

Mean density of inhomogeneous Boolean models with lower dimensional typical grain

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Abstract

The mean density of a random closed set Θ in \mathbb{R}^d with Hausdorff dimension n is the Radon-Nikodym derivative of the expected measure $\mathbb{E}[\mathcal{H}^n(\Theta \cap \cdot)]$ induced by Θ with respect to the usual d -dimensional Lebesgue measure. We consider here inhomogeneous Boolean models with lower dimensional typical grain. Under general regularity assumptions on the typical grain, related to the existence of its Minkowski content, and on the intensity measure of the underlying Poisson point process, we prove an explicit formula for the mean density. The proof of such formula provides as by-product estimators for the mean density in terms of the empirical capacity functional, which turns to be closely related to the well known random variable density estimation by histograms in the extreme case $n = 0$. Particular cases and examples are also discussed.

Keywords: Boolean model; random measure; mean density; Minkowski content.

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1 Introduction

In many real applications it is of interest to study random closed sets at different Hausdorff dimensions and their induced random measure; in particular several problems are related to the estimation of the density, said *mean density*, of the *expected measure*

$$\mathbb{E}[\mu_\Theta](B) := \mathbb{E}[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}$$

induced by a random closed set $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \rightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$ in \mathbb{R}^d with Hausdorff dimension n (\mathcal{H}^n , $\mathcal{B}_{\mathbb{R}^d}$, \mathbb{F} and $\sigma_{\mathbb{F}}$ denote here the n -dimensional Hausdorff measure, the Borel σ -algebra of \mathbb{R}^d , the class of the closed subsets in \mathbb{R}^d and the σ -algebra generated by the so-called hit-or-miss topology [16], respectively). While the extreme cases $n = d$ and $n = 0$ are easy to handle with elementary analytical tools, problems arise when $0 < n < d$. Indeed, while the expected measure induced by a d -dimensional random set is always absolutely continuous with respect to the d -dimensional Lebesgue measure and its density can be easily obtained by applying Fubini's theorem (as stated in the early work by Robbins [19]), and the mean density of a random point is given by its probability density function (and so the problem of its estimation has been largely solved since long in nowadays standard literature, by means of either histograms, or kernel estimators (e.g. see [21])), when we deal with a general lower dimensional random closed set it can be more demanding to check that the induced expected measure is absolutely continuous and to compute and estimate its density. Even if inhomogeneous random closed sets appear frequently in real

applications, only the stationary case has been extensively studied so far (e.g., see [7, 22] and reference therein). Indeed, although some results about the mean densities of certain inhomogeneous random closed sets are available in current literature (e.g., see [6, 14, 15, 20, 24]), mainly via tools from integral geometry and stereology, the pointwise estimation of the mean density of nontrivial lower dimensional inhomogeneous random closed sets is still an open problem, whenever gradient structures [13] or local stationarity cannot be assumed.

Our goal is to obtain a pointwise result for the mean density of a wide class of inhomogeneous random sets which might be of interest from a statistical point of view as well. We consider here inhomogeneous Boolean models with lower dimensional typical grain, being considered basic random set models in stochastic geometry [4], but we do not exclude that our approach could be applied to other kinds of inhomogeneous lower dimensional random closed sets in further developments. The main result of the paper is stated in Theorem 9, where we prove an explicit formula for the mean density; we point out that, even if such formula could be obtained in a more direct way via the well known Campbell's formula (see Remark 11), the proof we propose here provides as a by-product estimators of the mean density, answering, in the case of Boolean models, to an open problem in [1]. To this end, the classical Minkowski content notion plays a central role throughout the paper; as a matter of fact, the regularity assumptions on the typical grain we require are closely related to the well known general assumptions which guarantee the existence of the Minkowski content of deterministic closed sets.

The paper is organized as follows. In Section 2 we recall some basic definitions and preliminary results useful for the sequel. In Section 3 we prove a generalization of the Minkowski content of closed subsets of \mathbb{R}^d (Theorem 6); we will apply such result in the final part of the proof of Theorem 9. In Section 4 we introduce a wide class of Boolean models in \mathbb{R}^d with lower dimensional typical grain. We prove that the expected measures induced by the Boolean models we consider are absolutely continuous with respect to the usual Lebesgue measure (Lemma 10) and we provide an explicit formula for their density. Such formula simplifies in the special cases in which the Boolean model is assumed to be stationary (Corollary 12), or to have deterministic typical grain (Corollary 13). A simple example of inhomogeneous segment Boolean model and links with current literature are also provided. Finally, the problem of the estimation of the mean density is considered in Section 5; in particular, we define an estimator of the mean density (Proposition 15), which can be considered as the generalization to the case $0 < n < d$ of the classical random variable density estimation by histograms in the extreme case $n = 0$.

2 Basic notation and preliminaries

We recall that \mathcal{H}^0 is the usual counting measure, $\mathcal{H}^n(B)$ coincides with the classical n -dimensional measure of B for $1 \leq n < d$ integer if $B \in \mathcal{B}_{\mathbb{R}^d}$ is contained in a C^1 n -dimensional manifold embedded in \mathbb{R}^d , $\mathcal{H}^d(B)$ coincides with the usual d -dimensional Lebesgue measure of B for any Borel set $B \subset \mathbb{R}^d$. Throughout the paper dx stands for $\mathcal{H}^d(dx)$. A closed subset S of \mathbb{R}^d is said to be *countably \mathcal{H}^n -rectifiable* if there exist countably many n -dimensional Lipschitz graphs $\Gamma_i \subset \mathbb{R}^d$ such that $S \setminus \cup_i \Gamma_i$ is \mathcal{H}^n -negligible. (For definitions and basic properties of Hausdorff measure and rectifiable sets see, e.g., [3, 10, 11].) We call *Radon measure* in \mathbb{R}^d any nonnegative and σ -additive set function μ defined on $\mathcal{B}_{\mathbb{R}^d}$ which is finite on bounded sets, and we write $\mu \ll \mathcal{H}^n$ to say that μ is absolutely continuous with respect to \mathcal{H}^n . We will say that a random closed set Θ satisfies a certain property (e.g. Θ has Hausdorff dimension n) if $\Theta(\omega)$ satisfies that property for \mathbb{P} -a.e. $\omega \in \Omega$.

