

# SPECIFIC AREA AND MEAN SURFACE DENSITY OF INHOMOGENEOUS BOOLEAN MODELS

ELENA VILLA

Dept. of Mathematics, University of Milan, via Saldini 50, 20133 Milano, Italy

e-mail: elena.villa@unimi.it

## ABSTRACT

Starting from an open problem in (Matheron, 1975, p. 50-51) related to the existence of the so-called *specific area* of a random closed set, we consider here a class of  $d$ -dimensional inhomogeneous Boolean models in  $\mathbb{R}^d$  and we provide an explicit formula for the specific area. This turns out to be closely related to the notion of *outer Minkowski content* of sets. In particular, we show that the specific area may differ from the mean density of the topological boundary of the random set in general, and we provide sufficient conditions ensuring the equality.

Keywords: Boolean models, geometric measure theory, mean densities, outer Minkowski content.

## INTRODUCTION AND BASIC NOTATION

As stated in (Weil, 2001, p. 55), a problem of interest is to have explicit formulae for local densities of specific inhomogeneous Boolean models. In particular, about the notion of *mean surface density* of a  $d$ -dimensional random closed set  $\Theta$  in  $\mathbb{R}^d$ , the concept of *specific area*  $\sigma(x)$  of  $\Theta$  at a point  $x \in \mathbb{R}^d$  has been introduced in (Matheron, 1975, p. 50) and defined as the following limit

$$\sigma(x) := \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r}$$

whenever it exists. ( $\Theta_{\oplus r}$  denotes here the parallel set of  $\Theta$  at distance  $r$ , i.e.  $\Theta_{\oplus r} := \{x \in \mathbb{R}^d : \text{dist}(x, \Theta) \leq r\}$ .) It is mentioned in (Matheron, 1975) that the definition of  $\sigma(x)$  can be interpreted as the “translation” into probabilistic terms of the definition of *area*  $S(K) := \lim_{r \downarrow 0} \mathcal{H}^d(K_{\oplus r} \setminus K)/r$  of a compact set  $K \subset \mathbb{R}^d$ , where  $\mathcal{H}^d$  is the  $d$ -dimensional Hausdorff measure. Indeed, noticing that  $\mathbb{P}(x \in \Theta)$  is the density (the classical Radon-Nikodym derivative) of the mean volume measure  $\mathbb{E}[\mathcal{H}^d(\Theta \cap \cdot)]$  on  $\mathbb{R}^d$ , the existence of  $\sigma(x)$  might be related to the existence of the limit of  $\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \setminus \Theta)]/r$  as  $r$  goes to 0, known as *mean outer Minkowski content* of  $\Theta$ , introduced in (Ambrosio et al., 2008). More precisely, a straightforward application of Fubini’s theorem gives

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap B)]}{r} = \lim_{r \downarrow 0} \int_B \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} dx, \quad (1)$$

for all Borel subsets  $B$  of  $\mathbb{R}^d$ ; then it is clear that, whenever  $\Theta$  is stationary,  $\sigma$  is constant and given by

$$\sigma = \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap [0, 1]^d)]}{r}.$$

If  $\Theta$  satisfies a (local) Steiner formula (e.g., if  $\Theta$  is a Boolean model with convex grains), then the above limit can be studied in terms of the quermass densities (or Minkowski functionals) associated to  $\Theta$ , and so by means of tools from integral geometry mainly (e.g. see (Baddeley et al., 2007; Hug, 2000; Hug et al., 2006; Weil, 2001) and references therein).

The passage from stationary to nonstationary random closed sets, and from convex to more general grains gives rise to nontrivial problems. We provide here an overview of results in this direction recently obtained in (Villa, 2008b), by means of tools from geometric measure theory.

