sin \theta)}{ie^{i\theta}} \cdot ie^{i\theta} d\theta \\ &= \int_{\partial D(0,1)} \frac{R((z + 1/z)/2, (z - 1/z)/2i)}{iz} dz \\ &= 2\pi i \sum_{\substack{z_j \text{ pole for } f \\ z_j \in D(0,1)}} \text{Res}_f(z_j),\end{aligned}$$

and $f(z)$ is as in the statement. \square

Example 5.20. As an example we compute

$$I = \int_0^{2\pi} \frac{1}{a + \cos \theta} d\theta$$

for $|a| > 1$.

By the reasoning above we have that

$$I = 2\pi i \sum_{z_j \in D(0,1)} \text{Res}_f(z_j)$$

where

$$\begin{aligned}f(z) &= \frac{1}{iz} \cdot \frac{1}{a + (z + 1/z)/2} = \frac{2}{i} \cdot \frac{1}{z^2 + 2az + 1} \\ &= \frac{2}{i} \cdot \frac{1}{(z + a + \sqrt{a^2 - 1})(z + a - \sqrt{a^2 - 1})}.\end{aligned}$$

Notice that f has simple poles at the points $z_{\pm} = -a \pm \sqrt{a^2 - 1}$, of which only $z_+ = -a + \sqrt{a^2 - 1} \in D(0, 1)$, and where f has residue

$$\text{Res}_f(z_+) = \frac{1}{i\sqrt{a^2 - 1}}.$$

Therefore,

$$I = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

A third class of definite integrals we present is given by the *Fourier transform*.

Theorem 5.21. *Let $R(z)$ be a rational function having finitely many poles, none of which on the real line and such that $|R(z)| \leq C/|z|$ as $|z| \rightarrow +\infty$, for some $C > 0$. Then*

$$\int_{-\infty}^{+\infty} R(x)e^{ix} dx = \sum_{\operatorname{Im} z_j > 0} \operatorname{Res}_{R(z)e^{iz}}(z_j).$$

Proof. We integrate over $\partial\mathcal{R}$, where \mathcal{R} is the rectangle in \mathbf{R}^2 of vertices $(b, 0)$, $(b, b+iy)$, $(a, a+iy)$ and $(a, 0)$, where $a, b, y \in \mathbf{R}$, and we let $a \rightarrow -\infty$ and $b \rightarrow +\infty$ with $y > 0$ large enough so that \mathcal{R} contains all the poles of R in the upper half-plane \mathcal{U} .

Then, it suffices to show that

$$\lim_{a \rightarrow -\infty, b, y \rightarrow +\infty} \int_{\partial\mathcal{R}} R(z)e^{iz} dz = \int_{-\infty}^{+\infty} R(x)e^{ix} dx,$$

that is, that the limit of the integrals over the two lateral sides and over the top side of the rectangle are all 0.

We begin by showing that the integrals over the two vertical sides can be made arbitrarily small, independently of y . Let σ denote the right vertical segment. Then

$$\begin{aligned} \left| \int_{\sigma} R(z)e^{iz} dz \right| &= \left| \int_0^y R(b+it)e^{ib-t} dt \right| \leq \int_0^y \frac{C}{|b+it|} e^{-t} dt \\ &\leq \frac{C}{b} \int_0^y e^{-t} dt \leq \frac{C}{b}, \end{aligned}$$

with C independent of y .

The same argument gives analogous bound for the other vertical side.

Next, let σ_y denote the segment from $z = b + iy$ to $z = a + iy$. We have

$$\begin{aligned} \left| \int_{\sigma_y} R(z)e^{iz} dz \right| &= \left| \int_a^b R(t+iy)e^{it-y} dt \right| \leq \int_a^b \frac{C}{|t+iy|} e^{-y} dt \\ &\leq \frac{Ce^{-y}(b-a)}{y}. \end{aligned}$$

Therefore, given $\varepsilon > 0$ we can select a, b, y so that

$$\begin{aligned} \left| \int_{\partial\mathcal{R}} R(z)e^{iz} dz - \int_{-\infty}^{+\infty} R(x)e^{ix} dx \right| &\leq \frac{C}{b} + \frac{C}{|a|} + \frac{Ce^{-y}(b-a)}{y} \\ &\leq \varepsilon. \end{aligned}$$

This gives the conclusion. \square

We have a similar result also in the case of $R(z)$ having poles on the real line, but only if the poles are *simple*.

Example 5.22. As an example in case of a simple pole on the real line, we compute

$$I = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx,$$

and obtain as a consequence that

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \quad (5.4)$$

The other cases can be treated in a similar fashion.

We write I as

$$I = \lim_{\varepsilon \rightarrow 0^+, R \rightarrow +\infty} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^R \right) \frac{e^{ix}}{x} dx.$$

Let γ_R be the closed curve defined by $\gamma_{R,\varepsilon} = (\varepsilon, R) \cup \sigma_R \cup (-R, -\varepsilon) \cup \{-\sigma_\varepsilon\}$, where (a, b) denotes the segment from a to b , and σ_R the upper semi-circle, centered at the origin, of radius $r > 0$ with counter-clockwise orientation. Let U be the *interior* of $\gamma_{R,\varepsilon}$ (see Thm. 4.22 for terminology).

Since e^{iz}/z has no singularity in U , for all $R, \varepsilon > 0$,

$$\begin{aligned} 0 &= \int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z} dz \\ &= \left(\int_{\sigma_R} + \int_{(-R, -\varepsilon)} - \int_{\sigma_\varepsilon} + \int_{(\varepsilon, R)} \right) \frac{e^{iz}}{z} dz. \end{aligned}$$

Now,⁵

$$\lim_{R \rightarrow +\infty} \int_{\sigma_R} \frac{e^{iz}}{z} dz = 0,$$

since

$$\begin{aligned} \left| \int_{\sigma_R} \frac{e^{iz}}{z} dz \right| &\leq \int_0^\pi \frac{|e^{iz}|}{R} R d\theta = \int_0^\pi e^{-R \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \\ &\leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R} (1 - e^{-R}) \end{aligned}$$

that tends to 0 as $R \rightarrow +\infty$. Indeed, the last inequality in the display above holds true since, if $\theta \in [0, \pi/2]$, $\sin \theta > \frac{2}{\pi}\theta$, so that $e^{-R \sin \theta} \leq e^{-R \frac{2}{\pi}\theta}$.

Next,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\sigma_\varepsilon} \frac{e^{iz}}{z} dz &= \lim_{\varepsilon \rightarrow 0} \int_0^\pi e^{i\varepsilon(\cos \theta + i \sin \theta)} i d\theta \\ &= i\pi. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0, R \rightarrow +\infty} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^R \right) \frac{e^{ix}}{x} dx = i\pi,$$

that is,

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx = i\pi.$$

⁵This equality is sometimes referred to as *Jordan's Lemma*.

As a consequence we obtain that,

$$\begin{aligned} \int_0^{+\infty} \frac{\sin x}{x} dx &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx \\ &= \frac{1}{2i} \lim_{\varepsilon \rightarrow 0, R \rightarrow +\infty} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^R \right) \frac{e^{ix}}{x} dx \\ &= \frac{\pi}{2}. \end{aligned}$$

This proves (5.4).

We conclude this part with another example that will be used in the proof of Jensen's formula, Thm. 8.16.

Example 5.23. For all $0 < r \leq 1$ we have

$$\int_0^{2\pi} \log |1 - re^{i\theta}| d\theta = 0.$$

The function $f(z) = \frac{\log(1-rz)}{iz}$ when $r < 1$ is holomorphic on $\overline{D(0,1)}$ and

$$0 = \operatorname{Re} \int_{\partial D} f(z) dz = \int_0^{2\pi} \log |1 - re^{i\theta}| d\theta.$$

If $r = 1$, f is holomorphic in a nbgh of $D(0,1) \setminus D(1,\varepsilon)$. Let γ_ε be the curve boundary of such set, so that

$$\int_{\gamma_\varepsilon} f(z) dz = 0$$

for all $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$ and observing that the integral over the portion of $\partial D(1,\varepsilon)$ of f tends to 0, we obtain the desired conclusion.

Example 5.24. We have

$$\int_0^\pi \log \sin \theta d\theta = -\pi \log 2. \quad (5.5)$$

As a consequence we also re-obtain that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0.$$

For,

$$\begin{aligned} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta &= \frac{1}{2} \int_0^{2\pi} \log |1 - e^{i\theta}|^2 d\theta = \frac{1}{2} \int_0^{2\pi} \log (2(1 - \cos \theta)) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \log (4 \sin^2(\theta/2)) d\theta = \pi \log 4 + \frac{1}{2} \int_0^{2\pi} \log (\sin^2(\theta/2)) d\theta \\ &= 2\pi \log 2 + 2 \int_0^\pi \log(\sin t) dt = 0, \end{aligned}$$

if (5.5) holds.

In order to prove (5.5), we consider the function

$$1 - e^{2iz} = -2ie^{iz} \sin z.$$

Since

$$1 - e^{2iz} = 1 - e^{-2y}(\cos 2x + i \sin 2x),$$

this function is real and negative only when $x = n\pi$ and $y \leq 0$.

Now we integrate over the curve given by the rectangle with vertices the $A = 0$, $B = \pi$, $C = \pi + iR$, $D = iR$, for $R > 0$ and letting $R \rightarrow +\infty$. Actually we need to avoid the origin and the point π . So, near the origin and π , we follow clockwise circles centered at 0 and π resp., of radius ε , for angles of $\pi/4$. We leave the details to the reader as an exercise (or see [A], p. 160.)

5.4. The argument principle and Rouché's theorem. Consider a function f meromorphic in a domain Ω . Let $z_0 \in \Omega$ be either a zero or a pole for f . We define the order of f at z_0 , $\text{ord}(f; z_0)$ to be the order of vanishing of f at z_0 if z_0 is a zero, and *minus* the order of the pole, if z_0 is a pole for f .

Theorem 5.25. (The argument principle) *Let Ω be a domain, f be a meromorphic function in Ω . Let γ be a curve in Ω homologous to 0 in Ω , and suppose no zero or pole of f lies on γ . Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{z_j} n(\gamma, z_j) \text{ord}(f; z_j),$$

where the sum ranges over the zeros and poles of f in Ω .

Proof. Notice that there can be only finitely many zeros and poles in the bounded connected components of $\Omega \setminus \gamma$. Hence, we may assume that f has finitely many zeros and poles in Ω .

Suppose z_0 is a zero of order k for f . Then we can write

$$f(z) = (z - z_0)^k f_1(z),$$

where f_1 is holomorphic in a ngbh of z_0 (and meromorphic in Ω) and $f_1(z_0) \neq 0$. Then

$$f'(z) = k(z - z_0)^{k-1} f_1(z) + (z - z_0)^k f_1'(z),$$

so that

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{f_1'(z)}{f_1(z)} = \frac{k}{z - z_0} + h_1(z),$$

where h_1 is a function holomorphic in a ngbh of z_0 . Therefore, f'/f has a simple pole in z_0 with residue equal to k , with $k = \text{ord}(f; z_0)$.

Suppose now z_0 is a pole of order $m > 0$ for f . Then

$$f(z) = \frac{1}{(z - z_0)^m} f_2(z),$$

where f_2 is holomorphic in a ngbh of z_0 and $f_2(z_0) \neq 0$. Then,

$$f'(z) = -m \frac{1}{(z - z_0)^{m+1}} f_2(z) + \frac{1}{(z - z_0)^m} f_2'(z),$$

so that

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_0} + \frac{f_2'(z)}{f_2(z)} = \frac{-m}{z - z_0} + h_2(z),$$

where h_2 is holomorphic in a nbgh of z_0 . Hence, f'/f has a simple pole in z_0 with residue equal to $-m$, with $-m = \text{ord}(f; z_0)$.

The conclusion now follows from the residue theorem. \square

A simple, but far-reaching result is Rouché's theorem.

Theorem 5.26. (Rouché's Thm.) *Let γ be a curve in Ω , homologous to 0 in Ω and such that $n(\gamma, z) = 0$ or $n(\gamma, z) = 1$ for any point z not on γ . Let f, g be holomorphic functions in Ω such that*

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$. Then f and g have the same number of zeros in the interior of γ , counting multiplicity.

Proof. Notice that, by assumption, neither f nor g can vanish on γ . Moreover,

$$\left| \frac{g(z)}{f(z)} - 1 \right| < 1$$

for $z \in \gamma$. Let $F(z) = g(z)/f(z)$ and consider $\sigma = F \circ \gamma$. Then σ is a curve, whose image is contained in the circle centered at $z = 1$ and of radius 1. Notice also that,

$$\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}.$$

Therefore,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\sigma} \frac{1}{z} dz = \frac{1}{2\pi i} \int_a^b \frac{1}{F(\gamma(t))} F'(\gamma(t)) \gamma'(t) dt \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{F'(z)}{F(z)} dz \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} dz. \end{aligned}$$

The conclusion now follows from the argument principle. \square

5.5. Consequences of Rouché's theorem. We now show that if f is holomorphic in a nbh of a point z_0 and $f'(z_0) \neq 0$, then f admits a local inverse— that is, we prove Thm. 4.14.

Proof of Thm. 4.14. Without loss of generality, we may assume that $z_0 = 0$, $f(0) = 0$ and $f'(0) = 1$.⁶

Then,

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n.$$

Hence,

$$|f(z) - z| = |z|^2 |h_1(z)|,$$

for some holomorphic function h_1 ; that is,

$$|f(z) - z| \leq c_1 |z|^2,$$

for $|z| \leq R$ (where $R > 0$ is selected so that f is holomorphic in Ω and $\overline{D(0, R)} \subseteq \Omega$).

Let $\alpha \in \mathbf{C}$, $|\alpha| < \delta/2$, with $\delta > 0$ to be selected later. Set

$$f_{\alpha}(z) = f(z) - \alpha, \quad g_{\alpha}(z) = z - \alpha,$$

⁶Exercise: Show that we can make this reduction.

so that

$$|f_\alpha(z) - g_\alpha(z)| = |f(z) - z| \leq c_1|z|^2,$$

for $|z| \leq R$.

We wish to obtain the inequality

$$c_1|z|^2 = c_1\delta^2 < |g_\alpha(z)| \quad (5.6)$$

when $|z| = \delta$. But, $|g_\alpha(z)| = |z - \alpha| \geq \delta - |\alpha| > \delta/2$, so that, in order to obtain (5.6) it suffices that $c_1\delta^2 < \frac{\delta}{2}$, i.e.

$$\delta < \frac{1}{2c_1}$$

that we can always choose.

Thus, for such a δ and $|z| = \delta$,

$$|f_\alpha(z) - g_\alpha(z)| < |g_\alpha(z)|,$$

and by Rouché's Thm., f_α and g_α have the same number of zeros in $|z| < \delta$. It follows that the equation $f(z) = \alpha$ has exactly one solution in $\{|z| < \delta\}$ for all α with $|\alpha| < \delta/2$. Let

$$U = \{z : |z| < \delta, \text{ and } |f(z)| < \delta/2\}.$$

Then U is open and we have shown that $f : U \rightarrow D(0, \delta/2)$ is a bijection. The same argument also proves that f is an open mapping in U . Then $\varphi = f^{-1}$ is continuous. Finally, we show that φ is also holomorphic. We have,

$$\lim_{w \rightarrow w_1} \frac{\varphi(w) - \varphi(w_1)}{w - w_1} = \lim_{z \rightarrow z_1} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)},$$

since $f'(z_1) \neq 0$, for z_1 in a ngbh of $z_0 = 0$. \square

Theorem 5.27. *Let $\Omega \subseteq \mathbf{C}$ be a domain, $\{f_n\}$ a sequence of holomorphic functions converging uniformly on compact subsets of Ω to $f \not\equiv 0$. Then, for each $z_0 \in \Omega$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exists n_ε such that for all $n \geq n_\varepsilon$, f_n and f have the same number of zeros in $D(z_0, \varepsilon)$.*

Proof. Let $z_0 \in \Omega$. Since, $f \not\equiv 0$, there exists $\varepsilon_0 > 0$ such that f vanishes in $\overline{D(z_0, \varepsilon_0)}$ at most in z_0 . For each $0 < \varepsilon \leq \varepsilon_0$, let

$$m_\varepsilon = \inf_{|z - z_0| = \varepsilon} |f(z)|,$$

so that $m_\varepsilon > 0$ for $0 < \varepsilon \leq \varepsilon_0$. By the uniform convergence on compact subsets of Ω , there exists n_ε such that for $n \geq n_\varepsilon$,

$$|f(z) - f_n(z)| < \frac{m_\varepsilon}{2} \quad \text{for } z \in \overline{D(z_0, \varepsilon)}.$$

Hence, $|f(z) - f_n(z)| < |f(z)|$ for $|z - z_0| = \varepsilon$ and $n \geq n_\varepsilon$ and the result follows from Rouché's Thm. \square

The following corollary, known as Hurwitz's Thm., follows immediately.

Corollary 5.28. (Hurwitz's Thm.) *Let $\Omega \subseteq \mathbf{C}$ be a domain, $\{f_n\}$ a sequence of holomorphic functions converging uniformly on compact subsets of Ω to f . Suppose that f_n is nowhere vanishing in Ω . Then, either f is nowhere vanishing in Ω , or $f \equiv 0$.*

Another version of Hurwitz's Thm. is the following result.

Corollary 5.29. (Hurwitz's Thm., second version) *Let $\Omega \subseteq \mathbf{C}$ be a domain, $\{f_n\}$ a sequence of univalent (i.e. injective) holomorphic functions converging uniformly on compact subsets of Ω to f . Then, either f is univalent in Ω , or f is constant.*

Proof. Suppose f is non-constant. We wish to show that f is univalent. Let $z_1, z_2 \in \Omega$, $z_1 \neq z_2$, and, seeking a contradiction, assume that $f(z_1) = f(z_2)$.

Consider the sequence $h_n(z) = f_n(z) - f_n(z_2)$. Then $h_n \rightarrow h = f - f(z_2)$ uniformly on compact subsets of Ω . Since f is non-constant, $h \not\equiv 0$. By Thm. 5.27, given z_1 , there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists n_ε such that for $n \geq n_\varepsilon$, h_n and h have the same number of zeros in $D(z_1, \varepsilon)$. Let $\varepsilon > 0$ small so that $z_2 \notin D(z_1, \varepsilon)$. But then $h = f - f(z_2)$ has a zero (namely $z = z_1$) in $D(z_1, \varepsilon)$, so must $h_n = f_n - f_n(z_2)$, $n \geq n_\varepsilon$. But this implies that $f_n(\tilde{z}_n) = f_n(z_2)$ for some $\tilde{z}_n \neq z_2$, $n \geq n_\varepsilon$. This contradiction proves the result. \square

5.6. Exercises.

5.1. (i) Determine the Laurent expansion about $z_0 = 0$ of $f(z) = \frac{1}{(z-1)(z-2)}$ in $\{|z| < 1\}$, $\{1 < |z| < 2\}$ and in $\{|z| > 2\}$.