Throughout the paper we will consider countably \mathcal{H}^n -rectifiable random closed sets in \mathbb{R}^d , with $1 \leq n \leq d-1$ integer, such that $\mathbb{E}[\mu_\Theta]$ is a Radon measure; the particular cases $n=0$ and $n=d$ are trivial. (For a discussion about measurability of $\mathcal{H}^n(\Theta)$ we refer to [25, 5].) Whenever $\mathbb{E}[\mu_\Theta]$ is absolutely continuous with respect to \mathcal{H}^d , the following definition is given [8, 23]

Definition 1 (Absolute continuity in mean and mean density) *Let Θ be a countably \mathcal{H}^n -rectifiable random closed set in \mathbb{R}^d such that $\mathbb{E}[\mu_\Theta]$ is a Radon measure. We say that Θ is absolutely continuous in mean if $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$. In this case we call mean density of Θ , and denote by λ_Θ , the Radon-Nikodym derivative of $\mathbb{E}[\mu_\Theta]$ with respect to \mathcal{H}^d .*

Remark 2 In the case $n=0$ with $\Theta = X$ random point in \mathbb{R}^d , we have that Θ is absolutely continuous in mean if and only if X admits a probability density function f_X , and so $\lambda_\Theta = f_X$. On the other hand, it is easy to see (as an application of Fubini's theorem in $\Omega \times \mathbb{R}^d$ with the product measure $\mathbb{P} \times \mathcal{H}^d$) that any d -dimensional random closed set Θ in \mathbb{R}^d is absolutely continuous in mean with mean density $\lambda_\Theta(x) = \mathbb{P}(x \in \Theta)$ for \mathcal{H}^d -a.e $x \in \mathbb{R}^d$.

The problem of the approximation of the mean densities in the general setting of spatially inhomogeneous processes has been recently faced in [1], where an approximation, in weak form, of the mean density for sufficiently regular random closed sets is given in terms of their d -dimensional enlargement by Minkowski addition. More precisely, denoting by $S_{\oplus r} := \{x \in \mathbb{R}^d : \exists y \in S \text{ with } |x - y| \leq r\}$ the closed r -neighborhood of a closed set $S \subset \mathbb{R}^d$, and by $B_r(x)$ the closed ball centered in x with radius r , for any compact window $W \subset \mathbb{R}^d$ let $\Gamma_W(\Theta) : \Omega \rightarrow \mathbb{R}$ be the function so defined:

$$\Gamma_W(\Theta) := \sup\{\gamma \geq 0 : \exists \text{ a probability measure } \eta \ll \mathcal{H}^n \text{ such that} \\ \eta(B_r(x)) \geq \gamma r^n \quad \forall x \in \Theta \cap W_{\oplus 1}, r \in (0, 1)\}; \quad (1)$$

then the following theorem holds [1].

Theorem 3 *Let Θ be a countably \mathcal{H}^n -rectifiable random closed set in \mathbb{R}^d such that $\mathbb{E}[\mu_\Theta]$ is a Radon measure. Assume that for any compact window $W \subset \mathbb{R}^d$ there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $1/\Gamma_W(\Theta) \leq Y$ almost surely. If Θ is absolutely continuous in mean, then*

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} dx = \int_A \lambda_\Theta(x) dx \quad (2)$$

for any bounded Borel set $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(\partial A) = 0$.

Hence the above theorem gives a *weak* result for the mean density of very general lower dimensional random closed sets. In order to obtain a *pointwise* result, using the fact that A is arbitrary, we should prove that limit and integral in (2) can be exchanged; in such a way we could state that

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d, \quad (3)$$

and, as by-product, the right side of the above equation could suggest estimators for $\lambda_\Theta(x)$ in terms of the capacity functional of Θ . The proof of the validity of this formula for absolutely continuous in mean random sets seems to be a quite delicate problem, with the only exception of the stationary ones and the extreme cases $n=d$ and $n=0$. In Theorem 9 we prove that (3)

holds for a wide class of (inhomogeneous) *Boolean models* and, in particular, we give an explicit representation of the mean density λ_Θ in terms of the *intensity* and of the *typical grain* of the process. To this end we recall now the definition (typically standard) of Boolean model and we refer to [4, 9] for details about point processes and related concepts and results.

Definition 4 (Boolean model) *Let $\Psi = \{x_i\}_{i \in \mathbb{N}}$ be Poisson point process in \mathbb{R}^d with intensity f and let $\{Z_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. random compact sets in \mathbb{R}^d , which are also independent of the Poisson process Ψ . Denoting by Z_0 a further random compact set of the same distribution as the Z_i 's and independent of both of them and of Φ , the resulting random set*

$$\Theta := \bigcup_i (x_i + Z_i)$$

is said (inhomogeneous) Boolean model with intensity f and typical grain Z_0 .

In our assumptions Z_0 will be a lower dimensional random closed set uniquely determined by a random quantity in a suitable *mark space* \mathbf{K} , so that $Z_0(s)$ is a compact subset of \mathbb{R}^d containing the origin for any $s \in \mathbf{K}$. We remind that in common literature is usually assumed that

$$\mathbb{E}[\text{card}\{i : (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^d \quad (4)$$

(card stands for cardinality), and that a Boolean model as above can be described by a Poisson point process in $\mathbb{R}^d \times \mathbf{K}$ with intensity measure $\Lambda(d(x, s)) = f(x)dxQ(ds)$, where Q is a probability measure on \mathbf{K} representing the distribution of the typical grain.

3 A generalization of the Minkowski content of closed sets

Denoted by b_k the volume of the unit ball in \mathbb{R}^k , the n -dimensional *Minkowski content* $\mathcal{M}^n(S)$ of a closed set $S \subset \mathbb{R}^d$ is defined by

$$\mathcal{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n}r^{d-n}}$$

whenever the limit exists finite.

General results about the existence of the Minkowski content of closed subsets in \mathbb{R}^d are known in literature, related to rectifiability properties of the involved sets. In particular, the following theorem is proved in [3] (p. 110).

Theorem 5 *Let $S \subset \mathbb{R}^d$ be a countably \mathcal{H}^n -rectifiable compact set and assume that*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S, \forall r \in (0, 1) \quad (5)$$

holds for some $\gamma > 0$ and some Radon measure η in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^n . Then $\mathcal{M}^n(S) = \mathcal{H}^n(S)$.

Note that such theorem extends the well-known Federer's result ([11], p. 275) about the existence of $\mathcal{M}^n(S)$ for n -rectifiable compact sets $S \subset \mathbb{R}^d$ (i.e. S is representable as the image of a compact subset of \mathbb{R}^n by a Lipschitz function from \mathbb{R}^n to \mathbb{R}^d) to countably \mathcal{H}^n -rectifiable compact sets (see Remark 2.3 in [2]). Moreover, in many applications condition (5) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\tilde{S} \cap \cdot)$ for some closed set $\tilde{S} \supseteq S$, and for $n = d - 1$, it is not hard to check that such condition is satisfied by all sets with Lipschitz boundary (see [2, 3]).