Throughout the paper  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure,  $dx$  stands for  $\mathcal{H}^d(dx)$ ,  $\mathcal{B}_{\mathbb{R}^d}$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}^d$  and  $\mathcal{H}^n|_A$  denotes the restriction of  $\mathcal{H}^n$  to a  $\mathcal{H}^n$ -measurable set  $A \subset \mathbb{R}^d$  (i.e.  $\mathcal{H}^n|_A(B) = \mathcal{H}^n(A \cap B)$  for all  $B \in \mathcal{B}_{\mathbb{R}^d}$ ).  $B_r(x)$  will denote the closed ball with centre  $x$  and radius  $r$ . We remind that a compact set  $A \subset \mathbb{R}^d$  is called  *$n$ -rectifiable* ( $0 \leq n \leq d-1$  integer) if it is representable as the image of a compact subset of  $\mathbb{R}^n$  by a Lipschitz map from  $\mathbb{R}^n$  to  $\mathbb{R}^d$ ; more in general, a closed subset  $A$  of  $\mathbb{R}^d$  is called to be *countably  $\mathcal{H}^n$ -rectifiable* if there exist countably many  $n$ -dimensional Lipschitz graphs  $\Gamma_i \subset \mathbb{R}^d$  such that  $A \setminus \cup_i \Gamma_i$  is  $\mathcal{H}^n$ -negligible. (For definitions and basic properties of Hausdorff measure and rectifiable sets see, e.g., (Ambrosio et al., 2000; Falconer, 1985; Federer, 1969).) We call *Radon measure* in  $\mathbb{R}^d$  any nonnegative and  $\sigma$ -additive set function  $\mu$  defined on  $\mathcal{B}_{\mathbb{R}^d}$  which is finite on bounded sets.

In the sequel we shall deal with  $\mathbb{P}$ -a.s. countably  $\mathcal{H}^{d-1}$ -rectifiable random closed sets  $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \rightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$  ( $\mathbb{F}$  and  $\sigma_{\mathbb{F}}$  denote here the class of the closed subsets in  $\mathbb{R}^d$  and the  $\sigma$ -algebra generated by

the so-called hit-or-miss topology (Matheron, 1975), respectively). For a discussion about measurability of  $\mathcal{H}^{d-1}(\Theta)$  we refer to (Zähle, 1982). In particular, we shall consider Boolean models  $\Xi$  in  $\mathbb{R}^d$  (Baddeley *et al.*, 2007; Stoyan *et al.*, 1995) whose typical grain  $Z_0$  has topological boundary  $\partial Z_0$  with Hausdorff dimension  $d-1$   $\mathbb{P}$ -a.s. It is well known that Boolean models in  $\mathbb{R}^d$  can be described by marked Poisson point processes on  $\mathbb{R}^d$  with marks in the space of centred compact sets. Since in many examples and applications  $Z_0$  is uniquely determined by a random quantity in a suitable *mark space*  $\mathbf{K}$ , we shall consider (inhomogeneous) Boolean models

$$\Xi(\omega) = \bigcup_{(x_i, s_i) \in \Psi(\omega)} x_i + Z_0(s_i),$$

where  $Z_0(s)$  is a compact subset of  $\mathbb{R}^d$  containing the origin for any  $s \in \mathbf{K}$ , and  $\Psi$  is the marked Poisson point process in  $\mathbb{R}^d$  with marks in  $\mathbf{K}$  associated to  $\Xi$  with intensity measure

$$\Lambda(d(x, s)) = f(x)dxQ(ds).$$

The function  $f$  and the probability measure  $Q$  on  $\mathbf{K}$  are called *intensity of  $\Xi$*  and *mark distribution*, respectively;  $\mathbb{E}_Q$  will denote the expectation with respect to  $Q$ .