(ii) Determine the Laurent expansion of $g(z) = \frac{z}{(z-1)(z-3)(z-5)}$ about $z_1 = 1$, about $z_2 = 3$ and about $z_3 = 5$.

5.2. Compute the residue for the given function at the assigned point:

(i) $f(z) = \frac{z^2}{(z-2i)(z+3)}$, $z_0 = 2i$;

(ii) $f(z) = \frac{z^2+1}{z(z+3)^2}$, $z_0 = -3$;

(iii) $f(z) = \frac{e^z}{(z-i-1)^3}$, $z_0 = 1+i$;

(iv) $f(z) = \frac{\sin z}{z^3(z-2)(z+1)}$, $z_0 = 0$

5.3. Evaluate the following integrals:

(i) $\int_{|z|=2} \frac{z}{(9-z^2)(z+i)} dz$;

(ii) $\int_{\gamma} \frac{\cos z}{z(z^2+8)} dz$,

where $\gamma = \partial Q$ and Q is the square center at the origin and side length 4, oriented counter-clockwise;

(iii) $\int_0^{2\pi} \frac{1}{1+a \cos \theta} d\theta$, where $0 \leq a < 1$; $[2\pi/\sqrt{1-a^2}]$

(iv) $\frac{1}{2} \int_0^{2\pi} \sin^{2n} \theta d\theta$. $[\pi 2^{-2n} \binom{2n}{n}]$

5.4. Evaluate the following integrals:

(i) $\int_0^{\pi/2} \frac{dx}{a + \sin^2 x}$, where $a > 0$; $[\pi/2\sqrt{a(a+1)}]$

- (ii) $\int_{-\infty}^{+\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$;
- (iii) $\int_0^{+\infty} \frac{x^2}{(x^2 + a^2)^2} dx$, where $a \in \mathbf{R}$;
- (iv) $\int_0^{+\infty} \frac{x \sin x}{x^2 + a^2} dx$, where $a \in \mathbf{R}$;
- (v) $\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2 + 1} dx$, where $a \in \mathbf{R}$;
- (vi) $\int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x + 1} dx$, where $0 < a < 1$. $[\pi / \sin a\pi]$

5.5. Show that the set of the one-to-one and onto holomorphic maps of \mathbf{C} to itself (a group under composition) is

$$\{\varphi : \varphi(z) = az + b, \text{ where } a \in \mathbf{C} \setminus \{0\}, b \in \mathbf{C}\}.$$

5.6. Prove the following stronger version of Rouché theorem. Let Ω be a domain, f, g holomorphic in Ω γ a closed simple curve homologous to 0 in Ω . Suppose that

$$|f(z) - g(z)| < |f(z)| + |g(z)|$$

for all $z \in \gamma$. Then f and g have the same number of zeros in the interior of γ . [Hint: You may find useful Cor. 4.26.]

6. CONFORMAL MAPPINGS

A somewhat surprising and far reaching consequence of the maximum modulus principle is the *so-called* Schwarz's Lemma.

Theorem 6.1. (Schwarz's Lemma) *Let $D = D(0, 1)$ be the unit disk, $f : D \rightarrow \mathbf{C}$ be a holomorphic function such that*

- (i) $|f(z)| \leq 1$;
- (ii) $f(0) = 0$.

Then,

$$|f(z)| \leq |z|$$

for all $z \in D$ and $|f'(0)| \leq 1$.

If one of the two inequalities above is an equality, that is, if either $|f(z)| = |z|$ for some $z \in D \setminus \{0\}$ or $|f'(0)| = 1$, then f is a rotation, i.e. there exists $\alpha \in \mathbf{C}$, $|\alpha| = 1$ such that

$$f(z) = \alpha z.$$

Proof. Let $g(z) = f(z)/z$. Then g has a removable singularity at 0 and setting $g(0) = f'(0)$ makes g into a holomorphic function on D . Consider $g|_{D(0, r)}$ for $r < 1$. Then, for $|z| = r$,

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}, \quad |z| = r.$$

By the maximum modulus principle, $|g(z)| \leq 1/r$ for all z with $|z| \leq r$. Hence, letting $r \rightarrow 1^-$, we obtain $|g(z)| \leq 1$ for all $|z| < 1$, i.e.

$$|f(z)| \leq |z|$$

for all $z \in D$. Moreover, also $|g(0)| \leq 1$, that is, $|f'(0)| \leq 1$.

Next, suppose that $|f(z)| = |z|$ for some $z \neq 0$. Then, $|g(z)| = 1$ for some $z \in D$. By the maximum modulus principle, g is constant, $g(z) = \alpha$, with $|\alpha| = 1$. Thus, $f(z) = \alpha z$ for all $z \in D$.

Finally, if $|f'(0)| = 1$, then $|g(0)| = 1$, again implying that $g(z) = \alpha$ and $f(z) = \alpha z$ for all $z \in D$, with $|\alpha| = 1$. \square

Definition 6.2. Let Ω_1, Ω_2 be two domains in \mathbf{C} . We call a *biholomorphic* map between Ω_1 and Ω_2 a holomorphic map

$$f : \Omega_1 \rightarrow \Omega_2$$

such that f is a bijection between Ω_1 and Ω_2 (so that f^{-1} is also holomorphic). We say that Ω_1 and Ω_2 are biholomorphically equivalent.

If $\Omega_1 = \Omega_2 = \Omega$, a biholomorphic map between Ω and itself, f is called an *automorphism* of Ω . The set of all biholomorphic maps of a domain Ω onto itself is called the *automorphism group of Ω* and it will be denoted by $\text{Aut}(\Omega)$.

Notice that it is clear that $\text{Aut}(\Omega)$ is a group under the composition of functions.

Next we describe the automorphism group of the unit disk $D \equiv D(0, 1)$.

Theorem 6.3. For $a \in D$ and define

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Then φ_a is a biholomorphic map of D onto itself, i.e. $\varphi_a \in \text{Aut}(D)$, $\varphi_a^{-1} = \varphi_{-a}$, $\varphi_a(a) = 0$ and $\varphi_a(0) = -a$.

Finally,

$$\text{Aut}(D) = \{\varphi : \varphi(z) = e^{i\theta} \varphi_a(z), \theta \in \mathbf{R}\}.$$

Proof. Recall that (see Ex. 1.4) $\varphi_a : D \rightarrow D$ for $a \in \mathbf{C}$, $|a| < 1$ (and moreover, that $\varphi_a : \partial D \rightarrow \partial D$).

It is easy to see that also $\varphi_{-a} : D \rightarrow D$ and that $\varphi_{-a} = \varphi_a^{-1}$. This shows that $\varphi_a \in \text{Aut}(D)$ for all $a \in D$. Notice that $\varphi_a(a) = 0$ and $\varphi_a(0) = -a$.

Next, let $f \in \text{Aut}(D)$. Suppose first that $f(0) = 0$. Let $g = f^{-1}$. Then, Schwarz's Lemma applied to both f and g gives

$$|f'(0)| \leq 1, \quad \text{and} \quad |g'(0)| \leq 1.$$

But $f'(g(z))g'(z) = 1$, and since $g(0) = f^{-1}(0) = 0$, $f'(0)g'(0) = 1$. These facts imply that $|f'(0)| = |g'(0)| = 1$, so that $f(z) = \alpha z$, $|\alpha| = 1$, by Schwarz's Lemma again.

Suppose now that $f(0) = b$ and let $\varphi = \varphi_b \circ f$. Since $f, \varphi_b \in \text{Aut}(D)$, also $\varphi \in \text{Aut}(D)$. Moreover,

$$\varphi(0) = \varphi_b \circ f(0) = \varphi_b(b) = 0.$$

Thus, $\varphi_b \circ f(z) = \alpha z$, with $|\alpha| = 1$, by the previous argument. Since $\varphi_b^{-1} = \varphi_{-b}$ we have

$$\begin{aligned} f(z) &= \varphi_{-b}(\alpha z) = \frac{\alpha z + b}{1 + \bar{b}\alpha z} = \alpha \frac{z + b\bar{\alpha}}{1 + \bar{\alpha}bz} \\ &= \alpha \varphi_{-\bar{\alpha}b}(z). \quad \square \end{aligned}$$

We remark that from Exercise 5.5 it follows that

$$\text{Aut}(\mathbf{C}) = \{\varphi : \varphi(z) = az + b, a, b \in \mathbf{C}, a \neq 0\}. \quad (6.1)$$

A generalization of Schwarz Lemma is the following result.

Theorem 6.4. (Schwarz-Pick Lemma) Let $f : D \rightarrow D$ be holomorphic. Then for all $z \in D$ we have

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Proof. Let $a, b \in D$ be such that $f(a) = b$. Consider $g = \varphi_b \circ f \circ \varphi_{-a}$. Then, $g : D \rightarrow D$ and $g(0) = 0$. By Schwarz's Lemma, $|g'(0)| \leq 1$, i.e.

$$\left| \varphi'_b(f(\varphi_{-a}(0))) \cdot f'(\varphi_{-a}(0)) \cdot \varphi'_{-a}(0) \right| \leq 1.$$

Recalling that $\varphi'_c(z) = \frac{1 - |c|^2}{(1 - \bar{c}z)^2}$, we obtain

$$\left| \frac{1 - |b|^2}{(1 - \bar{b}f(a))^2} \cdot f'(a)(1 - |a|^2) \right| \leq 1,$$

so that,

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

Since a was arbitrary, the statement follows. \square

We now consider mappings between different domains in the complex plane. Recall that we denote by \mathcal{U} the upper half plane $\{z = x + iy : y > 0\}$.

Proposition 6.5. *Let*

$$f(z) = i \frac{1+z}{1-z}, \quad z \in D.$$

Then $f : D \rightarrow \mathcal{U}$ is a biholomorphic map.

Proof. This is a simple computation. For $z \in D$ we have

$$f(z) = i \frac{(1+z)(1-\bar{z})}{|1-z|^2} = i \frac{1-|z|^2 + 2i\operatorname{Im} z}{|1-z|^2},$$

so that $\operatorname{Im} f(z) = \frac{1-|z|^2}{|1-z|^2} > 0$, for $z \in D$.

We now show that f is invertible, by explicitly computing its inverse. Setting $w = f(z)$, then $w = i \frac{1+z}{1-z}$, so that $z = \frac{w-i}{w+i} \equiv g(w)$. Notice that $\operatorname{Im} w > 0$, then $|w-i| < |w+i|$, so that $z \in D$. Since $z = g(f(z))$, $z \in D$ and $w = f(g(w))$, $w \in \mathcal{U}$, it follows that f is one-to-one and also onto from D to \mathcal{U} . \square

We now show other examples of biholomorphic mappings.

Proposition 6.6. *The following are examples of biholomorphic mappings between the indicated domains:*

- (i) $z \mapsto z^2$ maps the first quadrant to \mathcal{U} , and the first quadrant intersect the unit disk onto \mathcal{U} intersect the unit disk.
- (ii) $z \mapsto z^2$ maps the first quadrant intersect the unit disk onto \mathcal{U} intersect the unit disk.
- (iii) $z \mapsto \log z$ maps \mathcal{U} intersect the unit disk onto the “half-strip” $\{\operatorname{Im} w \in (0, \pi)\} \cap \{\operatorname{Re} z < 0\}$.
- (iv) $z \mapsto \log z$ maps \mathcal{U} onto the strip $\{\operatorname{Im} w \in (0, \pi)\}$.
- (v) $z \mapsto \sin z$ maps the half-strip $\{-\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0\}$ onto \mathcal{U} .

Proof. (i)-(iv) are fairly simple.

Let $z = \rho e^{i\theta}$ and $w = z^2 = \rho^2 e^{i2\theta}$. If $\rho \in (0, +\infty)$ and $\theta \in (0, \pi/2)$, then $|w| \in (0, +\infty)$ and $\arg w \in (0, \pi)$. This shows (i). If $\rho \in (0, 1)$ and $\theta \in (0, \pi/2)$, then $|w| \in (0, 1)$ and $\arg w \in (0, \pi)$. This shows (ii).

Now let $z = \rho e^{i\theta}$ and $w = \log z = \log \rho + i\theta$. Then, $\rho \in (0, 1)$ if and only if $\operatorname{Re} w < 0$ and $\rho \in (0, +\infty)$ if and only if $\operatorname{Re} w \in \mathbf{R}$. Then (iii) and (iv) follow easily.

For (v), notice that, for $z = x + iy$,

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\ &= \sin x \cosh y + i \cos x \sinh y \\ &\equiv u + iv, \end{aligned}$$

as it is easy to check. Notice that $w = u + iv$ satisfies

$$\begin{cases} \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \\ \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1. \end{cases}$$

As a consequence, if we fix $y > 0$, then the mapping $z \mapsto \sin z$ maps the segment $\{z = x + iy, -\pi/2 < x < \pi/2\}$ onto the upper half of the ellipse of equation $\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$ (notice indeed that $v = \cos x \sinh y > 0$).

Moreover, if we fix $x \in (-\pi/2, \pi/2)$, the mapping $z \mapsto \sin z$ maps the half line $\{x + iy, x \neq 0, y > 0\}$ into points of the hyperbola $\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$, with $u = \sin x \cosh y > 0$ if $x > 0$ and $u = \sin x \cosh y < 0$ if $x < 0$.

Now it is easy to see that the mapping $z \mapsto \sin z$ is actually onto \mathcal{U} . \square

6.1. Fractional linear transformations. A fractional linear transformations is a map defined by

$$F(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbf{C}, \quad ad - bc \neq 0.$$

Notice that if $c = 0$ then $ad \neq 0$, so that F reduces to $F(z) = \frac{a}{d}z + \frac{b}{d}$, with $\frac{a}{d} \in \mathbf{C} \setminus \{0\}$. Therefore, $F \in \text{Aut}(\mathbf{C})$.

If $c \neq 0$ then

$$F : \mathbf{C} \setminus \{-d/c\} \rightarrow \mathbf{C} \setminus \{a/c\}.$$

Notice that

$$F'(z) = \frac{ad - bc}{(cz + d)^2}$$

so that F is constant if and only if $ad - bc = 0$. In this case F is also invertible its inverse $F^{-1} = G$ is also a fractional linear transformation given by

$$G(w) = \frac{dw - b}{-cw + a},$$

as it is immediate to check.

Recall that in Subsection we have defined the Riemann sphere, that is, the one-point compactification \mathbf{C}_∞ of the complex plane \mathbf{C} .

Given a fractional linear transformation F , we extend it to the Riemann sphere. We set

$$F(-d/c) = \infty, \quad F(\infty) = a/c,$$

if $c \neq 0$, and $F(\infty) = \infty$ if $c = 0$. In this way, F becomes a bijection of the Riemann sphere S^2 onto itself.

Proposition 6.7. *Every fractional linear transformation can be written as composition of the following transformations:*

- translations $T_b : z \mapsto z + b$, where $b \in \mathbf{C}$;
- inversion $J : z \mapsto 1/z$;
- dilations $D_a : z \mapsto az$, where $a \in \mathbf{C} \setminus \{0\}$.

Proof. This is elementary to check. Let

$$F(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbf{C}, \quad ad - bc \neq 0.$$

If $c = 0$ this is obvious.

If $c \neq 0$, then $ad \neq 0$ and

$$\begin{aligned} F(z) &= \frac{a}{c} \cdot \frac{z + \frac{b}{a}}{z + \frac{d}{c}} = \frac{a}{c} \left(\frac{\frac{b}{a} - \frac{d}{c}}{z + \frac{d}{c}} + 1 \right) \\ &= \frac{\alpha}{z + \delta} + \gamma \\ &= T_\gamma \circ D_\alpha \circ J \circ T_\delta(z). \quad \square \end{aligned}$$

Proposition 6.8. *A fractional linear transformation maps straight lines and circles into straight lines and circles.*

Proof. Recall that straight lines have complex equation

$$\bar{\alpha}z + \alpha\bar{z} + c = 0, \quad \alpha \in \mathbf{C} \setminus \{0\}, \quad c \in \mathbf{R}, \quad (6.2)$$

while circles have equation $|z - z_0|^2 = r^2$, that is, $|z|^2 - \bar{z}_0z - z_0\bar{z} + |z_0|^2 - r^2 = 0$, i.e.

$$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + C = 0, \quad |\alpha|^2 > C \in \mathbf{R}. \quad (6.3)$$

In order to prove the assertion it suffices to show that each of the transformation T_a, D_a, J as in Prop. 6.7 preserves straight lines and circles.

Translations obviously do. Let D_a be a dilation, then it clearly preserves straight lines, while the generic circle (6.3) is mapped onto the set $|a|^2 z\bar{z} + a\bar{\alpha}z + \bar{a}\alpha\bar{z} + C = 0$, that is,

$$z\bar{z} + \overline{\left(\frac{\alpha}{a}\right)}z + \frac{\alpha}{a}\bar{z} + \frac{C}{|a|^2} = 0,$$

where $|\alpha/a|^2 > C/|a|^2$.

It only remains to check for the inversion. The straight line (6.2) is mapped onto $\frac{\bar{\alpha}}{z} + \frac{\alpha}{\bar{z}} + c = 0$, that is,

$$cz\bar{z} + \alpha z + \bar{\alpha}\bar{z} = 0$$

which a straight line if $c = 0$ or a circle, since $|\alpha|^2 > 0$, if $c \neq 0$.

Finally, the circle (6.3) is mapped onto

$$\frac{1}{z\bar{z}} + \frac{\bar{\alpha}}{z} + \frac{\alpha}{\bar{z}} + C = 0,$$

that is,

$$Cz\bar{z} + \alpha z + \bar{\alpha}\bar{z} + 1 = 0$$

which a straight line if $C = 0$ and a circle if $C \neq 0$, since $|\alpha/C|^2 > 1/C$. \square

Theorem 6.9. *Given 3 distinct points z_1, z_2, z_3 on the Riemann sphere S^2 and other 3 distinct points w_1, w_2, w_3 on the Riemann sphere S^2 , there exists a unique fractional linear transformation F such that $F(z_j) = w_j$, $j = 1, 2, 3$.*

Proof. We begin with a few preliminary steps. We first show that if F has 3 fixed points, then F is the identity, that is, $F(z) = z$ for all $z \in \mathbf{C}$.

If ∞ is a fixed point for F , then $F \in \text{Aut}(\mathbf{C})$ and $F(z) = az + b$, $a \neq 0$. If z_1, z_2 are the other 2 distinct fixed points, they are solution of the system

$$\begin{cases} az_1 + b = z_1 \\ az_2 + b = z_2, \end{cases}$$

that implies $a = 1$ and $b = 0$.

If z_1, z_2, z_3 are generic, distinct, fixed points for F , consider a fractional linear transformation L mapping, say, z_3 to ∞ , e.g. $L(z) = 1/(z - z_3)$. Then

$$G = L \circ F \circ L^{-1}$$

has the 3 fixed points $L(z_1), L(z_2), L(z_3)$, that is, $L(z_1), L(z_2), \infty$. By the previous argument G is the identity; hence F is the identity.