We state now the main result of this section.

Theorem 6 Let μ be a positive measure in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^d with density f such that

- i) f is locally bounded (i.e. $\sup_{x \in K} f(x) < \infty$ for any compact $K \subset \mathbb{R}^d$);
- ii) the set of all discontinuity points of f is \mathcal{H}^n -negligible.

Let $S \subset \mathbb{R}^d$ be a countably \mathcal{H}^n -rectifiable compact set such that condition (5) holds for some $\gamma > 0$ and some probability measure η in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^n . Then

$$\lim_{r \downarrow 0} \frac{\mu(S_{\oplus r})}{b_{d-n} r^{d-n}} = \int_S f(x) \mathcal{H}^n(dx).$$

Remark 7 The above theorem may be seen as a generalization of Theorem 5; indeed, the classical Minkowski content follows as particular case by choosing $f \equiv 1$ and noticing that if a Radon measure η as in Theorem 5 exists, then it can be assumed to be a probability measure without loss of generality. Indeed, it is sufficient to consider the measure $\tilde{\eta}(\cdot) := \eta(W \cap \cdot) / \eta(W)$, where W is a compact subset of \mathbb{R}^d such that $S_{\oplus 1} \subset W$; it is clear that $\tilde{\eta}$ is a probability measure satisfying

$$\tilde{\eta}(B_r(x)) \geq \frac{\gamma}{\eta(W)} r^n \quad \forall x \in S, \forall r \in (0, 1).$$

Furthermore, a classical result from geometric measure theory (e.g., see [3] Theorem 2.56) tells us that if μ is a positive Radon measure on \mathbb{R}^d and $B \in \mathcal{B}_{\mathbb{R}^d}$ such that

$$\limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{b_n r^n} \geq t \in (0, \infty) \quad \forall x \in B,$$

then $\mu(\cdot) \geq t \mathcal{H}^n(B \cap \cdot)$. Hence, any set $S \subset \mathbb{R}^d$ as in Theorem 6 has finite \mathcal{H}^n -measure.

In order to make the proof of Theorem 6 more readable, we remind that Lemma 15 and Lemma 16 in [1] provide a local version of Theorem 5 and an upper bound for the Minkowski content of compact sets in \mathbb{R}^d , respectively; for our purpose we summarize as follows.

Lemma 8 If $S \subset \mathbb{R}^d$ is a countably \mathcal{H}^n -rectifiable compact set such that condition (5) holds for some $\gamma > 0$ and some finite measure η in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^n , then

$$\frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}} \leq \frac{\eta(\mathbb{R}^d)}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \quad \forall r < 2, \quad (6)$$

and

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r} \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap A) \quad (7)$$

for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^n(S \cap \partial A) = 0$.

Proof of Theorem 6. It is well known that, since $f \geq 0$, there exists an increasing sequence $\{f_k\}_{k \in \mathbb{N}}$ of step functions

$$f_k(x) = \sum_{j=1}^{N(k)} a_j^{(k)} \mathbf{1}_{A_j^{(k)}}(x), \quad a_j^{(k)} \geq 0, \quad N(k) \in \mathbb{N},$$

converging to f .

By ii), such sequence $\{f_k\}$ can be chosen so that $\mathcal{H}^n(S \cap \partial A_j^{(k)}) = 0$ for all j, k .

Let

$$g_k(r) := \frac{\int_{S_{\oplus r}} f_k(x) dx}{b_{d-n} r^{d-n}}.$$

We may observe that

$$(a) \quad \lim_{r \downarrow 0} g_k(r) = \sum_{j=1}^{N(k)} a_j^{(k)} \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r} \cap A_j^{(k)})}{b_{d-n} r^{d-n}} \stackrel{(7)}{=} \sum_{j=1}^{N(k)} a_j^{(k)} \mathcal{H}^n(S \cap A_j^{(k)}) = \int_S f_k(x) \mathcal{H}^n(dx).$$

Since S is compact with $\mathcal{H}^n(S) < \infty$ by Remark 7, and f is locally bounded, it follows that

$$\int_S f_k(x) \mathcal{H}^n(dx) < \infty.$$

$$(b) \quad \lim_{k \rightarrow \infty} g_k(r) = \frac{\int_{S_{\oplus r}} f(x) dx}{b_{d-n} r^{d-n}} \quad \text{uniformly in } (0, 1); \text{ indeed:}$$

we have that $f_k \uparrow f$ uniformly in $S_{\oplus r}$ for any $r > 0$ because f is bounded in $S_{\oplus r}$, being $S_{\oplus r}$ compact, i.e. for all $\varepsilon > 0$ there exists k_0 such that

$$\sup_{x \in S_{\oplus r}} |f_k(x) - f(x)| < \varepsilon \quad \forall k > k_0. \quad (8)$$

Hence, for any $\varepsilon > 0$, for all $k > k_0$

$$\left| g_k(r) - \frac{\int_{S_{\oplus r}} f(x) dx}{b_{d-n} r^{d-n}} \right| \leq \frac{\int_{S_{\oplus r}} |f_k(x) - f(x)| dx}{b_{d-n} r^{d-n}} \stackrel{(8)}{<} \varepsilon \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}} \stackrel{(6)}{\leq} \varepsilon \frac{1}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \quad \forall r \in (0, 1).$$

As a consequence of (a) and (b) $\lim_{r \downarrow 0} \lim_{k \rightarrow \infty} g_k(r) = \lim_{k \rightarrow \infty} \lim_{r \downarrow 0} g_k(r)$. Thus the following chain of equalities holds

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\mu(S_{\oplus r})}{b_{d-n} r^{d-n}} &= \lim_{r \downarrow 0} \frac{\int_{S_{\oplus r}} f(x) dx}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\int_{S_{\oplus r}} \lim_{k \rightarrow \infty} f_k(x) dx}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{S_{\oplus r}} f_k(x) dx}{b_{d-n} r^{d-n}} \\ &= \lim_{k \rightarrow \infty} \lim_{r \downarrow 0} \frac{\int_{S_{\oplus r}} f_k(x) dx}{b_{d-n} r^{d-n}} \stackrel{(a)}{=} \lim_{k \rightarrow \infty} \int_S f_k(x) \mathcal{H}^n(dx) = \int_S f(x) \mathcal{H}^n(dx). \end{aligned}$$

□

4 Mean density of inhomogeneous Boolean models

Let Θ the Boolean model so defined

$$\Theta(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,$$

with Φ Poisson process in $\mathbb{R}^d \times \mathbf{K}$ and Z_0 typical grain in \mathbb{R}^d , satisfying the usual condition (4) for Boolean models. To lighten the notations, from now on we denote by

- $Z^{x,s} := x - Z_0(s) \quad \forall (x, s) \in \mathbb{R}^d \times \mathbf{K}$;
- δ the random variable on \mathbf{K} so defined $\delta(s) := \text{diam} Z_0(s)$, where diam stands for diameter;
- \mathbb{E}_Q the expectation with respect to the probability measure Q on \mathbf{K} .