Now we show the statement about uniqueness. Suppose there exist two fractional linear transformations F and G such that $F(z_j) = G(z_j) = w_j$, $j = 1, 2, 3$. Then $F^{-1} \circ G$ has the 3 distinct fixed points z_1, z_2, z_3 , hence it is the identity, hence $F = G$.

Finally we show that there exists an F such that $F(z_j) = w_j$, $j = 1, 2, 3$. It suffices to construct a Φ such that

$$\Phi(z_1) = 0, \quad \Phi(z_2) = \infty, \quad \Phi(z_3) = 1,$$

and Ψ such that

$$\Psi(w_1) = 0, \quad \Psi(w_2) = \infty, \quad \Psi(w_3) = 1.$$

Once this is done we can define $F = \Psi^{-1} \circ \Phi$.

Therefore, we define

$$\Phi(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1},$$

and analogously for Ψ . \square

6.2. The Riemann mapping theorem. Goal of this part is to prove the Riemann mapping theorem that asserts that any simply connected domain $\Omega \neq \mathbf{C}$ is biholomorphically equivalent to the unit disk D , that is, there exists $f : \Omega \rightarrow D$ holomorphic, one-to-one and onto.

In order to prove this result we will need the notion of *normal families*.

We begin with the space of continuous functions on Ω , $C(\Omega)$. Let $U_1 \subseteq U_2 \subseteq \dots$ be a sequence of bounded open sets in Ω such that

- (i) $\overline{U_j} \subseteq U_{j+1}$, $j = 1, 2, \dots$ (here the “bar” denotes the topological closure);
- (ii) $\cup_{j=1}^{+\infty} U_j = \Omega$.

For each j we set

$$\|g\|_{\overline{U_j}} = \sup_{z \in \overline{U_j}} |g(z)|,$$

and define

$$d(f, g) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \cdot \frac{\|f - g\|_{\overline{U_j}}}{1 + \|f - g\|_{\overline{U_j}}}, \quad f, g \in C(\Omega).$$

Theorem 6.10. *The function d is a metric on $C(\Omega)$ and $C(\Omega)$ is complete in this metric. The convergence in such a metric is the uniform convergence on compact subsets of Ω .*

Proof. See Exercise 6.10. \square

Definition 6.11. A subset \mathcal{F} of $C(\Omega)$ (often called a *family*) is said to be *normal* if it is precompact in $C(\Omega)$ (that is, if its closure is compact).

Thus, since $C(\Omega)$ is a metric space, a family \mathcal{F} is normal if and only if every sequence $\{f_n\} \subseteq \mathcal{F}$ has a convergent subsequence $\{f_{n_k}\}$, where the convergence is in $C(\Omega)$, that is, uniformly on compact subsets of Ω .

Definition 6.12. A subset \mathcal{F} of $C(\Omega)$ is called *equicontinuous* on a set $K \subseteq \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $z_1, z_2 \in K$ and $|z_1 - z_2| < \delta$, then

$$|f(z_1) - f(z_2)| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

We now recall the celebrated theorem of Ascoli-Arzelà, for whose proof we refer to [A].

Theorem 6.13. (Ascoli-Arzelà) *Let \mathcal{F} be a family in $C(\Omega)$. Then \mathcal{F} is normal if and only if the following conditions hold:*

- (i) \mathcal{F} is equicontinuous on compact subsets of Ω ;
- (ii) for all $z \in \Omega$, the set $\{f(z) : f \in \mathcal{F}\}$ has compact closure in \mathbb{C} .

Definition 6.14. Let Ω be a domain and \mathcal{F} a family of functions in $C(\Omega)$. We say that \mathcal{F} is *locally bounded* (or *equibounded*) if for every $z_0 \in \Omega$ and $r_0 > 0$ such that $\overline{D(z_0, r_0)} \subseteq \Omega$ there exists $M > 0$ such that

$$\sup_{z \in \overline{D(z_0, r_0)}} |f(z)| \leq M$$

for all $f \in \mathcal{F}$.

Theorem 6.15. (Montel's theorem) *A family $\mathcal{F} \subseteq H(\Omega)$ is normal if and only if it is equibounded.*

Proof. Suppose \mathcal{F} is normal. Seeking a contradiction, suppose that \mathcal{F} is not equibounded. Then, there exists a compact $K \subseteq \Omega$ such that

$$\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = +\infty.$$

Then, there exists a sequence $\{f_n\} \subseteq \mathcal{F}$ such that

$$\|f_n\|_K = \sup\{|f_n(z)| : z \in K\} \geq n.$$

Since \mathcal{F} is normal, there exists a subsequence $\{f_{n_k}\}$ converging on compact subsets to f . In particular

$$\|f_{n_k} - f\|_K \rightarrow 0,$$

But this contradicts the condition $\|f_{n_k}\|_K \rightarrow +\infty$.

Conversely, suppose that \mathcal{F} is equibounded. Clearly, we use Ascoli-Arzelà theorem to prove that it is normal. Condition (ii) in Thm. 6.13 is obviously satisfied, so we only need to show that \mathcal{F} is equicontinuous on compact subsets of Ω . It suffices to verify the condition on the closed disks of the form $\overline{D(z_0, r_0/2)}$, where $\overline{D(z_0, r_0)} \subseteq \Omega$.⁷

Let $z_0 \in \Omega$, $r_0 > 0$ be such that $\overline{D(z_0, r_0)} \subseteq \Omega$. Let $M > 0$ be such that

$$\sup\{|f(z)| : z \in \overline{D(z_0, r_0)}, f \in \mathcal{F}\} \leq M.$$

⁷You should check this, see Exercise 6.12

Then, for $z, w \in \overline{D(z_0, r_0/2)}$, letting $\gamma = \partial D(z_0, r_0)$ we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - w} d\zeta \right| \\ &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{(z - w)f(\zeta)}{(\zeta - z)(\zeta - w)} d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + r_0 e^{i\theta})| \cdot |z - w|}{r_0^2/4} r_0 d\theta \\ &\leq \frac{4M}{r_0} \cdot |z - w|, \end{aligned}$$

which implies that \mathcal{F} is equicontinuous. \square

We are now ready for the main result of this part.

Theorem 6.16. (Riemann mapping theorem) *Let Ω be a simply connected domain, $\Omega \neq \mathbf{C}$. Then Ω is conformally equivalent to the unit disk D . More precisely, given $z_0 \in \Omega$ there exists a unique $f : \Omega \rightarrow D$ such that*

- (i) $f : \Omega \rightarrow D$ is biholomorphic;
- (ii) $f(z_0) = 0$;
- (iii) $f'(z_0) > 0$.

Proof. We begin by proving the uniqueness. Suppose there exist two mappings f and g with the required properties. Then

$$f \circ g^{-1} : D \rightarrow D$$

is an automorphism. Moreover, $f \circ g^{-1}(0) = 0$. Hence, $f \circ g^{-1}$ is a rotation, say $f \circ g^{-1}(z) = e^{i\theta} z$, for some fixed $\theta \in \mathbf{R}$ and all $z \in D$. Hence, $f(w) = e^{i\theta} g(w)$ for all $w \in \Omega$ and therefore

$$f'(z_0) = e^{i\theta} g'(z_0).$$

Since both $f'(z_0), g'(z_0) > 0$, it must be $e^{i\theta} = 1$, i.e. $f = g$.

Next we construct such a map f . We define

$$\mathcal{F} = \{f : \Omega \rightarrow D, \text{ holomorphic, univalent, } f(z_0) = 0, f'(z_0) > 0\}, \quad (6.4)$$

and set

$$\lambda = \sup\{f'(z_0) : f \in \mathcal{F}\}. \quad (6.5)$$

Then we are going to show:

- (1) $\mathcal{F} \neq \emptyset$;
- (2) \mathcal{F} is normal and that λ is actually the maximum, i.e. there exists $f \in \mathcal{F}$ such that $f'(z_0) = \lambda$;
- (3) the function f found in (2) is also onto, thus proving the theorem.

We begin the construction.

Let $\alpha \in \mathbf{C} \setminus \Omega$. Since Ω is simply connected, by Prop. 4.26, the function

$$h(z) = \sqrt{z - \alpha}$$

is well defined and holomorphic on Ω . Moreover, if $h(z_1) = \pm h(z_2)$ it follows that $z_1 - \alpha = z_2 - \alpha$, i.e. $z_1 = z_2$. Therefore, h is univalent, but actually much more!

If the image of Ω through h contains the disk $D_1 = D(h(z_0), r)$, that is, $h(\Omega) \supseteq D_1$, then it follows that

$$h(\Omega) \cap D(-h(z_0), r) = \emptyset. \quad (6.6)$$

Indeed, if $z_1 \in \Omega$ and $h(z_1) \in D(-h(z_0), r)$, then

$$|(-h(z_1)) - h(z_0)| = |h(z_1) - (-h(z_0))| < r,$$

that is, $-h(z_1) \in D_1$. Since $D_1 \subseteq h(\Omega)$, there exists $\zeta \in \Omega$ such that $h(\zeta) = -h(z_1)$, which implies $\zeta = z_1$, so that $h(z_1) = -h(z_1)$; hence $h(z_1) = 0$. This in turn implies $z_1 = \alpha \in \Omega$, against the assumption.

This contradiction shows that (6.6) holds. In particular

$$|h(z) + h(z_0)| \geq r$$

for all $z \in \Omega$.

The function h is univalent, defined on Ω , but does not take values in D yet. Then we set

$$h_1(z) = \frac{r}{2(h(z) + h(z_0))}, \quad z \in \Omega. \quad (6.7)$$

Then h_1 is univalent, defined on Ω , taking values in D . We can find an automorphism of D , $e^{i\theta}\varphi_a$, such that

$$f = e^{i\theta}\varphi_a \circ h_1 : \Omega \rightarrow D$$

and $f(z_0) = 0$, $f'(z_0) > 0$. For, it suffices to take $a = h_1(z_0) = r/4h(z_0)$ and θ such that

$$f'(z_0) = e^{i\theta}\varphi'_a(h_1(z_0))h'_1(z_0) > 0.$$

Thus, $f \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$, and (1) is proved.

Next, let λ be given by (6.5). Since \mathcal{F} is bounded, it is normal. Let $\{f_n\}$ be such that $f'_n(z_0) \rightarrow \lambda$. Then there exists a subsequence $\{f_{n_k}\}$ converging to a limit function f . Such function is holomorphic and non-constant, thus univalent by Hurwitz's theorem. Moreover, $f(z_0) = 0$ and $f'(z_0) = \lambda$. Then, (2) is also proved.

Finally, we show that the function f constructed above is actually onto D . This will complete the proof of the Riemann mapping theorem.

Seeking a contradiction, suppose there exists $\beta \in D$ such that $\beta \notin f(\Omega)$, i.e. $f(z) \neq \beta$ for all $z \in \Omega$. We are going to show that there exists $G \in \mathcal{F}$ such that $G'(z_0) > f'(z_0)$, thus finding a contradiction.

We set

$$F(z) = \sqrt{\frac{f(z) - \beta}{1 - \bar{\beta}f(z)}} = \sqrt{\varphi_\beta \circ f(z)}, \quad (6.8)$$

and the set

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \cdot \varphi_{F(z_0)} \circ F(z). \quad (6.9)$$

We will show that this function G provides the contradiction.

The function F is well defined and univalent on Ω , $F : \Omega \rightarrow D$, and

$$F'(z) = \frac{1}{2F(z)} \cdot \varphi'_\beta(f(z)) \cdot f'(z).$$

Hence,

$$F'(z_0) = \frac{1}{2F(z_0)} \cdot \varphi'_\beta(f(z_0)) \cdot f'(z_0) \neq 0. \quad (6.10)$$

Since,

$$\begin{aligned} G(z) &= \frac{|F'(z_0)|}{F'(z_0)} \cdot \varphi_{F(z_0)} \circ F(z) \\ &= \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}, \end{aligned}$$

notice that $G : \Omega \rightarrow D$, $G(z_0) = 0$, and it is univalent, since $\varphi_{F(z_0)}$ and F are.

Finally we show that $G'(z_0) > f'(z_0)$, thus obtaining a contradiction.

We have

$$\begin{aligned} G'(z) &= \frac{|F'(z_0)|}{F'(z_0)} \cdot \varphi'_\beta(F(z)) \cdot F'(z) \\ &= \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{1 - |F(z_0)|^2}{(1 - \overline{F(z_0)}F(z))^2} \cdot F'(z). \end{aligned}$$

Hence,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2}, \quad (6.11)$$

where

$$|F(z_0)|^2 = \left| \frac{f(z_0) - \beta}{1 - \overline{f(z_0)}\beta} \right| = |\beta|,$$

and

$$\begin{aligned} F'(z) &= \frac{1}{2F(z)} \cdot \varphi'_\beta(f(z)) \cdot f'(z) \\ &= \frac{1}{2F(z)} \cdot \frac{f'(z)(1 - |\beta|^2)}{(1 - \bar{\beta}f(z))^2}, \end{aligned}$$

so that

$$\begin{aligned} |F'(z_0)| &= \left| \frac{1}{2F(z_0)} \cdot \frac{f'(z_0)(1 - |\beta|^2)}{(1 - \bar{\beta}f(z_0))^2} \right| \\ &= f'(z_0) \cdot \frac{1 - |\beta|^2}{2\sqrt{|\beta|}}. \end{aligned}$$

Therefore, substituting into (6.11) we obtain

$$\begin{aligned} G'(z_0) &= f'(z_0) \cdot \frac{1 - |\beta|^2}{2\sqrt{|\beta|}} \cdot \frac{1}{1 - |\beta|} \\ &= f'(z_0) \cdot \frac{1 + |\beta|}{2\sqrt{|\beta|}} > f'(z_0), \end{aligned}$$

since $\frac{1+|\beta|}{2\sqrt{|\beta|}} > 1$, which is equivalent to $(1 - \sqrt{|\beta|})^2 > 0$, that is satisfied since $|\beta| < 1$.

This contradiction concludes the proof of the theorem. \square

6.3. Exercises. ⁸

6.1. Show that if f is a holomorphic function on $\mathcal{U} = \{z = x + iy : y > 0\}$ and if $\operatorname{Im} f(z) \geq 0$ then

- (i) $\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$;
- (ii) $\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y}$, where $z = x + iy$.

6.2. Show that if $f : \Omega \rightarrow \mathcal{D}$ is holomorphic, one-to-one and onto, Ω where \mathcal{D} are domains in \mathbf{C} . Show that f^{-1} is holomorphic on \mathcal{D} .

6.3. Let $f : D \rightarrow D$ be holomorphic. Let $a, b \in D$, $a \neq b$, and suppose that $f(a) = a$ and $f(b) = b$. Show that $f(z) = z$ for all $z \in D$.

6.4. Find a biholomorphic between the given domains Ω_1, Ω_2 .

- (i) $\Omega_1 = \mathcal{U} \cap D$ onto the first quadrant;
- (ii) $\Omega_1 = D \cap \{\text{the first quadrant}\}$ onto \mathcal{U} ;
- (iii) $\Omega_1 = D \setminus \{[a, 1)\}$ onto $\Omega_2 = D \setminus \{[0, 1)\}$, where $a > 0$.

6.5. Let $f(z) = (z + 1)/(z - 1)$. Find the image of the real line $\operatorname{Re} z = c$, $c \in \mathbf{R}$. If the image is a circle, find the center and the radius.

6.6. Answer the following questions.

- (i) Show that the transformation $w = iz + i$ maps the right half-plane $\{z = x + iy : x > 0\}$ onto the half-plane $\{w = u + iv : v > 1\}$.
- (ii) Find the image of the half-plane $y > 1$ under the transformation $w = (1 - i)z$.
- (iii) Find the fractional linear transformation that maps the points $\{2, i, -2\}$ onto the points $\{1, i, -1\}$, resp.

6.7. (i) Find all the biholomorphic mappings of the upper half-plane onto the unit disk. Do they map the boundary onto the boundary?

(ii) Describe the group $\operatorname{Aut}(\mathcal{U})$.

6.8. Find the fractional linear transformation

- (i) that maps the points $\{1, i, -1\}$ onto the points $\{0, 1, \infty\}$, resp.;
- (ii) that maps the points $\{0, 1, \infty\}$ onto the points $\{1, \infty, 0\}$, resp.

6.9. Let $f(z) = (z - i)/(z + i)$. Find the image of

⁸Unless otherwise specified, D denotes the unit disk, \mathcal{U} the upper half plane, \mathcal{R} the right half plane.

- (i) $\{it : t \geq 0\}$;
- (ii) $\{|z - 1| = 1\}$;
- (iii) $\{i + t : t \in \mathbf{R}\}$;
- (iv) $\{|z| = 2, \operatorname{Im} z \geq 0\}$;
- (v) $\{\operatorname{Re} z = 1, \operatorname{Im} z \geq 0\}$.

6.10. Prove Thm. 6.10 (and observe that $H(\Omega)$ is a closed subspace).

6.11. Show that if $f \in H(D)$, $f(z) = \sum_k a_k z^k$, where the power series converges in D , and $s_n(z) = \sum_{k=0}^n a_k z^k$, then $s_n \rightarrow f$ in $H(D)$.

6.12. A family $\mathcal{F} \subseteq C(\Omega)$ is said to be *equicontinuous at a point* $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon$$

for every $f \in \mathcal{F}$.

Show that \mathcal{F} is equicontinuous on compact subsets of Ω if and only if it is equicontinuous at every point $z_0 \in \Omega$.

Prove the assertion relative to the footnote in the proof of Montel's theorem, Thm. 6.15.

6.13. Let \mathcal{F} be a normal family of holomorphic on a domain Ω . Let

$$\mathcal{G} = \{g : g = f', f \in \mathcal{F}\}.$$

Show that \mathcal{G} is normal.

6.14. Let Ω be a domain, $\overline{D(z, r)} \subseteq \Omega$ and let f be holomorphic on Ω .

(i) Show that

$$|f(z)|^2 \leq \frac{1}{\pi r^2} \iint_{D(z, r)} |f(\zeta)|^2 dm(\zeta),$$

where dm denotes the Lebesgue measure in $\mathbf{C} \equiv \mathbf{R}^2$. (You may wish to use polar coordinates.)

(ii) For $M > 0$ let

$$\mathcal{F} = \{f \in H(\Omega) : \iint_{\Omega} |f(w)|^2 dm(w) \leq M\}.$$

Show that \mathcal{F} is normal.

6.15. Let $\mathcal{F} = \{f : f(z) = \tan(nz), n = 1, 2, \dots, z \in \mathcal{U}\}$. Show that \mathcal{F} is normal and find its unique limit function.

7. HARMONIC FUNCTIONS

In this section we return to one aspect of the theory that concerns the analysis of harmonic functions, subject often called *potential theory*.

Recall that a C^2 function u on an open set $A \subseteq \mathbf{R}^2$ is said to be harmonic on A if $\Delta u = 0$ on A , where $\Delta = \partial_x^2 + \partial_y^2$ is the *Laplacian*. The next lemma collects the first elementary but fundamental facts about the relation between harmonic and holomorphic functions.

Lemma 7.1. *If $f = u + iv$ is holomorphic on an open set $A \subseteq \mathbf{C}$ then its real and imaginary parts u and v are harmonic on A .*

If u is a real harmonic function on a simply connected open set \mathcal{D} , then there exists a real harmonic function v on \mathcal{D} such that $u + iv$ is holomorphic on \mathcal{D} . In this case, we will say that v is the harmonic conjugate of u on \mathcal{D} .

Proof. The first part follows from Subsection 1.2 .

Suppose now u is a real harmonic function on a simply connected open set \mathcal{D} . We wish to $v \in C^2(\mathcal{D})$ satisfying the CR-equations on \mathcal{D} , that is, such that

$$dv = (-\partial_y u)dx + (\partial_x u)dy.$$

The one on the right hand side is a closed differential since u is harmonic. Since \mathcal{D} is simply connected, it is an exact differential, so such a v exists. It immediately follows that $u + iv$ is holomorphic. \square

We remark that the hypothesis of \mathcal{D} being simply connected cannot be relaxed. As an example, consider $A = \mathbf{C} \setminus \{0\}$ and $u(x, y) = \frac{1}{2} \log(x^2 + y^2)$. Then u is real and harmonic. On $A \cap \{x + iy : x > 0\}$ is the real part of $\log z$, that cannot be extended to all of A . Hence, there exists no function holomorphic on A whose real part is u .

7.1. Maximum principle. We now prove the maximum principle for (real) harmonic functions.

Theorem 7.2. *Let $\Omega \subseteq \mathbf{C} \equiv \mathbf{R}^2$ be a domain (connected open set), $u : \Omega \rightarrow \mathbf{R}$ be harmonic. If there exists $z_0 \in \Omega$ and $r_0 > 0$ such that $D(z_0, r_0) \subseteq \Omega$ and $u(z_0) = \sup\{u(z) : z \in D(z_0, r_0)\}$, then u is constant on Ω .*

Proof. Let

$$\Omega' = \{z \in \Omega : \text{there exists } r_z > 0 \text{ such that for } w \in D(z, r_z), u(w) = u(z)\}.$$

We wish to show that Ω' is open, closed in Ω and non-empty, thus showing that $\Omega' = \Omega$.

On $D(z_0, r_0)$ we can find h holomorphic such that $\operatorname{Re} h = u$. Take $f = e^h$. Since $|f| = e^{\operatorname{Re} h} = e^u$, $|f|$ attains its maximum at z_0 . Hence f is constant on $D(z_0, r_0)$, so is u . Thus, $\Omega' \neq \emptyset$. Moreover, Ω' is open by construction.

Finally, let $z \in \overline{\Omega'}$. Let $D(z, r_z) \subseteq \Omega$. Since $z \in \overline{\Omega'}$, there exists some open disk on which u is constant. Let h_z be the holomorphic function on $D(z, r_z)$ whose real part is u . Then, h_z is constant on an open disk, hence on all of $D(z, r_z)$, so is u . Thus, $z \in \Omega'$, Ω' is closed, that is, $\Omega' = \Omega$. \square

Corollary 7.3. *Let u, Ω be as before. Suppose there exists $z_0 \in \Omega$ and $r_0 > 0$ such that $D(z_0, r_0) \subseteq \Omega$ and $u(z_0) = \inf\{u(z) : z \in D(z_0, r_0)\}$, then u is constant on Ω .*

Proof. It suffices to apply the maximum principle to $-u$. \square

Corollary 7.4. *Let u, Ω be as before. Furthermore, suppose that Ω is bounded and $u \in C(\overline{\Omega})$. Then*

$$\max\{u(z) : z \in \Omega\} = \max\{u(z) : z \in \partial\Omega\};$$

and

$$\min\{u(z) : z \in \Omega\} = \min\{u(z) : z \in \partial\Omega\}.$$

Proof. Since $\overline{\Omega}$ is compact and u is continuous on $\overline{\Omega}$, max and min are attained at some points. By Thm. 7.2 and Cor. 7.3, these points cannot be in the interior, unless u is constant, in which case the result still holds. \square

Theorem 7.5. (The mean value property) *Let $A \subseteq \mathbf{C}$ be open, $\overline{D(z_0, r)} \subseteq A$, u be harmonic on A . Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

Proof. Let $s > r$ be such that $D(z_0, s) \subseteq A$ and let h be holomorphic on $D(z_0, s)$ and such that $u = \operatorname{Re} h$, $h = u + iv$. We can apply Cauchy's formula to h on $\gamma = \partial D(z_0, r)$, $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$. We have

$$\begin{aligned} h(z_0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{h(\zeta)}{\zeta - z_0} d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{h(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta + i \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

By passing to the real and imaginary part we obtain the conclusion. \square

Although worth be to stated separately, the mean value property can be obtained as a particular case of the next result.

Theorem 7.6. (The Poisson formula for the disk) *Let $A \subseteq \mathbf{C}$ be open, $\overline{D(0, R)} \subseteq A$, u be harmonic on A . Then for every $z \in D(0, R)$ we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta.$$

Remark 7.7. Before proving the theorem, we make a few remarks.

(i) The function

$$P_z(\zeta) = \frac{1}{2\pi} \cdot \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}, \quad (7.1)$$

defined for $z \in D(0, R)$ and $\zeta \in \partial D(0, R)$ is called the *Poisson kernel* for the disk $D(0, R)$. In polar coordinates it has the expression

$$\begin{aligned} P_{re^{i\eta}}(Re^{i\theta}) &= \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\eta}|^2} \\ &= \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \eta) + r^2}. \end{aligned} \quad (7.2)$$

(ii) The Poisson kernel $P_{re^{i\eta}}(Re^{i\theta})$ is a positive kernel, that is,

$$P_{re^{i\eta}}(Re^{i\theta}) > 0$$

for all $0 < r < R$, $\eta, \theta \in [0, 2\pi]$.

(iii) If $\zeta = Re^{i\theta}$ and $z = re^{i\eta}$, then

$$P_{re^{i\eta}}(Re^{i\theta}) = \frac{1}{2\pi} \operatorname{Re} \frac{\zeta + z}{\zeta - z} = \frac{1}{2\pi} \operatorname{Re} \frac{Re^{i\theta} + re^{i\eta}}{Re^{i\theta} - re^{i\eta}}. \quad (7.3)$$

This follows at once from (7.1), since

$$\frac{\zeta + z}{\zeta - z} = \frac{\zeta + z}{\zeta - z} \cdot \frac{\bar{\zeta} - \bar{z}}{\bar{\zeta} - \bar{z}} = \frac{|\zeta|^2 - |z|^2 + (\bar{\zeta}z - \zeta\bar{z})}{|\zeta - z|^2}.$$

(iv) Finally, since the constant function $u(z) = 1$ is harmonic, from the reproducing property in Thm. 7.6 we see that

$$\frac{1}{2\pi} \int_0^{2\pi} P_z(Re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta = 1.$$

Proof of Thm. 7.6. Let $s > R$ and h be the holomorphic function on $D(0, s)$ such that $u = \operatorname{Re} h$. For $z \in D(0, R)$, by Cauchy's formula, letting $\gamma(\theta) = Re^{i\theta}$, with $\theta \in [0, 2\pi]$, we have

$$\begin{aligned} h(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{h(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \frac{Re^{i\theta}}{Re^{i\theta} - z} d\theta. \end{aligned} \quad (7.4)$$

Moreover, if we set $w = R^2/\bar{z}$, we observe that $w = \frac{R^2}{r^2}z$, where $z = re^{i\eta}$, and that the function

$$\zeta \mapsto \frac{h(\zeta)}{\zeta - w}$$

is holomorphic on $\overline{D(0, R)}$, since $|w| = \frac{R^2}{r} > R$. Therefore,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\gamma} \frac{h(\zeta)}{\zeta - w} d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \frac{Re^{i\theta}}{Re^{i\theta} - \frac{R^2}{\bar{z}}} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \frac{\bar{z}}{\bar{z} - Re^{-i\theta}} d\theta. \end{aligned} \quad (7.5)$$

Subtracting (7.5) from (7.4), we see that

$$\begin{aligned} h(z) &= \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \left(\frac{Re^{i\theta}}{Re^{i\theta} - z} - \frac{\bar{z}}{\bar{z} - Re^{-i\theta}} \right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta. \end{aligned}$$

By passing to real and imaginary parts we obtain

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta,$$

which is what we wanted to show. \square

7.2. The Dirichlet problem. The Dirichlet problem for the Laplacian on a domain $\Omega \subseteq \mathbf{C}$ is the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \quad (7.6)$$

where f is an assigned, continuous function on $\partial\Omega$, called the *data*.

We begin by studying the solution of (7.6) on the unit disk D .

Theorem 7.8. *Let f be a continuous function on the unit circle ∂D . Set*

$$u(z) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \cdot \frac{1-|z|^2}{|z-e^{i\theta}|^2} d\theta & z \in D \\ f(z) & z \in \partial D. \end{cases}$$

Then, u is continuous on the closed disk \overline{D} and harmonic on D . Hence, it solves the Dirichlet problem (7.6) for the unit disk.

Proof. We first show that u is harmonic on D . It suffices to write the Poisson kernel $P_z(e^{i\theta})$ as

$$\frac{1-|z|^2}{|z-e^{i\theta}|^2} = \frac{e^{i\theta}}{e^{i\theta}-z} + \frac{e^{-i\theta}}{e^{-i\theta}-\bar{z}} - 1.$$

Then,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta}-z} d\theta + \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{e^{-i\theta}}{e^{-i\theta}-\bar{z}} d\theta - \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

To check that u is indeed harmonic, it suffices to check that each term is harmonic. The first term is harmonic since it is holomorphic, the second term is harmonic since it is anti-holomorphic, while the third term is constant, hence harmonic.

Next we wish to show that $u \in C(\overline{\Omega})$. Clearly, u is continuous inside, since it is harmonic. Thus, we only need to check the continuity at the boundary.

Fix $z_0 = e^{i\eta_0} \in \partial D$. Recall that $u(z_0) = f(z_0)$ by definition. We need to show that, for a given $\varepsilon > 0$, we can find $\delta > 0$ such that $|z - z_0| < \delta$ implies $|u(z) - u(z_0)| < \varepsilon$. Using (iv) in Remark 7.7 we write

$$\begin{aligned} |u(z) - u(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta - f(e^{i\eta_0}) \right| \\ &= \left| \frac{1}{2\pi} \int_0^{2\pi} [f(e^{i\theta}) - f(e^{i\eta_0})] \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta \right| \\ &\leq \left| \frac{1}{2\pi} \int_{|\theta-\eta_0| \leq 2\sigma} [f(e^{i\theta}) - f(e^{i\eta_0})] \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta \right| \\ &\quad + \left| \frac{1}{2\pi} \int_{|\theta-\eta_0| > 2\sigma} [f(e^{i\theta}) - f(e^{i\eta_0})] \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta \right| \\ &= I + II, \end{aligned}$$

for some $\sigma > 0$ to be determined later.

By uniform continuity of $f(e^{i\theta})$, given $\varepsilon > 0$ we can choose $\sigma > 0$ such that $|\theta - \eta_0| \leq 2\sigma$ implies $|f(e^{i\theta}) - f(e^{i\eta_0})| < \varepsilon$. Therefore,

$$\begin{aligned} |I| &\leq \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta \\ &= \varepsilon. \end{aligned}$$

To estimate II , notice that if $|re^{i\eta} - e^{i\eta_0}| < \delta$, then

$$\begin{aligned} |e^{i\theta} - re^{i\eta}| &= |e^{i\theta} - e^{i\eta_0} - (re^{i\eta} - e^{i\eta_0})| \\ &\geq |e^{i\theta} - e^{i\eta_0}| - \delta, \end{aligned}$$

and that, for $\frac{\pi}{4} \geq |\theta - \eta_0| > 2\sigma$,

$$\begin{aligned} |e^{i\theta} - e^{i\eta_0}|^2 &= |1 - e^{i(\eta_0 - \theta)}|^2 = 2(1 - \cos(\eta_0 - \theta)) \\ &\geq \frac{|\eta_0 - \theta|^2}{2} > 2\sigma^2. \end{aligned}$$

If $|\theta - \eta_0| > \frac{\pi}{4}$ the above estimate holds trivially (for σ sufficiently small).

Therefore, for $|re^{i\eta} - e^{i\eta_0}| < \delta$ and $|\theta - \eta_0| \geq 2\sigma$,

$$|e^{i\theta} - re^{i\eta}| \geq \sqrt{2}\sigma - \delta \geq \sigma,$$

if we take $\delta \leq (\sqrt{2} - 1)\sigma$.

Therefore, if $\sup_{\theta \in [0, 2\pi]} |f(e^{i\theta})| \leq M$,

$$\begin{aligned} |II| &\leq \frac{1}{2\pi} \int_{|\theta - \eta_0| \geq 2\sigma} 2 \sup_{\theta \in [0, 2\pi]} |f(e^{i\theta})| \frac{1-r^2}{|re^{i\eta} - e^{i\theta}|^2} d\theta \\ &\leq \frac{M}{\pi} \int_{|\theta - \eta_0| \geq 2\sigma} \frac{1-r^2}{\sigma^2} d\theta \\ &\leq \frac{4M}{\sigma^2} (1-r) \\ &\leq \frac{4M}{\sigma^2} \delta; \end{aligned}$$

this last inequality, that is geometrically obvious, follows since

$$1-r = |e^{i\eta} - re^{i\eta}| \leq |e^{i\eta_0} - re^{i\eta}| < \delta.$$

Thus,

$$|u(z) - u(z_0)| \leq |I| + |II| < 2\varepsilon,$$

if δ is chosen less than $\varepsilon\sigma^2/4M$. \square

Other important properties of harmonic functions are expressed by Harnack's inequality and principle.

Theorem 7.9. (Harnack's inequality) *Let $A \subseteq \mathbf{C}$ be an open set, $\overline{D(0, R)} \subseteq A$, u harmonic on A , $u \geq 0$. Then, for every $z \in D(0, R)$*

$$\frac{R - |z|}{R + |z|} u(0) \leq u(z) \leq \frac{R + |z|}{R - |z|} u(0).$$

Proof. By Poisson formula in Thm. 7.6 and (7.1) we have

$$\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \leq \frac{R^2 - |z|^2}{(R - |z|)^2} = \frac{R + |z|}{R - |z|},$$

while

$$\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \geq \frac{R^2 - |z|^2}{(R + |z|)^2} = \frac{R - |z|}{R + |z|}.$$

Hence,

$$\begin{aligned} \frac{R - |z|}{R + |z|} u(0) &= \frac{R - |z|}{R + |z|} \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R - |z|^2}{|Re^{i\theta} - z|^2} u(Re^{i\theta}) d\theta \\ &= u(z) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R + |z|}{R - |z|} u(Re^{i\theta}) d\theta \\ &= \frac{R + |z|}{R - |z|} u(0). \quad \square \end{aligned}$$

We clearly also have the following.

Corollary 7.10. (Harnack's inequality) *Let $A \subseteq \mathbf{C}$ be an open set, $\overline{D(z_0, R)} \subseteq A$, u harmonic on A , $u \geq 0$. Then, for every $z \in D(z_0, R)$*

$$\frac{R - |z - z_0|}{R + |z - z_0|} u(z_0) \leq u(z) \leq \frac{R + |z - z_0|}{R - |z - z_0|} u(z_0).$$

The next result can be proved with an argument similar to one of Weierstrass' theorem 4.10.

Lemma 7.11. *Let $A \subseteq \mathbf{C}$ be an open set, $\{u_n\}$ a sequence of harmonic functions on A , converging to u uniformly on compact subsets of A . Then u is harmonic on A .*

Proof. Exercise VII.1. \square

Theorem 7.12. (Harnack's principle) *Let $u_1 \leq u_2 \leq \dots$ be a sequence of harmonic functions on a domain $\Omega \subseteq \mathbf{C}$. Then, either $u_n \rightarrow +\infty$ uniformly on compact subsets of Ω , or $u_n \rightarrow u$ uniformly on compact subsets of Ω to function u , which is necessarily harmonic.*

Proof. We will use the connectness of Ω . Set

$$\begin{aligned} \Omega_1 &= \{z \in \Omega : u_n(z) \rightarrow +\infty\}, \\ \Omega_2 &= \{z \in \Omega : u_n(z) \rightarrow \ell_z, \text{ finite}\}. \end{aligned}$$

We show that both Ω_1, Ω_2 are open. Since $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$, one of the two sets must be empty. The uniform convergence on compact subsets in both cases (to $+\infty$ or to $u(z)$) will follow.

Suppose $u_n(z_0) \rightarrow +\infty$. Let n_0 be such that $u_{n_0}(z_0) > 0$ and $r_0 > 0$ such that $u_{n_0}(z) > 0$ on $D(z_0, r_0)$, so that $u_n(z) > 0$ on $D(z_0, r_0)$. Clearly, we may assume that $\overline{D(z_0, r_0)} \subseteq \Omega$. By Harnack's inequality, for $z \in D(z_0, r_0/2)$, for $n > n_0$ we have

$$u_n(z) \geq \frac{r_0 - \frac{r_0}{2}}{r_0 + r_0} u_{n_0}(z_0) = \frac{u_{n_0}(z_0)}{4} \rightarrow +\infty.$$

Hence, $u_n \rightarrow +\infty$ uniformly on $\overline{D(z_0, r_0/2)}$, and Ω_1 is open.

Suppose now $u_n(z_1) \rightarrow \ell$, finite. Let $\overline{D(z_1, r_1)} \subseteq \Omega$, then for $z \in D(z_1, r_1/2)$

$$\begin{aligned} u_{n+m}(z) - u_n(z) &\leq \frac{r_1 + r_1}{r_1 - \frac{r_1}{2}} \left(u_{n+m}(z_1) - u_n(z_1) \right) \\ &= 4 \left(u_{n+m}(z_1) - u_n(z_1) \right) \rightarrow 0, \end{aligned}$$

for $n \rightarrow +\infty$, for any m . This shows that $\{u_n\}$ converges uniformly on $\overline{D(z_1, r_1)}$ and Ω_2 is open. \square

7.3. Exercises.

7.1. Prove Lemma 7.11.

7.2. Prove that for any fixed $\delta > 0$, the Poisson kernel $P_{re^{i\eta}}(Re^{i\theta})$ converges uniformly to 0 on the set $|\theta - \eta| \geq \delta$, as $r \rightarrow 1^-$; that is,

$$\sup_{|\theta - \eta| \geq \delta} P_{re^{i\eta}}(Re^{i\theta}) \rightarrow 0$$

as $r \rightarrow 1^-$.

8. ENTIRE FUNCTIONS

To begin our study of holomorphic functions in the entire plane, we discuss the notion of convergence for infinite products.

8.1. Infinite products. Let α_j be complex numbers, $j = 1, 2, \dots$. We want to give a meaning to the convergence of the infinite product $\prod_{j=1}^{+\infty} \alpha_j$.