Note that the condition (4) is equivalent to

$$\mathbb{E}[\Phi(\{(y, s) \in \mathbb{R}^d \times \mathbf{K} : (y + Z_0(s)) \cap B_R(0) \neq \emptyset\})] < \infty \quad \forall R > 0,$$

and so, in terms of the intensity measure Λ of Φ , to

$$\int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(dy \times ds) < \infty \quad \forall R > 0, \quad (9)$$

where $-Z_0$ is the symmetric of Z_0 with respect to the origin.

Assumptions: Let Φ have intensity measure $\Lambda(dy \times ds) = f(y)dyQ(ds)$ satisfying (9); further, let us assume that the following conditions on f and Z_0 are fulfilled:

(A1) $Z_0(s)$ is a countably \mathcal{H}^n -rectifiable compact set in \mathbb{R}^d for Q -a.e. $s \in \mathbf{K}$. Further there exist $\gamma > 0$ and a random closed set $\tilde{Z}_0 \supseteq Z_0$ with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)] < \infty$ such that, for Q -a.e. $s \in \mathbf{K}$,

$$\mathcal{H}^n(\tilde{Z}_0(s) \cap B_r(x)) \geq \gamma r^n \quad \forall x \in Z_0(s), \forall r \in (0, 1). \quad (10)$$

(A2) the set of all discontinuity points of f is \mathcal{H}^n -negligible and f is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K_{\oplus \delta}} f(y) \leq \xi_K \quad (11)$$

holds for some random variable ξ_K with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)\xi_K] < \infty$.

Theorem 9 (Main result) *Any Boolean model Θ as in Assumptions is absolutely continuous in mean with mean density λ_Θ given by*

$$\lambda_\Theta(x) = \int_{\mathbf{K}} \int_{Z^{x,s}} f(y) \mathcal{H}^n(dy) Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (12)$$

Before proving the above theorem, let us make a few observations about the above Assumptions in order to clarify their level of generality.

- Condition (11) is trivially satisfied whenever f is bounded, or f is locally bounded and $\text{diam} Z_0 \leq c \in \mathbb{R}_+$ Q -almost surely.
- Assumption (A1) is satisfied by a great deal of typical grains Z_0 with $\tilde{Z}_0 = Z_0$ or $\tilde{Z}_0 = Z_0 \cup \tilde{A}$, with \tilde{A} sufficiently regular random closed set to control the case when Z_0 can be arbitrarily small (see [1, 23]), and it assures that Z_0 is a countably \mathcal{H}^n -rectifiable random compact set with finite expected \mathcal{H}^n -measure.
- **Stationary case.** If Θ is stationary with $f \equiv c > 0$, then only the regularity assumption (A1) on the typical grain Z_0 is required. Indeed, the assumption (A2) is trivial and the usual condition (9), which in this case becomes

$$\mathbb{E}_Q[\mathcal{H}^d((Z_0)_{\oplus R})] < \infty \quad \forall R > 0,$$

is satisfied thanks to the assumption (A1): for all $R < 2$ by Lemma 8 with $\eta(\cdot) = \mathcal{H}^n(\tilde{Z}_0 \cap \cdot)$ we have that $\mathbb{E}_Q[\mathcal{H}^d((Z_0)_{\oplus R})] \leq \mathbb{E}[\mathcal{H}^n(\tilde{Z}_0)] 2^n 4^d b_d R^{d-n}/\gamma$, while for all $R \geq 2$, by repeating the same argument of the proof of the quoted proposition, it is easy to check that $\mathbb{E}_Q[\mathcal{H}^d((Z_0)_{\oplus R})] \leq \mathbb{E}[\mathcal{H}^n(\tilde{Z}_0)] 2^n (4R)^d b_d/\gamma$.

- **Deterministic typical grain.** If Θ has deterministic typical grain $Z_0 \subset \mathbb{R}^d$ the above Assumptions can be replaced by: Θ has intensity f and typical grain $Z_0 = S$ as in the hypotheses of Theorem 6. (See Corollary 13 for details.)

Lemma 10 *For any Boolean model Θ as in Assumptions, $\mathbb{E}[\mu_\Theta]$ is a locally finite measure absolutely continuous with respect to \mathcal{H}^d .*

Proof. For any $R > 0$ let $\mathcal{B}^R := \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : (y + Z_0(s)) \cap B_R(0) \neq \emptyset\}$; then

$$\mathbb{E}[\mathcal{H}^n(\Theta \cap B_R(0))] = \mathbb{E}[\mathbb{E}[\mathcal{H}^n(\Theta \cap B_R(0)) \mid \Phi(\mathcal{B}^R)]] \leq \mathbb{E}_Q[\mathcal{H}^n(Z_0)]\mathbb{E}[\Phi(\mathcal{B}^R)] < \infty$$

by (9) and condition (A1); so $\mathbb{E}[\mu_\Theta]$ is locally finite.

By contradiction, let $\mathbb{E}[\mu_\Theta]$ be not absolutely continuous with respect to \mathcal{H}^d ; then there exists $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(A) = 0$ such that $\mathbb{E}[\mathcal{H}^n(\Theta \cap A)] > 0$. In particular,

$$0 < \mathbb{P}(\mathcal{H}^n(\Theta \cap A) > 0) \leq \mathbb{P}\left(\sum_{(x_i, s_i) \in \Phi} \mathcal{H}^n((x_i + Z_0(s_i)) \cap A) > 0\right) = \mathbb{P}(\Phi(\mathcal{A}) > 0),$$

where

$$\mathcal{A} := \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : \mathcal{H}^n((y + Z_0(s)) \cap A) > 0\}.$$