Definition 8.1. We say that the infinite product $\prod_{j=1}^{+\infty} \alpha_j$ converges if

- (i) there exist at most finitely many $\alpha_j = 0$, say $\alpha_j \neq 0$ for $j \geq j_N$;
- (ii) for any $N_0 \geq j_N$, the limit

$$\lim_{N \rightarrow +\infty} \prod_{j=N_0}^N \alpha_j = \beta_{N_0}$$

exists finite and $\neq 0$.

Notice that, if condition (ii) is verified, we may compute the logarithm of β_{N_0} . Let $\beta = \beta_{N_0}$, $\alpha_N = \prod_{j=N_0}^N \alpha_j$, and let $D(\beta, \varepsilon)$ not contain the origin and let N_ε be such that $\alpha_N \in D(\beta, \varepsilon)$ for $N \geq N_\varepsilon$.

We may assume that β is not on the negative real axis and let \log denote the principal branch of the logarithm (otherwise, choose a different branch cut for the determination of the logarithm.) Then, we have

$$\begin{aligned} \log \beta_{N_0} &= \log \left(\lim_{N \rightarrow +\infty} \prod_{j=N_0}^N \alpha_j \right) = \lim_{N \rightarrow +\infty} \log \prod_{j=N_0}^N \alpha_j \\ &= \lim_{N \rightarrow +\infty} \sum_{j=N_0}^N \log_{(j)} \alpha_j, \end{aligned}$$

where $\log_{(j)}$ denotes some branch of the logarithm. Since the limit on the right hand side exists finite, $\log_{(j)} \alpha_j \rightarrow 0$ as $j \rightarrow +\infty$. Hence, in particular the branch of the logarithm must be the principal one, and $\alpha_j \rightarrow 1$. This is a necessary condition for the convergence of the infinite product.

Although the next result is not strictly necessary for what that follows, we state it for the sake of clarity.

Lemma 8.2. *Let α_j be non-zero complex numbers. Then $\prod_{j=1}^{+\infty} \alpha_j$ converges if and only if $\sum_{j=1}^{+\infty} \log \alpha_j$ converges, where \log denotes the principal branch.*

Proof. The previous argument shows that if $\prod_{j=1}^{+\infty} \alpha_j$ converges then also $\sum_{j=1}^{+\infty} \log \alpha_j$ converges.

Conversely, if $\sum_{j=1}^{+\infty} \log \alpha_j$ converges, then, since $e^{\sum_{j=1}^N \log \alpha_j} = \prod_{j=1}^N \alpha_j$ also $\prod_{j=1}^{+\infty} \alpha_j$ converges. \square

For simplicity of notation, we are going to write $\alpha_j = 1 + a_j$.

Lemma 8.3. *Let $a_j \in \mathbf{C}$ be such that $|a_j| < 1$. Let $Q_N = \prod_{j=1}^N (1 + |a_j|)$. Then*

$$e^{\frac{1}{2} \sum_{j=1}^N |a_j|} \leq Q_N \leq e^{\sum_{j=1}^N |a_j|}.$$

Proof. Since $1 + |a_j| \leq e^{|a_j|}$,

$$(1 + |a_1|) \cdots (1 + |a_N|) \leq e^{\sum_{j=1}^N |a_j|}.$$

On the other hand, since $e^x \leq 1 + 2x$ for $0 \leq x \leq 1$,

$$\begin{aligned} e^{\frac{1}{2} \sum_{j=1}^N |a_j|} &\leq (1 + 2(|a_1|/2)) \cdots (1 + 2(|a_N|/2)) \\ &= \prod_{j=1}^N (1 + |a_j|). \quad \square \end{aligned}$$

Proposition 8.4. *Let $a_j \in \mathbf{C}$ be such that $|a_j| < 1$. Then $\prod_{j=1}^{+\infty} (1 + |a_j|)$ converges if and only if $\sum_{j=1}^{+\infty} |a_j|$ converges.*

Proof. Suppose $\sum_{j=1}^{+\infty} |a_j| = M$. Then, by the previous lemma, $Q_N \leq e^M$, for all N . Since $Q_1 \leq Q_2 \leq \cdots$, the sequence of “partial products” $\{Q_N\}$ converges.

Conversely, if the infinite product converges to Q , then $Q \geq 1$ and $\sum_{j=1}^N |a_j| \leq 2 \log Q$ for all N . Then $\sum_{j=1}^{+\infty} |a_j|$ converges. \square

Proposition 8.5. *If the infinite product $\prod_{j=1}^{+\infty} (1 + |a_j|)$ converges, then also $\prod_{j=1}^{+\infty} (1 + a_j)$ converges. Hence, if the series $\sum_{j=1}^{+\infty} |a_j|$ converges, also $\prod_{j=1}^{+\infty} (1 + a_j)$ converges.*

Proof. Since the product $\prod_{j=1}^{+\infty} (1 + |a_j|)$ converges, then $|a_j| \rightarrow 0$, so that $1 + a_j \neq 0$ for $j \geq j_0$. We may assume $j_0 = 1$. Let

$$P_N = \prod_{j=1}^N (1 + a_j), \quad \text{and} \quad Q_N = \prod_{j=1}^N (1 + |a_j|).$$

Notice that, for a suitable choice of indices j_k ,

$$P_N = 1 + \sum_{n=1}^N \prod_{k=1}^n a_{j_k}.$$

Then,

$$\begin{aligned} |P_N - 1| &= \left| \sum_{n=1}^N \prod_{k=1}^n a_{j_k} \right| \\ &\leq \sum_{n=1}^N \prod_{k=1}^n |a_{j_k}| = Q_N - 1. \end{aligned} \tag{8.1}$$

Then, for $N, M > 1$, $N > M$,

$$\begin{aligned}
|P_N - P_M| &= \left| \prod_{j=1}^N (1 + a_j) - \prod_{j=1}^M (1 + a_j) \right| \\
&= \left| \prod_{j=1}^M (1 + a_j) \right| \cdot \left| 1 - \prod_{j=M+1}^N (1 + a_j) \right| \\
&\leq Q_M \left(\prod_{j=M+1}^N (1 + |a_j|) - 1 \right) \\
&= Q_N - Q_M.
\end{aligned} \tag{8.2}$$

Hence, $\{P_N\}$ is a Cauchy sequence, since $\{Q_N\}$ is, and it converges. We need to show that it does not converge to 0. By Lemma 8.3

$$\prod_{j=M}^N (1 + |a_j|) \leq e^{\sum_{j=M}^N |a_j|} \leq \frac{3}{2}$$

for $M \geq j_0$, and $N > M$. Then, arguing as in (8.1) we see that

$$\left| 1 - \prod_{j=M}^N (1 + a_j) \right| \leq \prod_{j=M}^N (1 + |a_j|) - 1 \leq \frac{1}{2},$$

for $M \geq j_0$, and $N > M$. Hence,

$$\left| \prod_{j=M}^N (1 + a_j) \right| \geq \frac{1}{2}$$

so that

$$\begin{aligned}
\lim_{N \rightarrow +\infty} |P_N| &= \lim_{N \rightarrow +\infty} \left| \prod_{j=1}^M (1 + a_j) \right| \cdot \left| \prod_{j=M}^N (1 + a_j) \right| \\
&\geq \frac{1}{2} \left| \prod_{j=1}^{j_0} (1 + a_j) \right|. \quad \square
\end{aligned}$$

We apply these results to the infinite product of functions.

Theorem 8.6. *Let $A \subseteq \mathbf{C}$ be an open set. Let $f_j : A \rightarrow \mathbf{C}$ be holomorphic, $j = 1, 2, \dots$. Suppose that $\sum_{j=1}^{+\infty} |f_j|$ converges uniformly on compact subsets of A . Let*

$$\prod_{j=1}^{+\infty} (1 + f_j(z)).$$

Then the partial products F_N converge uniformly on compact subsets of A to a holomorphic function F . Moreover, F vanishes at a point $z_0 \in A$ if and only if there exists j such that $f_j(z_0) = -1$ and the multiplicity of the zero of F equals the number of the zeros of $1 + f_j$ at z_0 .

Proof. We first consider the question of uniform convergence. Let $K \subseteq A$ be compact. Then there exists $C > 0$ such that for all integers N

$$\sup_{z \in K} \sum_{j=1}^N |f_j(z)| \leq C.$$

Then,

$$\sup_{z \in K} \prod_{j=1}^N (1 + |f_j(z)|) \leq e^C,$$

with C independent of N .

Let $\varepsilon > 0$ and j_0 be such that for $N > M \geq j_0$

$$\sup_{z \in K} \sum_{j=M}^N |f_j(z)| \leq \varepsilon.$$

Let

$$Q_N(z) = \prod_{j=1}^N (1 + |f_j(z)|).$$

Then, for $N > M \geq j_0$

$$\begin{aligned} |Q_N(z) - Q_M(z)| &\leq Q_M(z) \left| \prod_{j=M+1}^N (1 + |f_j(z)|) - 1 \right| \\ &\leq e^{\sum_{j=1}^M |f_j(z)|} \left(\prod_{j=M+1}^N (1 + |f_j(z)|) - 1 \right) \\ &\leq e^{\sum_{j=1}^M |f_j(z)|} \left(\exp \left\{ \sum_{j=M+1}^N |f_j(z)| \right\} - 1 \right) \\ &\leq e^C (e^\varepsilon - 1) \end{aligned}$$

that tends to 0 as $\varepsilon \rightarrow 0$.

Thus, using (8.2) we have

$$|F_N(z) - F_M(z)| \leq |Q_N(z) - Q_M(z)| \leq e^C (e^\varepsilon - 1),$$

so that the sequence $\{F_N\}$ converges uniformly on K , hence on compact subsets of A , to a limit function F . Clearly, F is holomorphic on A .

Next we turn to the zeros. Suppose $F(z_0) = 0$. By definition of infinite product, there exists j_0 such that

$$\lim_{N \rightarrow +\infty} \prod_{j=j_0+1}^N (1 + f_j(z_0)) \neq 0.$$

Let

$$G(z) = \lim_{N \rightarrow +\infty} \prod_{j=j_0+1}^N (1 + f_j(z)).$$

Then G is holomorphic, $G(z_0) \neq 0$ and

$$F(z) = G(z) \prod_{j=1}^{j_0} (1 + f_j(z))$$

and the statement follows. \square

8.2. The Weierstrass factorization theorem.

Definition 8.7. We define the *Weierstrass elementary factors* as $E(z, 0) = 1 - z$ and for $n = 1, 2, \dots$,

$$E(z, n) = (1 - z)e^{z + \frac{z^2}{2} + \dots + \frac{z^n}{n}}.$$

Lemma 8.8. For $|z| \leq 1$, $|1 - E(z, n)| \leq |z|^{n+1}$.

Proof. The case $n = 0$ is trivial. Let $n \geq 1$. Notice that

$$\begin{aligned} E'(z, n) &= -e^{z + \frac{z^2}{2} + \dots + \frac{z^n}{n}} + (1 - z)(1 + z + \dots + z^{n-1})e^{z + \frac{z^2}{2} + \dots + \frac{z^n}{n}} \\ &= (-1 + (1 - z^n))e^{z + \frac{z^2}{2} + \dots + \frac{z^n}{n}} \\ &= -z^n e^{z + \frac{z^2}{2} + \dots + \frac{z^n}{n}}. \end{aligned}$$

Then,

$$E'(z, n) = - \sum_{k=n}^{+\infty} b_k z^k,$$

for some coefficients $b_k > 0$ for all $k \geq n$. On the other hand, if $E(z, n) = \sum_{k=0}^{+\infty} a_k z^k$, then

$$E'(z, n) = \sum_{k=0}^{+\infty} (k+1) a_{k+1} z^k.$$

Comparing the coefficients we have

$$\begin{aligned} a_1 &= \dots = a_n = 0 \\ a_{n+j+1} &= -\frac{b_{n+j}}{n+j+1} < 0 \quad j = 0, 1, \dots, \end{aligned}$$

and moreover,

$$a_0 = E(0, n) = 1.$$

Then,

$$E(z, n) = 1 + \sum_{k=n+1}^{+\infty} a_k z^k \quad a_k < 0,$$

so that, for $|z| \leq 1$,

$$\begin{aligned}
 |1 - E(z, n)| &= \left| \sum_{k=n+1}^{+\infty} a_k z^k \right| = |z|^{n+1} \left| \sum_{k=n+1}^{+\infty} a_k z^{k-(n+1)} \right| \\
 &\leq |z|^{n+1} \sum_{k=n+1}^{+\infty} |a_k| = -|z|^{n+1} \sum_{k=n+1}^{+\infty} a_k \\
 &= |z|^{n+1} (1 - E(1, n)) \\
 &= |z|^{n+1}. \quad \square
 \end{aligned}$$

Let now $\{z_j\}$ be a sequence of non-zero complex numbers such that $\lim_{j \rightarrow +\infty} |z_j| = +\infty$. We claim that there exist integers $\{p_j\}$ such that the series

$$\sum_{j=1}^{+\infty} \left(\frac{r}{|z_j|} \right)^{p_j+1}$$

converges for all $r > 0$. Let $N = N(r)$ be such that $|z_j| > 2r$ for $j \geq N$. Then

$$\sum_{j=N}^{+\infty} \left(\frac{r}{|z_j|} \right)^j \leq \sum_{j=N}^{+\infty} \frac{1}{2^j} < +\infty.$$

Therefore, any sequence $\{p_j\}$ such that $p_j \geq j - 1$ will do the job.

Consider now the infinite product, called *Weierstrass product*,

$$\prod_{j=1}^{+\infty} E(z/z_j, p_j) = \prod_{j=1}^{+\infty} \left(1 - \frac{z}{z_j} \right) e^{z/z_j + \frac{(z/z_j)^2}{2} + \dots + \frac{(z/z_j)^{p_j}}{p_j}}. \quad (8.3)$$

Theorem 8.9. *Let $\{z_j\} \subseteq \mathbf{C}$, $\{p_j\} \subseteq \mathbf{N}$ be chosen as above. Then the Weierstrass product*

$$\prod_{j=1}^{+\infty} E(z/z_j, p_j)$$

converges uniformly on every set $\{|z| \leq r\}$, $r > 0$, to a holomorphic entire function F . The zeros of F are precisely the points $\{z_j\}$ counted with the corresponding multiplicity.

Proof. Let $r > 0$ be fixed. Let j_0 be such that $|z_j| > r$ for $j \geq j_0$. Thus,

$$|E(z/z_j, p_j) - 1| \leq \left| \frac{z}{z_j} \right|^{p_j+1} \leq \left(\frac{r}{|z_j|} \right)^{p_j+1}.$$

By the hypothesis on the p_j 's,

$$\sum_{j=j_0}^{+\infty} |E(z/z_j, p_j) - 1| \leq \sum_{j=j_0}^{+\infty} \left(\frac{r}{|z_j|} \right)^{p_j+1} < +\infty.$$

Weierstrass's M -test implies that $\sum_{j=j_0}^{+\infty} |E(z/z_j, p_j) - 1|$ converges uniformly on $\{|z| \leq r\}$, for any $r > 0$. Thm. 8.6 now implies that

$$\prod_{j=1}^{+\infty} E(z/z_j, p_j) = \prod_{j=1}^{j_0-1} E(z/z_j, p_j) \prod_{j=j_0}^{+\infty} E(z/z_j, p_j)$$

converges uniformly on compact subsets of \mathbf{C} to an entire function F whose zeros are precisely the zeros of the $E(z/z_j, p_j)$'s. \square

Corollary 8.10. *Let $\{z_j\}$ be a sequence such that $|z_j| \rightarrow +\infty$. Then there exists an entire function F whose zeros are precisely the $\{z_j\}$, counting multiplicity.*

Proof. We may assume that $z_1 = \dots = z_k = 0$, and $z_j \neq 0$ for $j > k$. Let $p_j = j - 1$.

Let $r > 0$ be fixed. Let $N = N(r)$ be such that $|z_j| > 2r$ for $j \geq N$. Then

$$\sum_{j=N}^{+\infty} \left(\frac{r}{|z_j|} \right)^j \leq \sum_{j=N}^{+\infty} \frac{1}{2^j} < +\infty.$$

Thus, by Thm. 8.9, the function

$$F(z) = z^k \prod_{j=k+1}^{+\infty} E(z/z_j, j-1)$$

is the desired entire function. \square

Theorem 8.11. (Weierstrass' Factorization Theorem) *Let f be an entire function. Suppose that f vanishes of order k at the origin. Let $\{z_j\}$ be the other zeros of f , counting multiplicity. Then there exists an entire function g such that*

$$f(z) = z^k e^{g(z)} \prod_{j=1}^{+\infty} E(z/z_j, j-1).$$

Proof. By the Cor. 8.10, the function $h(z) = z^k \prod_{j=1}^{+\infty} E(z/z_j, j-1)$ is entire and has the same zeros as f . Hence, the function f/h can be extended to an entire function, with no zero. Since \mathbf{C} is simply connected, $\log(f/h) = g$ is well defined and entire.

Hence, $e^g = f/h$, that is,

$$f(z) = h(z) e^{g(z)} = z^k e^{g(z)} \prod_{j=1}^{+\infty} E(z/z_j, j-1). \quad \square$$

We now apply this result to describe an identity that describes the factorization of $\sin z$.

Proposition 8.12. *We have*

$$\begin{aligned} \sin \pi z &= \pi z \prod_{n \neq 0} E(z/n, 1) \\ &= \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n} \right) e^{z/n} = \pi z \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{n^2} \right). \end{aligned} \quad (8.4)$$

In order to prove the above identity we need a preliminary result.

Lemma 8.13. *We have the following identities:*

$$(i) \quad \pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right);$$

$$(ii) \frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbf{Z}} \frac{1}{(z - n)^2}.$$

Proof. (i) The function

$$f_1(z) = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z - n} + \frac{1}{n} \right)$$

is meromorphic in \mathbf{C} and having simple poles at the integers, with residues all equal to 1. The function

$$f_2(z) = \pi \frac{\cos \pi z}{\sin \pi z}$$

is also meromorphic in \mathbf{C} and having simple poles at the integers, with residues all equal to 1.

Hence, $h(z) = f_1(z) - f_2(z)$ is entire. It is immediate to check that

$$h'(z) = - \sum_{n \in \mathbf{Z}} \frac{1}{(z - n)^2} + \frac{\pi^2}{\sin^2 \pi z}$$

is periodic of period 1 (and it would not be so obvious that h is periodic of period 1).

We wish to prove that $h(z) \equiv 0$. We begin by showing that h' is constant, and equal to 0. In order to show that h' is constant, we show that h' is bounded and then invoke Liouville's theorem. Being periodic of period 1, h' is bounded if and only if it is bounded in the strip $\{z = x + iy : 0 \leq x \leq 1\}$. But, on the compact set $\{z = x + iy : 0 \leq x \leq 1, |y| \leq 1\}$ h' is certainly bounded. For $|y| > 1$ and $0 \leq x \leq 1$, the sum

$$\sum_{n \in \mathbf{Z}} \frac{1}{|x + iy - n|^2}$$

is finite. Moreover, since $\frac{1}{|x + iy - n|^2} \leq \frac{1}{y^2 + n^2} \leq \frac{1}{1 + n^2}$, for $n < 0$, while $\frac{1}{|x + iy - n|^2} \leq \frac{1}{1 + (n-1)^2}$ for $n \geq 2$, we can apply Lebesgue's dominated convergence theorem to obtain that

$$\sum_{n \in \mathbf{Z}} \frac{1}{|x + iy - n|^2} \rightarrow 0 \quad \text{as } |y| \rightarrow +\infty.$$

The same is true for the function $\frac{\pi^2}{\sin^2 \pi z}$. Recall that

$$|\sin \pi z| = \frac{|e^{i\pi z} - e^{-i\pi z}|}{2} \geq \frac{e^{\pi|y|} - e^{-\pi|y|}}{2} = \sinh \pi|y| \rightarrow +\infty$$

as $|y| \rightarrow +\infty$. Then, $\left| \frac{\pi^2}{\sin^2 \pi z} \right| \rightarrow 0$ as $|y| \rightarrow +\infty$ in the set $\{z = x + iy : 0 \leq x \leq 1, |y| > 1\}$.

This proves that h' is bounded, hence constant. But, since h' tends to 0 as $|y| \rightarrow +\infty$, the constant must be 0. This proves (ii).

Thus, h is constant, and it is 0, since h vanishes at the integers. This proves (i), and we are done. \square

Proof of Prop. 8.12. Notice that the last equality in (8.4) follows at once.

For $n \in \mathbf{Z}$, let $z_n = n$. By Thm. 8.9 the function

$$f(z) = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n} \right) e^{z/n}$$

is entire, having simple zeros at the integers. Let $z_0 \in \mathbf{C} \setminus \mathbf{Z} = \Omega$. Since $f(z_0) \neq 0$, there exists a disk $\overline{D(z_0, r_0)} \subseteq \Omega$ on which $\log f(z)$ is well defined and holomorphic. On such disk,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{d}{dz} \log f(z) = \frac{d}{dz} \left[\log \pi z + \sum_{n \neq 0} \left(\log(1 - z/n) + \frac{z}{n} \right) \right] \\ &= \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z - n} + \frac{1}{n} \right) \\ &= \pi \cot \pi z, \end{aligned}$$

by the previous lemma. On the same disk $\overline{D(z_0, r_0)}$ $\log \sin \pi z$ is well defined and its derivative equals $\pi \cot \pi z$. Then, there exists a constant C such that $\log f(z) = \log \sin \pi z + C$, that is, $f(z) = C_1 \sin \pi z$ on $\overline{D(z_0, r_0)}$; hence on Ω and therefore on all of \mathbf{C} . Since $\lim_{z \rightarrow 0} \frac{f(z)}{\sin \pi z} = 1$, $C_1 = 1$ and we are done. \square

8.3. The Mittag-Leffler Theorem. We now address another question. Suppose we are given a sequence of distinct points $\{z_j\}$ such that $\lim_{j \rightarrow +\infty} |z_j| = +\infty$ and a sequence of complex values $\{c_j\}$. Does there exist an entire function f such that $f(z_j) = c_j$, for $j = 1, 2, \dots$? If the points and values are finitely many, say n , then the answer is positive and the function f is actually a polynomial Q given by *Lagrange's interpolation formula*. Indeed, let

$$P(z) = (z - z_1) \cdots (z - z_n),$$

and

$$Q(z) = P(z) \sum_{j=1}^n \frac{c_j}{P'(z_j)(z - z_j)}.$$

It is easy to check that $Q(z_k) = c_k$, $k = 1, 2, \dots, n$, since

$$\begin{aligned} \lim_{z \rightarrow z_k} Q(z) &= \lim_{z \rightarrow z_k} \sum_{j=1}^n \frac{c_j P(z)}{P'(z_j)(z - z_j)} = \lim_{z \rightarrow z_k} \left(\frac{c_k P(z)}{P'(z_k)(z - z_k)} + \sum_{j \neq k} \frac{c_j P(z)}{P'(z_j)(z - z_j)} \right) \\ &= c_k. \end{aligned}$$

Theorem 8.14. *Let $\{z_j\}$ be a sequence of distinct points such that $\lim_{j \rightarrow +\infty} |z_j| = +\infty$ and let $\{c_j\}$ be a sequence of complex values. Let F be an entire function having as zeros exactly the z_j 's. Let $\{q_j\}$ be a sequence of positive integers such that*

$$\sum_{j=1}^{+\infty} \frac{|c_j|}{|F'(z_j)|} \frac{1}{2^{q_j}} < +\infty.$$

Set

$$f(z) = F(z) \sum_{j=1}^{+\infty} \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j}.$$

Then f can be defined at the points z_j by setting $f(z_j) = c_j$ so that f is entire.

Proof. We begin by showing that the series

$$\sum_{j=1}^{+\infty} \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j},$$

converges uniformly on compact subsets of $\mathbf{C} \setminus \{z_1, z_2, \dots\}$. For fixed $R > 0$, there exists $\delta > 0$ such that the disks $D(z_j, \delta)$ with $|z_j| \leq 2R$ are all disjoint. Consider the compact set

$$K = \{|z| \leq R\} \setminus \bigcup_{j=1}^{+\infty} D(z_j, \delta/2).$$

Then, for $z \in K$ we write

$$\sum_{j=1}^{+\infty} \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j} = \sum_{|z_j| \leq 2R} \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j} + \sum_{|z_j| > 2R} \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j}.$$

The second sum converges uniformly on K by Weierstrass' M -test, since, for $z \in K$, $|z/z_j| \leq \frac{1}{2}$ and

$$\left| \frac{c_j}{F'(z_j)(z - z_j)} \left(\frac{z}{z_j} \right)^{q_j} \right| \leq \frac{|c_j|}{|F'(z_j)|R} \frac{1}{2^{q_j}},$$

which is the term of a converging sequence, by assumption. The first sum is a finite sum, and it is finite since $|z - z_j| \geq \delta/2$ for $z \in K$, $j = 1, 2, \dots$.

Therefore, the series represents a meromorphic function in \mathbf{C} , having simple poles at the z_j 's. Therefore, f has only removable singularities in \mathbf{C} , at the points $\{z_j\}$. At these points

$$\lim_{z \rightarrow z_j} f(z) = \lim_{z \rightarrow z_j} \frac{c_j F(z)}{F'(z_j)(z - z_j)} = c_j,$$

as we wished to prove. \square

We conclude this part by answering another question. Suppose we are given a sequence of distinct points $\{z_j\}$ such that $\lim_{j \rightarrow +\infty} |z_j| = +\infty$. Does there exist a function f holomorphic in $\mathbf{C} \setminus \{z_1, z_2, \dots\}$, having as singular points exactly the z_j 's and at these points assigned singular parts?

This can be rephrased as follows. Given the sequence $\{z_j\}$ as above, does there exist a function f having Laurent expansion at z_j given by

$$f(z) = \sum_{k=1}^{\infty} \frac{c_k}{(z - z_j)^k} + h_j(z),$$

with h_j holomorphic in a ngbh of z_j ? This is equivalent to say, given entire functions g_j with $g_j(0) = 0$, then

$$f(z) = g_j \left(\frac{1}{z - z_j} \right) + h_j(z)$$

in a ngbh of z_j , with h_j holomorphic in such a ngbh.

Theorem 8.15. (Mittag-Leffler's Theorem) *Let $\{z_j\}$ be a sequence of distinct points such that $\lim_{j \rightarrow +\infty} |z_j| = +\infty$ and let $\{g_j\}$ be a sequence of entire functions such that $g_j(0) = 0$, $j = 1, 2, \dots$. Then, there exists F holomorphic in $\mathbf{C} \setminus \{z_1, z_2, \dots\}$, having $\{z_1, z_2, \dots\}$ as isolated singularities, with singular part at z_j given by*

$$g_j \left(\frac{1}{z - z_j} \right)$$

for $j = 1, 2, \dots$.

Proof. We notice that, if the z_j 's were finitely many, say N , then

$$\sum_{j=1}^N g_j \left(\frac{1}{z - z_j} \right)$$

would do the job, since the g_j 's have isolated singularities at the z_j 's.

We may assume no $z_j = 0$. The function $g_j(1/(z - z_j))$ is holomorphic for $|z| < |z_j|$. Then

$$g_j \left(\frac{1}{z - z_j} \right) = \sum_{k=0}^{+\infty} a_k^{(j)} z^k, \quad |z| < |z_j|.$$

Let $M_j > 0$ be such that $\sum_{j=1}^{+\infty} M_j < +\infty$. Then, we can select integers N_j be such that

$$\left| g_j \left(\frac{1}{z - z_j} \right) - \sum_{k=0}^{N_j} a_k^{(j)} z^k \right| < M_j$$

for $|z| \leq |z_j|/2$.

Now we set

$$F(z) = \sum_{j=1}^{+\infty} \left(g_j \left(\frac{1}{z - z_j} \right) - \sum_{k=0}^{N_j} a_k^{(j)} z^k \right) =: \sum_{j=1}^{+\infty} f_j(z).$$

We wish to show that the function F has the required properties, that is, is holomorphic in $\mathbf{C} \setminus \{z_1, z_2, \dots\}$ and at each point z_j has singular part equal to $g_j(1/(z - z_j))$.

Notice that each term f_j in the series above is regular (i.e. holomorphic) in $\mathbf{C} \setminus \{z_j\}$ (i.e., \mathbf{C} minus a singleton), and has singular part at z_j given by $g_j(1/(z - z_j))$. Thus, it suffices to show that the series $\sum_{j=1}^{+\infty} f_j$ converges uniformly on compact subsets of $\mathbf{C} \setminus \{z_1, z_2, \dots\}$.

Let $R > 0$ be fixed. We write

$$F(z) = \sum_{|z_j| < 2R} f_j(z) + \sum_{|z_j| \geq 2R} f_j(z).$$

The first sum is a finite sum, and it is holomorphic in $\mathbf{C} \setminus \{z_j : |z_j| < 2R\}$ (and has the prescribed singularities at each z_j , with $|z_j| < 2R$).

For the second sum, notice that if $|z| \leq R$ and $|z_j| \geq 2R$, then $|z| \leq |z_j|/2$ for such j . Then,

$$|f_j(z)| \leq \left| g_j \left(\frac{1}{z - z_j} \right) - \sum_{k=0}^{N_j} a_k^{(j)} z^k \right| < M_j,$$

and by Weierstrass' M -test the series $\sum_{|z_j| \geq 2R} f_j(z)$ converges uniformly on $\{|z| \leq R\}$ to a function holomorphic on $\{|z| \leq R\}$. This proves the theorem. \square

8.4. Jensen's formula. The following is one of the most important results in the analysis of the relation between the growth of the modulus of a holomorphic function and the number of its zeros. We will see its applications in the next section.

Theorem 8.16. (Jensen's formula) *Let $r > 0$ and let f be holomorphic in a ngbh of $\overline{D(0, r)}$. Let z_1, \dots, z_n be the zeros of f in $\overline{D(0, r)}$, counting multiplicity. Assume that f does not vanish at the origin. Then*

$$\log |f(0)| + \log \prod_{j=1}^n \frac{r}{|z_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta. \quad (8.5)$$

Moreover, setting $n(t) = \{\text{number of zeros of } f \text{ in } \overline{D(0, t)}\}$, for $0 \leq t \leq r$, then

$$\log |f(0)| + \int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta. \quad (8.6)$$

Proof. We begin with (8.5) and observe that it holds true if f does not vanish in $\overline{D(0, r)}$ since $\log |f(z)|$ is then harmonic and by the mean value property

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

Assume now that f has exactly one zero $z_1 = re^{i\theta_1}$ on the circle $\{|z| = r\}$ (and none in the interior). Then, we consider $g(z) = f(z)/(z - z_1)$ and apply the mean value property to the harmonic function $\log |g|$ and obtain

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \left(\log |f(re^{i\theta})| - \log |re^{i\theta} - re^{i\theta_1}| \right) d\theta.$$

Since $\log |g(0)| = \log |f(0)| - \log r$, we have

$$\begin{aligned} \log |f(0)| &= \frac{1}{2\pi} \int_0^{2\pi} \left(\log |f(re^{i\theta})| - \log |re^{i\theta} - re^{i\theta_1}| \right) d\theta + \log r \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |re^{i\theta} - re^{i\theta_1}| d\theta + \log r \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta, \end{aligned}$$

using the periodicity of $e^{i\theta}$. Since from Example 5.23 we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0,$$

(8.5) holds also when f has exactly one zero on the circle $\{|z| = r\}$ and none in the interior.

By induction⁹ one can easily see that the conclusion holds also when f has exactly n zero z_1, \dots, z_n on the circle $\{|z| = r\}$ and none in the interior.

Suppose now f has zeros z_1, \dots, z_n inside or on the circle $\{|z| = r\}$. Define

$$F(z) = f(z) \prod_{j=1}^n \frac{r^2 - \overline{z_j}z}{r(z - z_j)}.$$

⁹Make sure you agree with this assertion.

Then F is holomorphic in a nbgh $\overline{D(0, r)}$ of and has no zeros inside the disk $D(0, r)$. Moreover, $|F(z)| = |f(z)|$ on the circle $\{|z| = r\}$ (see Exercise 1.4), then

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

while

$$|F(0)| = |f(0)| \prod_{j=1}^n \frac{r}{|z_j|}.$$

This proves (8.5).

In order to prove (8.6) observe that, we may assume $0 < |z_1| \leq |z_2| \leq \cdots |z_n|$ and we set

$$r_0 = 0, \quad r_j = |z_j|, \text{ for } j = 1, \dots, n, \quad r_{n+1} = r,$$

then $n(t) = j$ for $t \in (r_j, r_{j+1})$, $j = 0, \dots, n$. Therefore,

$$\begin{aligned} \int_0^r \frac{n(t)}{t} dt &= \sum_{j=1}^n \int_{r_j}^{r_{j+1}} \frac{n(t)}{t} dt = \sum_{j=1}^n j \log \left(\frac{r_{j+1}}{r_j} \right) = \log \prod_{j=1}^n \left(\frac{r_{j+1}}{r_j} \right)^j \\ &= \log \prod_{j=1}^n \frac{r}{|z_j|}. \end{aligned}$$

This proves (8.6), hence the theorem. \square

8.5. Entire functions of finite order. In this part we study the relation between the growth of the modulus of an entire function and the distribution of its zeros. We begin by introducing the notion of order of an entire function.

Definition 8.17. An entire function f is said to be of finite *order* ρ , $0 \leq \rho < +\infty$, if

$$\rho = \inf \left\{ \lambda \geq 0 : \sup_{|z|=r} |f(z)| = \mathcal{O}(e^{r^\lambda}) \text{ as } r \rightarrow +\infty \right\}.$$

If there exists no finite $\lambda > 0$ such that $\sup_{|z|=r} |f(z)| = \mathcal{O}(e^{r^\lambda})$, f is said to have *infinite order*. We will write $\text{ord}(f) = \rho$, $0 \leq \rho \leq \infty$.

Notice that for an entire function f of finite order ρ we also have the identity to

$$\rho = \inf \left\{ \lambda \geq 0 : \text{there exist } A, B > 0 : |f(z)| \leq Ae^{B|z|^\lambda} \text{ for all } z \in \mathbf{C} \right\}.$$

A few examples are now in order.

- (1) Polynomials have order 0. For, let N be the degree of $p(z)$. Then

$$|p(z)| \leq Cr^N \leq C_\varepsilon e^{r^\varepsilon} \quad \text{as } r \rightarrow +\infty$$

for all $\varepsilon > 0$, and a suitable constant $C_\varepsilon > 0$.

- (2) The exponential e^z has order 1, and more generally, e^{z^n} have order n :

$$|e^{z^n}| = e^{\text{Re } z^n} \leq e^{|z|^n} \leq e^{r^n},$$

and no smaller power of r would suffice.

- (3) $\sin z$, $\cos z$, $\cosh z$, $\sinh z$ have order 1.
 (4) $\exp\{\exp z\}$ has infinite order.

Definition 8.18. Let $\{z_j\}$ be a sequence of non-zero complex numbers. We call *exponent of convergence* of the sequence, the positive number b , if it exists,

$$b = \inf \left\{ \rho > 0 : \sum_{j=1}^{+\infty} \frac{1}{|z_j|^\rho} < +\infty \right\}.$$

Notice that $\{n\}$ has exponent of convergence equal to 1.

The next result relates the growth of the zeros of an entire function to its order.

Theorem 8.19. *Let f be an entire function of finite order $\rho > 0$. Then, its zeros $\{z_j\}$ different from 0 have finite exponent of convergence b such that*

$$b \leq \rho.$$

Proof. We may assume that $f(0) \neq 0$. For, otherwise we write $f(z) = z^k g(z)$, where g is entire and $g(0) \neq 0$ and $\text{ord}(g) = \text{ord}(f)$, as it is easy to check. Moreover, we may assume that $0 < |z_1| \leq |z_2| \leq \dots$.

Recall that we denote by $n(t)$ the function $n(t) = \{\text{number of zeros of } f \text{ in } \overline{D(0, t)}\}$. Notice that

$$n(r) \log 2 = \int_r^{2r} \frac{n(t)}{t} dt \leq \int_r^{2r} \frac{n(t)}{t} dt \leq \int_0^{2r} \frac{n(t)}{t} dt.$$

Therefore, by Jensen's formula (8.6), since f is of order ρ , for any $\varepsilon > 0$,

$$\begin{aligned} n(r) &\leq \frac{1}{\log 2} \int_0^{2r} \frac{n(t)}{t} dt \leq C \left(\int_0^{2\pi} \log |f(2re^{i\theta})| d\theta - \log |f(0)| \right) \\ &\leq C \left(\int_0^{2\pi} \log (Ae^{(2r)^{\rho+\varepsilon}}) d\theta - \log |f(0)| \right) \\ &\leq Cr^{\rho+\varepsilon}, \end{aligned}$$

for r large enough. If we choose $r = |z_j|$ we have

$$j \leq n(r) \leq Cr^{\rho+\varepsilon} = C|z_j|^{\rho+\varepsilon}.$$

Therefore, for every $\lambda > 1$

$$\sum_{j=1}^{+\infty} \frac{1}{|z_j|^{\lambda(\rho+\varepsilon)}} \leq C \sum_{j=1}^{+\infty} \frac{1}{j^\lambda} < \infty.$$

Then, the exponent of convergence b of $\{z_j\}$ is such that

$$b \leq \lambda(\rho + \varepsilon)$$

for all $\lambda > 1$ and $\varepsilon > 0$; hence $b \leq \rho$, as we wished to prove. \square

Theorem 8.20. *Let $\{z_j\}$ be a sequence of non-zero complex numbers having exponent of convergence ρ_1 . Then, there exists a positive integer p such that the Weierstrass product*

$$\prod_{j=1}^{+\infty} E(z/z_j, p)$$

converges, uniformly on compact sets, to an entire function.