Denoting by $\mathcal{A}_s := \{y \in \mathbb{R}^d : (y, s) \in \mathcal{A}\}$ the section of \mathcal{A} at $s \in \mathbf{K}$, we may apply Fubini's theorem to get

$$\int_{\mathcal{A}_s} \mathcal{H}^n((y + Z_0(s)) \cap A) dy = \int_{\mathcal{A}_s} \left(\int_{Z_0(s)} \mathbf{1}_{A-y}(x) \mathcal{H}^n(dx) \right) dy = \int_{Z_0(s)} \left(\int_{\mathcal{A}_s} \mathbf{1}_{A-x}(y) dy \right) \mathcal{H}^n(dx) = 0,$$

because $\mathcal{H}^d(A) = 0$. Being the function $y \mapsto \mathcal{H}^n((y + Z_0(s)) \cap A)$ strictly positive in \mathcal{A}_s , we conclude that $\mathcal{H}^d(\mathcal{A}_s) = 0$ for all $s \in \mathbf{K}$. Then it follows

$$\mathbb{E}[\Phi(\mathcal{A})] = \int_{\mathcal{A}} \Lambda(dy \times ds) = \int_{\mathbf{K}} \left(\int_{\mathcal{A}_s} f(y) dy \right) Q(ds) = 0;$$

but this is impossible, because $\mathbb{P}(\Phi(\mathcal{A}) > 0) > 0$ implies $\mathbb{E}[\Phi(\mathcal{A})] > 0$. \square

Proof of Theorem 9. Clearly Θ is a countably \mathcal{H}^n -rectifiable random closed set in \mathbb{R}^d , and by Lemma 10 it is absolutely continuous in mean, so $\mathbb{E}[\mu_\Theta] = \lambda_\Theta \mathcal{H}^d$ for some integrable function λ_Θ on \mathbb{R}^d .

Let W be a fixed compact subset of \mathbb{R}^d . For any $\omega \in \Omega$ let us consider the set

$$\tilde{\Theta}(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + \tilde{Z}_0(s_i)$$

and the probability measure η_W absolutely continuous with respect to \mathcal{H}^n so defined

$$\eta_W(B) := \frac{\mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 1} \cap B)}{\mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 1})} \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Note that for any $x \in \Theta(\omega) \cap W_{\oplus 1}$ there exists $(\bar{x}, \bar{s}) \in \Phi(\omega)$ such that $x \in \bar{x} + Z_0(\bar{s})$, and so

$$\eta_W(B_r(x)) \geq \frac{\mathcal{H}^n(\tilde{Z}_0(\bar{s}) \cap B_r(x - \bar{x}))}{\mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 1})} \stackrel{(10)}{\geq} \frac{\gamma}{\mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 1})} r^n \quad \forall r \in (0, 1).$$

Thus, here the function $\Gamma_W(\Theta)$ defined in (1) is such that $\Gamma_W(\Theta) \geq \gamma/\mathcal{H}^n(\tilde{\Theta} \cap W_{\oplus 1})$; by the same argument in the proof of Lemma 10 it is easy to check that $\mathbb{E}[\mathcal{H}^n(\tilde{\Theta} \cap W_{\oplus 1})] < \infty$, then Theorem 3 applies and we get

$$\mathbb{E}[\mu_{\Theta}](A) = \lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n}r^{d-n}} dx \quad (13)$$

for any bounded set $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^d(\partial A) = 0$.

Let us denote by $\mathcal{Z}^{x,r}$ the subset of $\mathbb{R}^d \times \mathbf{K}$ so defined

$$\mathcal{Z}^{x,r} := \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : x \in (y + Z_0(s))_{\oplus r}\} = \{(y, s) \in \mathbb{R}^d \times \mathbf{K} : y \in Z_{\oplus r}^{x,s}\},$$

and observe that $\forall x \in A, \forall r < 2$

$$\begin{aligned} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n}r^{d-n}} &= \frac{\mathbb{P}(\Phi(\mathcal{Z}^{x,r}) > 0)}{b_{d-n}r^{d-n}} = \frac{1 - e^{-\Lambda(\mathcal{Z}^{x,r})}}{b_{d-n}r^{d-n}} \\ &\leq \frac{\Lambda(\mathcal{Z}^{x,r})}{b_{d-n}r^{d-n}} = \frac{1}{b_{d-n}r^{d-n}} \int_{\mathbf{K}} \int_{Z_{\oplus r}^{x,s}} f(y) dy Q(ds) \\ &\leq \int_{\mathbf{K}} \frac{\mathcal{H}^d(Z_{\oplus r}^{x,s})}{b_{d-n}r^{d-n}} \sup_{y \in A_{\oplus \delta(s)+2}} f(y) Q(ds) \\ &\stackrel{(6),(11)}{\leq} \frac{2^n 4^d b_d}{\gamma b_{d-n}} \int_{\mathbf{K}} \mathcal{H}^n(\tilde{Z}_0(s)) \xi_{A_{\oplus 2}}(s) Q(ds) \stackrel{(A2)}{<} \infty. \end{aligned} \quad (14)$$

Then by the dominated convergence theorem we can exchange limit and integral in Eq. (13).

Similarly, we have that for all $r < 2$

$$\frac{\int_{Z_{\oplus r}^{x,s}} f(y) dy}{b_{d-n}r^{d-n}} \leq \frac{\mathcal{H}^d((Z_0(s))_{\oplus r})}{b_{d-n}r^{d-n}} \sup_{y \in Z_{\oplus r}^{x,s}} f(y) \leq \frac{2^n 4^d b_d}{\gamma b_{d-n}} \mathcal{H}^n(\tilde{Z}_0(s)) \xi(s)$$

for some random variable ξ with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)\xi] < \infty$, by (6) and (A2). Then, the dominated convergence theorem implies that

$$\lim_{r \downarrow 0} \frac{\Lambda(\mathcal{Z}^{x,r})}{b_{d-n}r^{d-n}} = \int_{\mathbf{K}} \lim_{r \downarrow 0} \frac{\int_{Z_{\oplus r}^{x,s}} f(y) dy}{b_{d-n}r^{d-n}} Q(ds) = \int_{\mathbf{K}} \int_{Z^{x,s}} f(y) \mathcal{H}^n(dy) Q(ds), \quad (15)$$

where the last equality follows by applying Theorem 6 with $\mu = f\mathcal{H}^d$ and $S = Z^{x,s}$.