Proof. If $p + 1 > \rho_1$, the series $\sum_{j=1}^{+\infty} (1/|z_j|^{p+1})$ converges. Then, $\sum_{j=1}^{+\infty} (|z|/|z_j|^{p+1})$ converges for every $z \in \mathbb{C}$ and therefore, by Thm. 8.9, $\prod_{j=1}^{+\infty} E(z/z_j, p)$ represents an entire function. \square

Definition 8.21. Let $\{z_j\}$ be a sequence of non-zero complex numbers having finite exponent of convergence. Then the Weierstrass product

$$\prod_{j=1}^{+\infty} E(z/z_j, p)$$

is called a *canonical* Weierstrass product, and the smallest integer p such that

$$\sum_{j=1}^{+\infty} \frac{1}{|z_j|^{p+1}} < +\infty$$

is called the *genus* of the canonical product.

Remark 8.22. Notice that, if $\{z_j\}$ has finite order of convergence b , then its genus p is such that:

- if b is not an integer, $p < b < p + 1$;
- if b is an integer, then $p = b$ if $\sum_{j=1}^{+\infty} (1/|z_j|^b)$ diverges, and $p = b - 1$ if $\sum_{j=1}^{+\infty} (1/|z_j|^b)$ converges.

Next we determine the order of a Weierstrass canonical product.

Theorem 8.23. Let $\{z_j\}$ be a sequence of finite order of convergence b . Then the canonical Weierstrass product has finite order b .

Proof. From Thm. 8.19 we know that $\text{ord}(f) = \rho \geq b$. Then, it suffices to prove that $\rho \leq b$ and in particular that

$$\log \left| \prod_{j=1}^{+\infty} E(z/z_j, p) \right| \leq C|z|^{b+\varepsilon}$$

for some constant $C > 0$ and all $\varepsilon > 0$. Now,

$$\log \left| \prod_{j=1}^{+\infty} E(z/z_j, p) \right| = \sum_{j=1}^{+\infty} \log |E(z/z_j, p)|$$

and for $|z| = r$ we decompose the sum as

$$\sum_{j=1}^{+\infty} \log |E(z/z_j, p)| = \sum_{|z_j| \leq 2r} \log |E(z/z_j, p)| + \sum_{|z_j| > 2r} \log |E(z/z_j, p)| =: \Sigma_1 + \Sigma_2.$$

We begin with Σ_2 . We observe that, for $|w| \leq r < 1$ we have

$$\log |E(w, p)| \leq \frac{|w|^{p+1}}{1-r}, \quad (8.7)$$

For,

$$\begin{aligned} \log |E(w, p)| &= \operatorname{Re} \left(\log(1 - w) + w + \cdots + \frac{z^p}{p} \right) = \operatorname{Re} \left(- \sum_{k=p+1}^{+\infty} \frac{w^k}{k} \right) \\ &\leq |w|^{p+1} \sum_{k=0}^{+\infty} \frac{r^k}{k + p + 1} \leq \frac{|w|^{p+1}}{1 - r}. \end{aligned}$$

Therefore, for $|z| = r$ and $|z_j| = r_j$,

$$\Sigma_2 \leq 2 \sum_{|z_j| > 2r} |z/z_j|^{p+1} \leq 2 \sum_{r_j > 2r} (r/r_j)^{p+1}.$$

Now, recalling that $p \leq b \leq p + 1$, if $b = p + 1$ we immediately have $\Sigma_2 \leq Cr^b$. If $b < p + 1$, let $\varepsilon > 0$ be such that $b + \varepsilon < p + 1$, then

$$\Sigma_2 \leq 2 \sum_{r_j > 2r} (r/r_j)^{b+\varepsilon} \leq 2r^{b+\varepsilon}.$$

Hence, in either case

$$\Sigma_2 \leq Cr^{b+\varepsilon}. \quad (8.8)$$

The estimate companion of (8.7) that we need is, for $|w| \geq r$,

$$\log |E(w, p)| \leq C_r |w|^{\max(p, \varepsilon)}. \quad (8.9)$$

For, if $p > 0$,

$$\log |E(w, p)| = \operatorname{Re} \left(\log(1 - w) + w + \cdots + \frac{z^p}{p} \right) \leq \log(1 + |w|) + \sum_{k=1}^p \frac{|w|^k}{k} \leq C|w|^p,$$

while, if $p = 0$, $\log |E(w, p)| \leq \log(1 + |w|) \leq C|w|^\varepsilon$. Therefore, recalling that $p \leq b$ so that $\max(p, \varepsilon) \leq b + \varepsilon$,

$$\begin{aligned} \Sigma_1 &\leq \sum_{|z_j| \leq 2r} \log |E(z/z_j, p)| \leq C \sum_{r_j \leq 2r} (r/r_j)^{b+\varepsilon} \\ &\leq Cr^{b+\varepsilon} \sum_{r_j \leq 2r} (1/r_j)^{b+\varepsilon} \leq Cr^{b+\varepsilon}. \end{aligned} \quad (8.10)$$

The conclusion now follows from (8.10) and (8.8). \square

In the case of entire functions of finite order it is possible to refine the description contained in the Weierstrass factorization theorem.

Theorem 8.24. (Hadamard's Factorization Thm.) *Let f be an entire function of finite order ρ and let p be the genus of its zeros $\{z_j\}$ different from 0. Let k be the order of zero of f at the origin. Then*

$$f(z) = z^k e^{g(z)} \prod_{j=1}^{+\infty} E(z/z_j, p)$$

where g is a polynomial of degree $\leq \rho$.

Key technical fact needed in the proof of Hadamard's theorem is the following estimate from below for the modulus of a canonical Weierstrass product.

Lemma 8.25. *Let $\{z_j\}$ be a sequence of finite order of convergence b and let p be the genus of the canonical product $\prod_{j=1}^{+\infty} E(z/z_j, p)$. Then, for z outside the closure of the disks $D(z_j, |z_j|^{-p-1})$ we have*

$$\left| \prod_{j=1}^{+\infty} E(z/z_j, p) \right| \geq e^{-r^{b+\varepsilon}},$$

for all r sufficiently large.

We also need the following result.

Lemma 8.26. *Suppose g is entire function and $u = \operatorname{Re}(g)$ satisfies $u(z) \leq Cr^s$ whenever $|z| = r_j$ for a sequence of positive real numbers $\{r_j\}$ that tends to infinity. Then g is a polynomial of degree $\leq s$.*

For the proofs of these lemmas we refer to [SS], Lemma 5.3 and 5.5.

Assuming these lemmas, we prove the theorem.

Proof of Thm. 8.24. By the Weierstrass factorization theorem it suffices to prove that g is a polynomial of degree $\leq \rho$, where

$$e^{g(z)} = \frac{f(z)}{z^k \prod_{j=1}^{+\infty} E(z/z_j, p)}.$$

Consider the circular projections of the circles $D(z_j, |z_j|^{-p-1})$ onto the positive real axis, obtained by rotating them about the origin and taking the intersection with the positive real axis. These projections have length $2|z_j|^{-p-1}$ and their sum is $\leq \sum_{j=1}^{+\infty} 2|z_j|^{-p-1} < +\infty$.

On these sets we have

$$\begin{aligned} |e^{g(z)}| &= \left| \frac{f(z)}{z^k \prod_{j=1}^{+\infty} E(z/z_j, p)} \right| \\ &\leq \frac{e^{r^{\rho+\varepsilon}}}{r^k e^{-r^{b+\varepsilon}}} \leq e^{r^{\rho+b+2\varepsilon}}, \end{aligned}$$

for r sufficiently large, $\varepsilon > 0$.

Therefore, on these sets, when $|z| = r_j$, $j = 1, 2, \dots$, where $r_j = |z_j|$, we have

$$e^{\operatorname{Re} g(z)} \leq C e^{r^{\rho+\varepsilon'}},$$

that is, $\operatorname{Re} g(z) \leq r^{\rho+\varepsilon'}$ when $|z| = r_j$, $j = 1, 2, \dots$. This implies that g is a polynomial of degree $\leq \rho + \varepsilon'$, for all $\varepsilon' > 0$, that is, the degree of g is $\leq \rho$, as we wish to prove. \square

8.6. Exercises.

8.1. Show that $\prod_{n=1}^{+\infty} |1 + i/n|$ converges while $\prod_{n=1}^{+\infty} (1 + i/n)$ does not converge.

8.2. Show that the infinite product $(1+z) \prod_{n=1}^{+\infty} (1+z^{2^n})$ converges for $|z| < 1$ and does not converge when $|z| > 1$. Show that, for $|z| < 1$, the product equals $1/(1-z)$. [Hint: Induction.]

8.3. Show that

$$\cos \pi z = \prod_{n=1}^{+\infty} \left(1 - \frac{4z^2}{(2n-1)^2} \right).$$

Find a factorization for $\sinh z$ and $\cosh z$.

8.4. Let f be an entire function and n a positive integer. Show that there exists an entire function h such that $h^n = f$ if and only if the orders of the zeros of f are all divisible by n .

8.5. Let f, g be entire functions.

- (i) Show that there exist entire f_1, g_1 and h such that $f = hf_1$, $g = hg_1$ and f_1 and g_1 have no common zero.
- (ii) For f, g and h as above, show that there exist entire functions A and B such that $Af + Bg = h$. [Hint: Mittag-Leffler.]

8.6. Let $\{f_n\}$ be a sequence of holomorphic functions on a given domain Ω . Suppose that $\prod_{n=1}^{+\infty} f_n$ converges uniformly on compact subsets of Ω to f .

- (i) Show that

$$\sum_{k=1}^{+\infty} \left(f'_k(z) \prod_{n \neq k} f_n(z) \right)$$

converges uniformly on compact subsets of Ω , and that such limit equals f' .

- (ii) Suppose that f is non-zero on a given compact set $K \subseteq \Omega$. Show that

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{+\infty} \frac{f'_n(z)}{f_n(z)}$$

and that the convergence is uniform on K .

8.7. Show that if f is an entire function for which

$$\sup_{|z|=r} |f(z)| \leq Cr^n$$

for some integer n and all $r > 0$, then f is a polynomial.

8.8. Show that if f is an entire function of finite order ρ and we write $M(r) = \sup_{|z|=r} |f(z)|$, then

$$\rho = \limsup_{r \rightarrow +\infty} \frac{\log \log M(r)}{\log r}.$$

8.9. Given entire functions f, g of finite order $\text{ord}(f), \text{ord}(g)$ resp., show that $f + g$ and fg have finite order $\leq \max(\text{ord}(f), \text{ord}(g))$. Moreover, if $\text{ord}(f) \neq \text{ord}(g)$, show that $\text{ord}(f + g) = \max(\text{ord}(f), \text{ord}(g))$.

8.10. Show that f is an entire function of finite order ρ if and only if f' is an entire function of finite order ρ .

8.11. Let $f(z) = e^{e^z} - 1$. Find the zeros $\{z_j\}$ of f and determine the exponent of convergence, if it exists finite, of $\{z_j\}$.

8.12. Determine the exponent of convergence, if it exists finite, of the sequence of points $\{m+in : n, m \in \mathbf{Z}, (m, n) \neq (0, 0)\}$.

9. ANALYTIC CONTINUATION

In this section we analyze a phenomenon that is peculiar of analytic functions, as a consequence of the identity principle; once a holomorphic function is assigned in a ngbh of a given point, its values are uniquely determined in its natural region of holomorphicity.

9.1. The monodromy theorem. Let f be a holomorphic function in a ngbh of z_0 . Then f admits power series expansion in disk $D(z_0, r_0)$ and we say that f is *analytic* in that disk. If $z_1 \in D(z_0, r_0)$, then f admits power series expansion in a disk $D(z_1, r_1)$, and it is well possible that $D(z_1, r_1) \not\subseteq D(z_0, r_0)$. We have the following.

Definition 9.1. Let $D(z_0, r_0), D(z_1, r_1)$ be as above, and let f_0, f_1 be the analytic functions given by the power series expansion centered at z_0 and z_1 resp.

Then we call each pair (f_0, D_0) and (f_1, D_1) an *analytic element* and say that (f_1, D_1) is direct analytic continuation of (f_0, D_0) .

Notice that, in the above situation, i.e. with $f_0 = f_1$ on $D_0 \cap D_1$, setting

$$f(z) = \begin{cases} f_0(z) & z \in D_0 \\ f_1(z) & z \in D_1, \end{cases}$$

we obtain a holomorphic function on $D_0 \cup D_1$.

Now we want to define analytic continuation along *paths*. We call *path* the (image of) an injective continuous function

$$\gamma : [a, b] \rightarrow \mathbf{C}.$$

Let $\gamma : [a, b] \rightarrow \mathbf{C}$ a path and let $a_0 = a < a_1 < \dots < a_n = b$ be a partition of $[a, b]$. We say that the sequence of disks $\{D_0, D_1, \dots, D_n\}$ is *connected* along γ if $\gamma([a_j, a_{j+1}]) \subseteq D_j$. In this case, $\gamma(a_{j+1}) \in D_j \cap D_{j+1}$.

Definition 9.2. Let (f_0, D_0) be an analytic element, $\{D_0, D_1, \dots, D_n\}$ a sequence of disks *connected* along a path γ . A sequence of pairs

$$\{(f_0, D_0), (f_1, D_1), \dots, (f_n, D_n)\}$$

is called *analytic continuation* of (f_0, D_0) along γ if (f_{j+1}, D_{j+1}) is direct analytic continuation of (f_j, D_j) for $j = 0, \dots, n-1$.

Theorem 9.3. Let $\{(f_0, D_0), (f_1, D_1), \dots, (f_n, D_n)\}$ be an analytic continuation of (f_0, D_0) along γ . Suppose that $g_0 = f_0$ in a ngbh of z_0 , with $z_0 = \gamma(a)$, and that $\{(g_0, E_0), \dots, (g_m, E_m)\}$ is an analytic continuation of (g_0, E_0) along γ .

Then (g_m, E_m) is direct analytic continuation of (f_n, D_n) .

Proof. Assume first that each E_j has the same center as D_j , $j = 0, 1, \dots, n = m$. Then $f_0 = f_1$ on $D_0 \cap D_1$ and $g_0 = g_1$ on $E_0 \cap E_1$.

The holomorphic function f defined by f_0 on D_0 and by f_1 on D_1 coincides with the holomorphic function g defined by g_0 on E_0 and by g_1 on E_1 on $(D_0 \cup D_1) \cap (E_0 \cup E_1)$, which is connected and non-empty, since $\gamma(a_1) \in D_0 \cap D_1 \cap E_0 \cap E_1$. Then, $f_1 = g_1$ on $D_1 \cap E_1$, i.e. (g_1, E_1) is direct analytic continuation of (f_1, D_1) . Now we proceed by induction to prove the statement in this case.

Next we show that the continuation is independent of the choice of the partition on γ . Since given any two partitions there exists a common refinement, it suffices to prove that the continuation is independent of whether it is done with respect to a given partition or to any refinement of the given partition.

Let then $\{D_0, \dots, D_n\}$ be connected along γ , $a_0 = a < a_1 < \dots < a_n = b$ being the corresponding partition of $[a, b]$. Let $a_0 = a < a_1 < \dots < a_j < a_* < a_{j+1} < \dots < a_n = b$ be a refinement. Consider the analytic continuations

$$\{(f_0, D_0), (f_1, D_1), \dots, (f_n, D_n)\}$$

and

$$\{(g_0, E_0), (g_1, E_1), \dots, (g_j, E_j), (g_*, E_*), (g_{j+1}, E_{j+1}), \dots, (g_n, E_n)\},$$

with $f_0 = g_0$ in some ngbh of z_0 . From the first part of the proof we have that (g_k, E_k) is direct analytic continuation of (f_k, D_k) , for $k \leq j$.

Next, since $g_* = g_j$ on $E_* \cap E_j$ and $f_j = g_j$ on $D_j \cap E_j$, it follows that

$$g_* = f_j \quad \text{on } D_j \cap E_j \cap E_*,$$

where $D_j \cap E_j \cap E_*$ is non-empty since it contains $\gamma(a_*)$. Finally, we repeat the first part of the proof on the portion of γ from a_* to $a_n = b$. \square

Theorem 9.4 (Monodromy theorem). *Let Ω be a domain, f a holomorphic function in a ngbh of $z_0 \in \Omega$. Let γ, σ be two paths from z_0 to $w_0 \in \Omega$, contained in Ω . Suppose that*

- (i) γ is homotopic to σ in Ω ;
- (ii) f can be analytically continued along any path in Ω .

Let $(f_\gamma, D_\gamma), (f_\sigma, D_\sigma)$ be the analytic continuation of (f_0, D_0) along γ and σ , resp. Then $f_\gamma = f_\sigma$ in a ngbh of w_0 .

Proof. Let Ψ be a homotopy of γ with σ , i.e.

$$\Psi : [a, b] \times [0, 1] \rightarrow \Omega$$

with Ψ continuous, $\Psi(\cdot, 0) = \gamma$, $\Psi(\cdot, 1) = \sigma$, $\Psi(a, s) = z_0$, $\Psi(b, s) = w_0$ for all $s \in [0, 1]$, and $\Psi(\cdot, s) = \gamma_s$ a path for all $s \in [0, 1]$.

Let

$$A = \{s \in [0, 1] : f_{\Psi(\cdot, s)} = f_\gamma \text{ in some ngbh of } w_0\}.$$

We wish to show that: (i) A is open and non-empty in $[0, 1]$ and (ii) closed in $[0, 1]$ — thus showing that $A = [0, 1]$; hence the theorem.

(i) A is clearly non-empty. Let $s_0 \in A$, $s_0 > 0$. Let D_0, D_1, \dots, D_n be connected along $\Psi(\cdot, s_0) = \gamma_{s_0}$ and let $\{(f, D_0), (f_1, D_1), \dots, (f_n, D_n)\}$ be an analytic continuation along γ_{s_0} (which exists by the hypothesis on f). Let

$$\varepsilon_0 = \text{dist} \left(\gamma_{s_0}([a, b]), (D_0 \cup \dots \cup D_n)^c \right).$$

Let $\delta_0 > 0$ be such that $|\Psi(\cdot, s) - \Psi(\cdot, s_0)| < \varepsilon_0/2$ for $|s - s_0| < \delta_0$. This implies that (f_{n_s}, D_{n_s}) is direct analytic continuation of $(f_{n_{s_0}}, D_{n_{s_0}})$. Then, it follows that $(s_0 - \delta_0, s_0 + \delta_0) \subseteq A$.

(ii) Let $s_k \in A$, $k = 1, 2, \dots$, and let $s_k \rightarrow s_*$. Let (f_*, D_*) be analytic continuation of (f, D_0) along $\gamma_{s_*} = \Psi(\cdot, s_*)$. Again, let

$$\varepsilon = \text{dist} \left(\gamma_{s_*}([a, b]), (E_0 \cup \dots \cup E_m)^c \right),$$

where E_0, \dots, E_m are connected along γ_{s_*} and f_* is the holomorphic function defined by the analytic continuation along γ_{s_*} . Let k_0 be such that for $k \geq k_0$ $|\Psi(\cdot, s_k) - \Psi(\cdot, s_{s_*})| < \varepsilon/2$.

We wish to show that $f_* = f_\gamma$ in a ngbh of w_0 . As before, (f_*, E_*) is direct analytic continuation of $(f_{n_{s_k}}, E_{n_{s_k}})$, for $k \geq k_0$. The desired conclusion now follows. \square

9.2. The gamma function. The subject of this and of the next section is to introduce probably the two most famous and studied non-elementary functions: the Euler *gamma function* $\Gamma(z)$ and the Riemann *zeta function* $\zeta(s)$.

Definition 9.5. For $\operatorname{Re} z > 0$ we set

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

We first state a general result about the holomorphicity of functions defined by integrals. For its proof we refer to [L].

Proposition 9.6. *Let I be an interval on the real line \mathbf{R} and let $A \subseteq \mathbf{C}$ be an open set. Let $F = F(z, t) : A \times I \rightarrow \mathbf{C}$ be continuous. Furthermore assume that:*

(i) *for every compact set $K \subseteq A$ the integral*

$$\int_I F(z, t) dt$$

exists finite (possibly as an improper integral) uniformly for $z \in K$;

(ii) *the function $A \ni z \mapsto F(z, t)$ is holomorphic in A , for all $t \in I$.*

Then, setting

$$f(z) = \int_I F(z, t) dt,$$

f is holomorphic function on A and

$$f'(z) = \int_I \partial_z F(z, t) dt.$$

Theorem 9.7. *The function $\Gamma(z)$ is holomorphic for $\operatorname{Re} z > 0$. Moreover, it can be analytically continued in the domain $\Omega = \mathbf{C} \setminus \{0, -1, -2, \dots\}$. At the non-positive integers $z = -n$, with $n = 0, 1, 2, \dots$, the function $\Gamma(z)$ has simple poles with residues $(-1)^n/n!$.*

Proof. It follows from the previous proposition that $\Gamma(z)$ is holomorphic for $\operatorname{Re} z > 0$, since for $t > 0$, $|t^z| = t^x$ so that the integral defining $\Gamma(z)$ converges absolutely.

Next we notice that, integrating by parts we have

$$\begin{aligned} \int_0^{+\infty} t^{z-1} e^{-t} dt &= \lim_{a \rightarrow 0^+, b \rightarrow +\infty} \int_a^b t^{z-1} e^{-t} dt \\ &= \lim_{a \rightarrow 0^+, b \rightarrow +\infty} \left(\frac{1}{z} t^z e^{-t} \Big|_a^b + \frac{1}{z} \int_a^b t^z e^{-t} dt \right) \\ &= \frac{1}{z} \int_0^{+\infty} t^z e^{-t} dt. \end{aligned}$$

Notice that we have obtained the identity

$$z\Gamma(z) = \Gamma(z+1), \quad (9.1)$$

valid when $\operatorname{Re} z > 0$.

The expression $\frac{1}{z} \int_0^{+\infty} t^z e^{-t} dt$ on the right hand side above defines a function holomorphic on $\{\operatorname{Re} z > -1\} \setminus \{z = 0\}$ that coincides with $\Gamma(z)$ on the set $\{\operatorname{Re} z > 0\}$. Hence, the function Γ can be analytically continued on the set $\{\operatorname{Re} z > -1\} \setminus \{z = 0\}$.

Assume by induction that, for $n \geq 2$,

$$\Gamma(z) = \frac{1}{z(z+1) \cdots (z+n-1)} \int_0^{+\infty} t^{z+n-1} e^{-t} dt,$$

for $z \in \{\operatorname{Re} z > -n\} \setminus \{0, -1, \dots, -n+1\}$.

Arguing as before, integrating by parts again we obtain

$$\begin{aligned} \Gamma(z) &= \frac{1}{z(z+1) \cdots (z+n-1)(z+n)} \int_0^{+\infty} t^{z+n} e^{-t} dt \\ &= \left(\prod_{j=0}^n \frac{1}{z+j} \right) \Gamma(z+n+1), \end{aligned}$$

for $\operatorname{Re} z > -n-1$ and $z \neq 0, -1, \dots, -n$.

This shows that, $\Gamma(z)$ is holomorphic for $z \in \mathbf{C} \setminus \{0, -1, -2, \dots\}$. Moreover, in the non-positive integers Γ has simple poles with residues given by

$$\begin{aligned} \lim_{z \rightarrow -n} (z+n)\Gamma(z) &= \lim_{z \rightarrow -n} (z+n) \prod_{j=0, \dots, n} \frac{1}{z+j} \int_0^{+\infty} t^{z+n} e^{-t} dt \\ &= \prod_{j=0, \dots, n} \frac{1}{j-n} \int_0^{+\infty} e^{-t} dt \\ &= \frac{(-1)^n}{n!}. \quad \square \end{aligned}$$

In the next proposition we collect a few facts that emerged from the previous proof.

Proposition 9.8. *Let $\Omega = \mathbf{C} \setminus \{0, -1, -2, \dots\}$. The gamma function $\Gamma(z)$ satisfies the following properties:*

- (i) $z\Gamma(z) = \Gamma(z+1)$ for all $z \in \Omega$;
- (ii) $\Gamma(n+1) = n!$;
- (iii) $\Gamma(1/2) = \sqrt{\pi}$.

Proof. Equation (9.1) gives (i).

Since $\Gamma(1) = 1$, (ii) follows from (i) inductively.

Condition (iii) follows from the well-known identity $\int_{-\infty}^{+\infty} e^{-t^2/2} dt = \sqrt{\pi}$ and the change of variables $x = \sqrt{t}$. \square

Theorem 9.9. Set γ be the (positive) constant¹⁰ given by the equality

$$e^\gamma = \prod_{n=1}^{+\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}. \quad (9.2)$$

Let $\Omega = \mathbf{C} \setminus \{0, -1, -2, \dots\}$. Then, for all $z \in \Omega$,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}. \quad (9.3)$$

Proof. We begin with some remark on the constant γ . We know that the infinite product converges, and by taking the logarithm on both sides of (9.2) it follows that

$$\gamma = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right). \quad (9.4)$$

Let $G(z)$ be the function on the right hand side in (9.3), that is,

$$G(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

We wish to show that $G = \Gamma$ on Ω .

Consider the (canonical) Weierstrass product

$$H(z) = \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Then H is an entire function having simple zeros at the negative integers. Therefore, $1/H$ is a meromorphic function having simple poles at the negative integers. Then,

$$\begin{aligned} \frac{e^{-\gamma z}}{zH(z)} &= \frac{e^{-\gamma z}}{z \prod_{k=1}^{+\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}} = \lim_{n \rightarrow +\infty} \frac{e^{-\gamma z}}{z \prod_{k=1}^n \left(1 + \frac{z}{k}\right) e^{-z/k}} \\ &= \lim_{n \rightarrow +\infty} \frac{e^{-\gamma z}}{z} \prod_{k=1}^n \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \\ &= \lim_{n \rightarrow +\infty} \frac{e^{-\gamma z}}{z} \prod_{k=1}^n \frac{ke^{z/k}}{z+k} \\ &= \lim_{n \rightarrow +\infty} \frac{e^{-\gamma z} n! e^{(1+\frac{1}{2}+\dots+\frac{1}{n})z}}{z(z+1)\cdots(z+n)} \\ &= \lim_{n \rightarrow +\infty} \frac{e^{-(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n)z} n! e^{(1+\frac{1}{2}+\dots+\frac{1}{n})z}}{z(z+1)\cdots(z+n)} \\ &= \lim_{n \rightarrow +\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}, \end{aligned}$$

where we have used (9.4). The convergence is uniform on compact subsets of Ω , as it is easy to check.

¹⁰ γ is called the *Euler-Mascheroni constant*.

Therefore, uniformly on compact subsets of Ω , $G(z) = \lim_{n \rightarrow +\infty} g_n(z)$, where

$$g_n(z) = \frac{n!n^z}{z(z+1) \cdots (z+n)}. \quad (9.5)$$

On the other hand, we now make the following claims:

(C.1) $\Gamma(z) = \lim_{n \rightarrow +\infty} f_n(z)$ uniformly on compact subsets of $\{\operatorname{Re} z > 0\}$, where

$$f_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt;$$

(C.2) for $x \geq 1$,

$$f_n(x) = \frac{n!n^x}{x(x+1) \cdots (x+n)};$$

(C.3) $f_n(x) = g_n(x)$, for $x \geq 1$.

These three facts imply that $\Gamma(z) = G(z)$ for $z = x$ and $x \geq 1$; hence the desired conclusion. Thus, we prove the claims.

In order to prove (C.1) notice that, for $0 \leq t \leq n$

$$\left(1 - \frac{t}{n}\right)^n \leq e^{-t};$$

it suffices to recall that, for $0 < s < 1$, $\log(1-s) = -\sum_{k=1}^{+\infty} s^k$. Hence, we can use Lebesgue's Dominated Convergence Thm. to obtain the conclusion.

(C.2) follows easily by integrating by parts since, for $x \geq 1$, the boundary terms vanish:

$$\begin{aligned} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt &= \frac{1}{x} \left(1 - \frac{t}{n}\right)^n t^x \Big|_0^n + \frac{n}{nx} \int_0^n \left(1 - \frac{t}{n}\right)^{n-1} t^x dt \\ &= \frac{n(n-1)}{n^2 x(x+1)} \int_0^n \left(1 - \frac{t}{n}\right)^{n-2} t^{x+1} dt \\ &= \frac{n!}{n^n x(x+1) \cdots (x+n-1)} \int_0^n t^{x+n-1} dt \\ &= \frac{n!n^x}{x(x+1) \cdots (x+n)}. \end{aligned}$$

Finally, (C.3) follows from (9.5), and we are done. \square

Corollary 9.10. *For all $z \in \Omega$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof. It follows from the Thm. that

$$\begin{aligned} \Gamma(z)\Gamma(-z) &= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \cdot \frac{e^{\gamma z}}{-z} \prod_{n=1}^{+\infty} \left(1 - \frac{z}{n}\right)^{-1} e^{-z/n} \\ &= -\frac{1}{z^2} \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{n^2}\right)^{-1} \\ &= -\frac{\pi}{z \sin \pi z}. \end{aligned}$$

Therefore, $-z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin \pi z}$, and the conclusion follows from Prop. 9.8 (i). \square

10. THE RIEMANN ZETA FUNCTION AND THE PRIME NUMBERS THEOREM

10.1. The Riemann zeta function. The *Riemann zeta function* is defined as

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}. \quad (10.1)$$

Since for $z = x + iy$, $|1/n^z| = 1/n^x$, so that the series converges for all z with $\operatorname{Re} z > 1$.

We now emphasise the relation between the gamma and the zeta function.

Proposition 10.1. *For $\operatorname{Re} z > 1$ we have*

$$\zeta(z)\Gamma(z) = \int_0^{+\infty} (e^t - 1)^{-1} t^{z-1} dt.$$

Proof. Making the change the variables $t = nu$ we see that, when $\operatorname{Re} z > 0$,

$$\Gamma(z) = n^z \int_0^{+\infty} u^{z-1} e^{-nu} du.$$

Therefore, when $\operatorname{Re} z > 1$,

$$\begin{aligned} \zeta(z)\Gamma(z) &= \sum_{n=1}^{+\infty} \frac{1}{n^z} \Gamma(z) = \sum_{n=1}^{+\infty} \int_0^{+\infty} t^{z-1} e^{-nt} dt \\ &= \int_0^{+\infty} (e^t - 1)^{-1} t^{z-1} dt. \end{aligned}$$

The last equality follows since

$$\sum_{n=1}^{+\infty} |t^{z-1} e^{-nt}| = \sum_{n=1}^{+\infty} t^{x-1} e^{-nt} = (e^t - 1)^{-1} t^{x-1},$$

which is absolutely integrable in t on $(0, +\infty)$, for $x > 1$. \square

We wish to use the identity in Prop. 10.1 to analytically continue the zeta function to a larger domain.

Theorem 10.2. *The zeta function can be analytically continued to be meromorphic in plane with a simple pole in $z = 1$, with residue 1. Moreover, for $-1 < \operatorname{Re} z < 0$ it satisfies Riemann functional equation*

$$\zeta(z) = 2(2\pi)^{z-1} \Gamma(1-z) \zeta(1-z) \sin(\pi z/2). \quad (10.2)$$

Proof. Notice that, the function $(e^t - 1)^{-1} - t^{-1}$ remains bounded in a ngbh of $t = 0$. Therefore, the integral

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$$

converges uniformly for z in compact subsets of $\{\operatorname{Re} z > 0\}$.

Hence, by Prop. 10.1,

$$\begin{aligned} \zeta(z)\Gamma(z) &= \int_0^{+\infty} (e^t - 1)^{-1} t^{z-1} dt \\ &= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt + \frac{1}{z-1} + \int_1^{+\infty} \frac{t^{z-1}}{e^t - 1} dt. \end{aligned} \quad (10.3)$$

The right hand side makes sense for $\operatorname{Re} z > 0$ and $z \neq 1$, so we can extend $\zeta(z)$ by setting

$$\zeta(z) = \Gamma(z)^{-1} R(z),$$

where $R(z)$ denotes the right hand side in (10.3), for $\{\operatorname{Re} z > 0, z \neq 1\}$.

Next, suppose $0 < \operatorname{Re} z < 1$. Then,

$$\frac{1}{z-1} = - \int_1^{+\infty} t^{z-2} dt,$$

so that identity (10.3) becomes

$$\zeta(z)\Gamma(z) = \int_0^{+\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt, \quad (10.4)$$

which in turn can be written as, for $0 < \operatorname{Re} z < 1$,

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt - \frac{1}{2z} + \int_1^{+\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt. \quad (10.5)$$

Now, both integrals converge when $-1 < \operatorname{Re} z < 1$, $z \neq 0$. Then equation (10.5) can be used to define $\zeta(z)$ when $-1 < \operatorname{Re} z < 1$, $z \neq 0$, and actually also for $z = 0$, since in order to define $\zeta(z)$ one must divide the right hand side in (10.5) by $\Gamma(z)$ that has a simple pole at $z = 0$; hence $1/\Gamma(z)$ has a simple zero. This allows to see that the zeta function is actually regular at $z = 0$.

This fact, combined with its original definition, shows that $\zeta(z)$ can be defined in $\{\operatorname{Re} z > -1, z \neq 1\}$, having a simple pole in $z = 1$.

We now claim that, for $-1 < \operatorname{Re} z < 0$,

$$(C.1) \quad \zeta(z)\Gamma(z) = 2 \int_0^{+\infty} \left(\sum_{n=1}^{+\infty} \frac{1}{t^2 + 4n^2\pi^2} \right) t^z dt;$$

$$(C.2) \quad \int_0^{+\infty} \left(\sum_{n=1}^{+\infty} \frac{1}{t^2 + 4n^2\pi^2} \right) t^z dt = (2\pi)^{z-1} \zeta(1-z) \int_0^{+\infty} \frac{t^z}{t^2 + 1} dt;$$

$$(C.3) \quad \int_0^{+\infty} \frac{t^z}{t^2 + 1} dt = \frac{\pi}{2} \sec(\pi z/2).$$

Assuming the claims for now, using Cor. 9.10 we obtain (10.2). For, if $-1 < \operatorname{Re} z < 0$,

$$\begin{aligned} \zeta(z) &= \frac{1}{\Gamma(z)} 2(2\pi)^{z-1} \zeta(1-z) \frac{\pi}{2} \sec(\pi z/2) \\ &= \frac{\Gamma(1-z) \sin \pi z}{\pi} 2(2\pi)^{z-1} \zeta(1-z) \frac{\pi}{2} \sec(\pi z/2) \\ &= \frac{\Gamma(1-z) 2 \sin(\pi z/2) \cos(\pi z/2)}{\pi} 2(2\pi)^{z-1} \zeta(1-z) \frac{\pi}{2} \sec(\pi z/2) \\ &= 2(2\pi)^{z-1} \Gamma(1-z) \zeta(1-z) \sin(\pi z/2). \end{aligned}$$

Notice that the right hand side is analytic for $\operatorname{Re} z < 0$. Thus, we may analytically continue ζ to the whole left-half plane; hence to all of \mathbf{C} , taken away $z = 1$, its only pole.

We only need to prove the claims. For them, we momentarily refer to [C], p. 191-192.

This concludes the proof. \square

We now draw some conclusions from Riemann functional equation (10.2). Since $\Gamma(1-z)$ has simple poles at $z = 2, 3, \dots$ and $\zeta(z)$ is regular at those points, we must have

$$\zeta(1-z) \sin(\pi z/2) = 0$$

for $z = 2, 3, \dots$. Since $\sin(\pi z/2)$ has simple zeros at the even integers, $\zeta(1-z) = 0$ for $z = 3, 5, \dots$. Hence,

$$\zeta(z) = 0 \quad \text{for } z = -2, -4, \dots$$

In the same way, we see that the zeta function has no other zeros outside the *critical strip* $\{0 \leq \operatorname{Re} z \leq 1, z \neq 1\}$.

The zeros $z = -2, -4, \dots$, are called the *trivial zeros* for the zeta function.

The Riemann Hypothesis. *The zeros of the zeta function in the critical strip all lie on the line $\operatorname{Re} z = \frac{1}{2}$.*

Theorem 10.3. (Euler's Theorem) *If $\operatorname{Re} z > 1$, then*

$$\zeta(z) = \prod_{n=1}^{+\infty} \frac{1}{1 - p_n^{-z}},$$

where $\{p_n\}$ is the sequence of prime numbers.

Proof. We first use the geometric series to write

$$\frac{1}{1 - p_n^{-z}} = \sum_{m=0}^{+\infty} p_n^{-mz}.$$

Next,

$$\begin{aligned} \prod_{k=1}^n \frac{1}{1 - p_k^{-z}} &= \prod_{k=1}^n \sum_{m=0}^{+\infty} p_k^{-mz} \\ &= \sum_{j=1}^{+\infty} n_j^{-z}, \end{aligned}$$

where the sum is over the integers n_1, n_2, \dots that can be factored in terms of the primes p_1, p_2, \dots, p_n alone. Letting $n \rightarrow +\infty$ we obtain the conclusion. \square

The product expansion for the zeta function is usually written as

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \operatorname{Re} s > 1. \quad (10.6)$$

Equation (10.6) is called *Euler's formula*.

Notice that the argument in the previous proof also shows that the product of the factors $\frac{1}{1-1/p_k}$ for the first k primes p_1, \dots, p_n equals

$$\prod_{k=1}^n \frac{1}{1 - \frac{1}{p_k}} = \sum_{j=1}^{+\infty} \frac{1}{n_j},$$

where the sum is over the integers n_1, n_2, \dots that can be factored in terms of the primes p_1, p_2, \dots, p_n alone. By passing to the limit for $n \rightarrow +\infty$ we see that

$$\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p}} = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.$$

Hence, there have to be infinitely many prime numbers.

10.2. The prime numbers theorem. The theory of the zeta functions can be used to estimate the number of primes that are less or equal to a given positive number x . For $x > 0$, set

$$\pi(x) = \text{number of primes } \leq x.$$

Theorem 10.4. *For $x \rightarrow +\infty$ we have*

$$\pi(x) \sim \frac{x}{\log x}.$$

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