Summarizing, we have that for any bounded set $A \in \mathcal{B}_{\mathbb{R}^d}$ with $\mathcal{H}^d(\partial A) = 0$

$$\begin{aligned} \mathbb{E}[\mu_{\Theta}(A)] &= \lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta)}{b_{d-n}r^{d-n}} dx = \int_A \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta)}{b_{d-n}r^{d-n}} dx \\ &\stackrel{(14)}{=} \int_A \lim_{r \downarrow 0} \frac{1 - e^{-\Lambda(\mathcal{Z}^{x,r})}}{b_{d-n}r^{d-n}} dx = \int_A \lim_{r \downarrow 0} \frac{\Lambda(\mathcal{Z}^{x,r}) + o(r^{d-n})}{b_{d-n}r^{d-n}} dx \\ &\stackrel{(15)}{=} \int_A \left(\int_{\mathbf{K}} \int_{Z^{x,s}} f(y) \mathcal{H}^n(dy) Q(ds) \right) dx. \end{aligned}$$

We conclude that $\mathbb{E}[\mu_{\Theta}]$ has density

$$\lambda_{\Theta}(x) = \int_{\mathbf{K}} \int_{Z^{x,s}} f(y) \mathcal{H}^n(dy) Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

□

Remark 11 Formula (12) could also be obtained in a more direct way by the well-known Campbell's formula (e.g., see [4], p.28, and [14] for a similar application), after having shown that $\mathbb{E}[\mu_\Theta](\cdot) = \mathbb{E}[\sum_i \mathcal{H}^n((x_i + Z_i) \cap \cdot)]$ for any Boolean model Θ as in Assumptions, being zero the probability that different grains overlap in a subset of \mathbb{R}^d of positive \mathcal{H}^n -measure. (To show such property, one could proceed arguing similarly to the final part of the proof of Lemma 10.) On the other hand, a proof via Campbell's formula seems not to give any hint to provide computable estimators for the mean density (see Section 5 for a more detailed discussion).

In the particular cases in which Θ is stationary, or the typical grain Z_0 is a deterministic subset of \mathbb{R}^d satisfying the hypotheses of Theorem 5 (with η probability measure by Remark 7), Theorem 9 specializes as follows.

Corollary 12 (Stationary case) *Let Θ be a stationary Boolean model with intensity $f \equiv c > 0$ and typical grain Z_0 satisfying the assumption (A1). Then Θ is absolutely continuous in mean with mean density*

$$\lambda_\Theta(x) = c\mathbb{E}_Q[\mathcal{H}^n(Z_0)] \quad \forall x \in \mathbb{R}^d.$$

Proof. We have observed that Θ satisfies the Assumptions and λ_Θ is constant on \mathbb{R}^d . Then the thesis follows by Theorem 9, noticing that $\mathcal{H}^n(Z^{x,s}) = \mathcal{H}^n(Z_0(s))$ for all $(x, s) \in \mathbb{R}^d \times \mathbf{K}$ and that λ_Θ is constant being $\mathbb{E}[\mu_\Theta]$ translation invariant. \square

Corollary 13 (Deterministic typical grain) *Let Θ be a Boolean model in \mathbb{R}^d with deterministic typical grain $Z_0 \subset \mathbb{R}^d$. If the underlying Poisson process Φ in \mathbb{R}^d has intensity f locally bounded and such that the set of all its discontinuity points is \mathcal{H}^n -negligible, and if Z_0 is a countably \mathcal{H}^n -rectifiable compact set such that condition (5) holds for some $\gamma > 0$ and some probability measure η in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^n , then Θ is absolutely continuous in mean with mean density*

$$\lambda_\Theta(x) = \int_{Z_0} f(x-y)\mathcal{H}^n(dy) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. Since Z_0 is compact $\text{diam} Z_0 = \delta < \infty$, and we know by Remark 7 that $\mathcal{H}^n(Z_0)$ is finite. Then the assumption (A2) and the condition (9) are easily checked. It follows that $\mathbb{E}[\mu_\Theta]$ is locally finite and, by proceeding as in Lemma 10, again we have that $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$.

For any $y \in \mathbb{R}^d$ let η^y be the measure on \mathbb{R}^d so defined

$$\eta^y(B) := \eta(B - y) \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Let W be a fixed compact subset of \mathbb{R}^d and for any $\omega \in \Omega$ consider the measure

$$\tilde{\eta}(B) := \frac{\sum_{x_i \in \Phi(\omega)} \eta^{x_i}(B) \mathbf{1}_{(x_i + Z_0) \cap W_{\oplus 1} \neq \emptyset}}{\text{card}\{x_i \in \Phi(\omega) : (x_i + Z_0) \cap W_{\oplus 1} \neq \emptyset\}} \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Note that

- $\tilde{\eta}$ is a probability measure absolutely continuous with respect to \mathcal{H}^n ;
- for any $x \in \Theta(\omega) \cap W_{\oplus 1}$ there exists $\bar{x} \in \Phi(\omega)$ such that $x \in \bar{x} + Z_0$, and so

$$\tilde{\eta}(B_r(x)) \geq \frac{\gamma}{\text{card}\{x_i \in \Phi(\omega) : (x_i + Z_0) \cap W_{\oplus 1} \neq \emptyset\}} r^n \quad \forall r \in (0, 1).$$

Then Θ satisfies the hypotheses of Theorem 3 with $Y = \text{card}\{x_i \in \Phi(\omega) : (x_i + Z_0) \cap W_{\oplus 1} \neq \emptyset\}/\gamma$, which has finite expectation thanks to (9), and so again we have that Eq. (13) holds for any bounded set $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^d(\partial A) = 0$. Finally, by proceeding as in the final part of the proof of Theorem 9 with deterministic Z_0 , we conclude that (3) still holds and

$$\lambda_{\Theta}(x) = \int_{x-Z_0} f(y) \mathcal{H}^n(dy) = \int_{Z_0} f(x-y) \mathcal{H}^n(dy) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

□

Remark 14 (The special case $n = d - 1$) A well known tool in stochastic geometry is the so-called *local spherical contact distribution function*, defined as $H_{\Theta}(r, x) := \mathbb{P}(x \in \Theta_{\oplus r} \mid x \notin \Theta)$ for all $(r, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ (e.g. see [15]). If Θ is a Boolean model as in Assumptions with $(d-1)$ -dimensional typical grain, then $\mathbb{P}(x \in \Theta) = 0$ for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$, and so the mean density λ_{Θ} can be written in terms of H_{Θ} as well

$$\lambda_{\Theta}(x) \stackrel{(3)}{=} \frac{1}{2} \frac{\partial}{\partial r} H_{\Theta}(r, x)|_{r=0} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Therefore known results about H_{Θ} could be applied to $(d-1)$ -dimensional Boolean models to obtain further information on their mean density.

We conclude this section by giving a very simple example of inhomogeneous Boolean model in \mathbb{R}^d , in order to show how the mean density can be easily computed by applying Theorem 9.

Example 1 (Segment Boolean model) For sake of simplicity we consider a Boolean model Θ of segments in \mathbb{R}^2 , but a similar example can be done in \mathbb{R}^d with $d > 2$. So, let $\mathbf{K} = \mathbb{R}_+ \times [0, 2\pi]$ and for all $s = (l, \alpha) \in \mathbf{K}$ let $Z_0(s)$ be the segment with length l and orientation α so defined

$$Z_0(s) := \{(u, v) \in \mathbb{R}^2 : u = \tau \cos \alpha, v = \tau \sin \alpha, \tau \in [0, l]\}.$$

We consider the case in which both length and orientation are random. Denoting by L the \mathbb{R}_+ -valued random variable representing the length of Z_0 and by $\mathbb{P}_L(dl)$ its probability law, let Φ be the marked Poisson process in $\mathbb{R}^d \times \mathbf{K}$ having intensity measure $\Lambda(dy \times ds) = f(y)dyQ(ds)$ with $f(u, v) = u^2 + v^2$ and $Q(ds) = \frac{1}{2\pi}d\alpha\mathbb{P}_L(dl)$ such that $\int_{\mathbb{R}_+} l^3\mathbb{P}_L(dl) < \infty$. (This last assumption is to guarantee that the usual condition (4) is satisfied; for a different intensity f we might have a different condition on the moments of L .) It is easily shown [1, 23] that Θ satisfies the assumption (A1); by noticing that

$$\int_{(-Z_0(s))_{\oplus R}} f(y)dy \leq (l+R)^2(2lR + \pi R^2) \quad \forall s = (l, \alpha) \in \mathbf{K},$$

it follows that

$$\int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(dy \times ds) \leq \int_{\mathbb{R}_+} (l+R)^2(2lR + \pi R^2)\mathbb{P}_L(dl) < \infty \quad \forall R > 0,$$

so condition (9) (and similarly the assumption (A2)) is satisfied. Hence Theorem 9 applies and we

get

$$\begin{aligned}
\lambda_{\Theta}(x_1, x_2) &\stackrel{(12)}{=} \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \int_0^l f(x_1 - \tau \cos \alpha, x_2 - \tau \sin \alpha) d\tau d\alpha \mathbb{P}_L(dl) \\
&= \int_0^\infty \frac{1}{2\pi} \int_0^{2\pi} \left((x_1^2 + x_2^2)l - (x_1 \cos \alpha + x_2 \sin \alpha)l^2 + \frac{1}{3}l^3 \right) d\alpha \mathbb{P}_L(dl) \\
&= (x_1^2 + x_2^2)\mathbb{E}[L] + \frac{1}{3}\mathbb{E}[L^3].
\end{aligned}$$

Note that in the particular case $f \equiv c > 0$, Θ is stationary and by Corollary 12 we obtain the well known result $\lambda_{\Theta}(x) = c\mathbb{E}[L] \forall x \in \mathbb{R}^d$ (cf. [7], p. 42).

5 On the estimation of the mean density

While the problem of the mean density estimation has been widely examined in the stationary case, the inhomogeneous one has been mainly faced by assuming local stationarity. For instance, Θ is assumed to have a gradient structure, i.e. it is considered to be homogeneous perpendicularly to a particular gradient direction (see [13]), or it is assumed to be homogeneous in certain subregions of \mathbb{R}^d , so that the known results in the homogeneous case can be applied to estimate a stepwise approximation of the mean density λ_{Θ} . We also mention that the stationary Boolean model Θ is often assumed to have unknown constant intensity $c > 0$ and known mark distribution Q , so that only the parameter c has to be estimated, being $\lambda_{\Theta} = c\mathbb{E}_Q[\mathcal{H}^n(Z_0)]$ in this case. A series of results about the estimation of the intensity c of the underlying Poisson point process associated to Θ , related to the estimation of λ_{Θ} , can be found in [7] §3.4 (see also [17, 22]).

Having now an explicit formula for the mean density of inhomogeneous Boolean models as in Assumptions, in all situations in which it is possible to estimate the intensity f and the mark distribution Q of the typical grain Z_0 , an estimation of λ_{Θ} could be obtained. Actually, the estimation of f and Q as well as the computing of the mean density $\lambda_{\Theta}(x)$ at a given point $x \in \mathbb{R}^d$ might be quite hard. In this section we introduce an estimator for the mean density $\lambda_{\Theta}(x)$ of Boolean models as in Assumptions in the general case of f and Q unknown.

In the proof of Theorem 9 we have shown, in particular, that λ_{Θ} is also given by the limit in (3) for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$. Noticing that $\mathbb{P}(x \in \Theta_{\oplus r}) = T_{\Theta}(B_r(x))$, where T_{Θ} is the capacity (or hitting) functional of Θ [16], a natural estimator of $\lambda_{\Theta}(x)$ can be given in terms of the *empirical capacity functional* of Θ , without estimating the intensity f and the distribution of Z_0 separately. We recall that the empirical capacity functional \hat{T}_{Ξ}^N based on an i.i.d. random sample Ξ_1, \dots, Ξ_N of a random closed set Ξ is defined as (see, e.g., [12])

$$\hat{T}_{\Xi}^N(K) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\Xi_i \cap K \neq \emptyset}, \quad \forall \text{ compact } K \subset \mathbb{R}^d,$$

and that the strong law of large numbers implies that $\hat{T}_{\Xi}^N(K)$ converges almost surely to $T_{\Xi}(K)$ for any compact subset K of \mathbb{R}^d .

Let Θ be an inhomogeneous Boolean model in \mathbb{R}^d as in Assumptions and $\Theta_1, \dots, \Theta_N$ be an i.i.d. random sample of Θ ; for any fixed $r > 0$ we have that

$$\mathbb{E}\left[\frac{\hat{T}_{\Theta}^N(B_r(x))}{b_{d-n}r^{d-n}}\right] = \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n}r^{d-n}}, \quad \text{var}\left(\frac{\hat{T}_{\Theta}^N(B_r(x))}{b_{d-n}r^{d-n}}\right) = \frac{\mathbb{P}(x \in \Theta_{\oplus r})(1 - \mathbb{P}(x \in \Theta_{\oplus r}))}{N(b_{d-n}r^{d-n})^2}. \quad (16)$$

Hence, (3) and (16) suggest that we take

$$\hat{\lambda}_\Theta^N(x) := \frac{\sum_{i=1}^N \mathbf{1}_{\Theta_i \cap B_{R_N}(x) \neq \emptyset}}{Nb_{d-n}R_N^{d-n}}, \quad (17)$$

with R_N such that

$$\lim_{N \rightarrow \infty} R_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} NR_N^{d-n} = \infty, \quad (18)$$

as asymptotically unbiased and consistent estimator of the mean density $\lambda_\Theta(x)$ of Θ at point x .

Proposition 15 *Let Θ be a Boolean model in \mathbb{R}^d as in Assumptions and $\{\Theta_i\}_{i \in \mathbb{N}}$ be a sequence of random closed sets i.i.d. as Θ ; then*

$$\lim_{N \rightarrow \infty} \hat{\lambda}_\Theta^N(x) = \lambda_\Theta(x) \quad \text{in probability,} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. By the definition of $\hat{\lambda}_\Theta^N$ it follows

$$\lim_{N \rightarrow \infty} \mathbb{E}[\hat{\lambda}_\Theta^N(x)] = \lim_{N \rightarrow \infty} \frac{\mathbb{P}(x \in \Theta_{\oplus R_N})}{b_{d-n}R_N^{d-n}} \stackrel{(3)}{=} \lambda_\Theta(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

We check now that the variance of $\hat{\lambda}_\Theta^N(x)$ goes to 0. Since

$$\lim_{N \rightarrow \infty} \mathbb{P}(x \in \Theta_{\oplus R_N}) = \mathbb{P}(x \in \Theta) = 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d \text{ because } n < d,$$

$$\lim_{N \rightarrow \infty} \frac{\mathbb{P}(x \in \Theta_{\oplus R_N})}{b_{d-n}R_N^{d-n}} = \lambda_\Theta(x) \in \mathbb{R} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d \text{ by (3),}$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{NR_N^{d-n}} = 0 \quad \text{by (18),}$$

we have that for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$

$$\lim_{N \rightarrow \infty} \text{var}(\hat{\lambda}_\Theta^N(x)) = \lim_{N \rightarrow \infty} \frac{N\mathbb{P}(x \in \Theta_{\oplus R_N})(1 - \mathbb{P}(x \in \Theta_{\oplus R_N}))}{(Nb_{d-n}R_N^{d-n})^2} = 0.$$

Hence the thesis follows. \square

Then, a problem of statistical interest could be to find the optimal *width* R_N satisfying condition (18) which minimizes the mean squared error of $\hat{\lambda}_\Theta^N(x)$ (i.e. $\mathbb{E}[(\hat{\lambda}_\Theta^N(x) - \lambda_\Theta(x))^2]$). To investigate this problem is not the aim of the present paper and we leave this as open problem for further developments; we point out here that $\hat{\lambda}_\Theta^N$ can be seen as the generalization to the case of n -dimensional random closed sets, of the well known estimator of the probability density of a random point, which is a particular 0-dimensional random closed set.

Remark 16 (The special case $n = 0$) Even if the particular case $n = 0$ can be handle with much more elementary tools, it is easy to check that if $\Theta = X$ is a random point in \mathbb{R}^d with probability density function f_X , Eq. (3) holds with $\lambda_X = f_X$ (it is sufficient to observe that $\mathbb{E}[\mathcal{H}^0(X \cap \cdot)] = \mathbb{P}(X \in \cdot)$) and the estimator $\hat{\lambda}_X^N$ turns to be closely related to the well known definition of histogram (see [23] for details). Let us consider the case of a random variable X with density f_X ; then Proposition 15 applies with $d = 1$ and $n = 0$, making explicit the correspondence

with the usual *density estimation by means of histograms* (e.g., see [18] §VII.13). Indeed, if $\{X_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables with the same distribution of X , we define

$$\hat{f}_X(x) := \hat{\lambda}_X^N(x) \stackrel{(17)}{=} \frac{\sum_{i=1}^N \mathbf{1}_{B_{R_N}(x)}(X_i)}{Nb_1 R_N} = \frac{\text{card}\{i : X_i \in I_x\}}{N|I_x|},$$

where I_x is the interval in \mathbb{R} centered in x with length $|I_x| = 2R_N$ with the usual condition

$$|I_x| \longrightarrow 0 \quad \text{and} \quad N|I_x| \longrightarrow \infty \quad \text{as } N \rightarrow \infty.$$

Therefore, statistical problems and techniques related to the choice of the “optimal width” R_N in the general case $N > 0$ could be investigated starting from available results for random variables.

We conclude observing how the following corollary of Theorem 9, which could lead to further developments on this topic and on the estimation of the mean density, is consistent with a couple of available results in literature.

Corollary 17 *Any Boolean model Θ as in Assumptions is absolutely continuous in mean with mean density*

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{E}[\text{card}\{(x_i, s_i) \in \Phi : (x_i + Z_0(s_i)) \cap B_r(x) \neq \emptyset\}]}{b_{d-n} r^{d-n}} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (19)$$

Proof. The assertion follows directly by (15), noticing that $\Lambda(\mathcal{Z}^{x,r}) = \mathbb{E}[\text{card}\{(x_i, s_i) \in \Phi : (x_i + Z_0(s_i)) \cap B_r(x) \neq \emptyset\}]$. \square

By Proposition 21 in [1] we get that for a locally finite union Θ of i.i.d. random closed sets E_i with Hausdorff dimension $n < d$ it holds

$$\lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\mathbb{E}[\text{card}\{E_i : E_i \cap B_r(x) \neq \emptyset\}]}{b_{d-n} r^{d-n}} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d, \quad (20)$$

provided that at least one of the two limits exists. On the other hand, in [20] the mean density of a class of nonstationary *n-flat processes* is studied and a similar result to (19) is obtained. Namely, we remind that a *n-flat process* in \mathbb{R}^d (with n integer less than d) is a point process X on \mathcal{E}_n^d , the space of n -dimensional planes in \mathbb{R}^d ; it is proved that if the intensity measure of X has a continuous density h with respect to some translation-invariant, locally finite measure on \mathcal{E}_n^d , then the n -dimensional random closed set $\Theta := \bigcup_{E \in X} E$ has continuous mean density

$$\lambda_\Theta(x) = \int_{\mathcal{L}_n^d} h(x + L) \Psi(dL),$$

where \mathcal{L}_n^d is the Grassmannian of n -dimensional linear subspaces in \mathbb{R}^d and Ψ is a finite measure on \mathcal{L}_n^d coming from a decomposition result of the intensity measure of X , and in particular it is claimed that (see [20], p. 142, or [4], p. 179)

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{E}[\text{card}\{E \in X : E \cap B_r(x) \neq \emptyset\}]}{b_{d-n} r^{d-n}}.$$

We may like to notice that Theorem 3 applies to *n-flat process*; hence we are lead to conjecture that the exchange between limit and integral in (2) may hold for further processes $\Theta = \bigcup_i E_i$, union of i.i.d. n -dimensional random closed sets, so that Proposition 15 and, by (20), Corollary 17, could be extended to this kinds of random closed sets.

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