Dealing with Boolean models, it is commonly assumed that the mean number of grains hitting any compact subset of  $\mathbb{R}^d$  is finite; in terms of  $\Lambda$ , this is equivalent to assume that

$$\int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(dy \times ds) < \infty \quad \forall R > 0. \quad (2)$$

**Definition 1 (Mean surface density)** *If the measure  $\mathbb{E}[\mathcal{H}^{d-1}_{|\partial\Xi}]$  is absolutely continuous with respect to  $\mathcal{H}^d$ , we denote by  $\lambda_{\partial\Xi}$  its Radon-Nikodym derivative and we also call it the mean surface density of  $\Xi$ .*

From now on we set  $Z^x := x - Z_0 \forall x \in \mathbb{R}^d$ . It is easily seen (Villa, 2008b) that

$$\begin{aligned} \mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi) &= \exp \left\{ -\mathbb{E}_Q \left[ \int_{Z^x} f(y) dy \right] \right\} \cdot \\ &\cdot \left( 1 - \exp \left\{ -\mathbb{E}_Q \left[ \int_{Z^x_{\oplus r} \setminus Z^x} f(y) dy \right] \right\} \right) \end{aligned} \quad (3)$$

Note that in the particular case in which  $\Xi$  is stationary (say  $f \equiv c > 0$ ) with deterministic typical grain, the above equation becomes

$$\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi) = e^{-c\mathcal{H}^d(Z_0)} (1 - e^{-c\mathcal{H}^d(Z_{0\oplus r} \setminus Z_0)});$$

therefore the existence of  $\sigma(x)$ , in this case independent of  $x$ , as expected, strongly depends on the existence of the so-called *outer Minkowski content* of  $Z_0$ .

## OUTER MINKOWSKI CONTENT OF SETS

The right derivative at  $r=0$  of the volume function  $V(r) := \mathcal{H}^d(A_{\oplus r})$  of a Borel set  $A \subset \mathbb{R}^d$  is also named the *outer Minkowski content* of  $A$ , defined as (Ambrosio *et al.*, 2008)

$$\mathcal{M}(A) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r},$$

provided that the limit exists finite. Note that if  $A$  is lower dimensional, then  $\mathcal{M}(A) = 2\mathcal{M}^{d-1}(A)$ , where  $\mathcal{M}^{d-1}(A)$  is the  $(d-1)$ -dimensional Minkowski content of  $A$  (see, e.g., (Ambrosio *et al.*, 2000; Federer, 1969)), whereas if  $A$  is a  $d$ -dimensional set, closure of its interior, then  $A_{\oplus r} \setminus A$  coincides with the outer Minkowski enlargement at distance  $r$  of  $\partial A$ .

It is known (e.g., by the Steiner formula) that  $\mathcal{M}(A) = \mathcal{H}^{d-1}(\partial A)$  if  $A$  is a  $d$ -dimensional convex body; more in general, it can be proved (Villa, 2008a) that the same general conditions which guarantee the existence of  $\mathcal{M}^{d-1}(\partial A)$  (see, e.g., (Ambrosio *et al.*, 2000, Th. 2.104) and (Federer, 1969, Th.3.2.39)) imply the existence of the outer Minkowski content of  $A$ , but  $\mathcal{M}(A)$  may differ from  $\mathcal{H}^{d-1}(\partial A)$ . In particular, the *d-dimensional density* (briefly, *density*) of  $A$  at its boundary points, defined as (Ambrosio *et al.*, 2000)

$$\theta_d(A, x) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A \cap B_r(x))}{\mathcal{H}^d(B_r(x))},$$

whenever the limit exists, plays a central role in the determination of the value of the outer Minkowski content of  $A$ . It is clear that  $\theta_d(A, x)$  equals 1 for all  $x$  into the interior of  $A$ , and 0 for all  $x$  into the interior of the complement set of  $A$ , while different values can be assumed at its boundary points. The following definition is given (Ambrosio *et al.*, 2000):

**Definition 2 (Essential boundary)** *For every  $t \in [0, 1]$  and every  $\mathcal{H}^d$ -measurable set  $A \subset \mathbb{R}^d$  let*

$$A^t := \{x \in \mathbb{R}^d : \theta_d(A, x) = t\}.$$

*The set of points  $\partial^* A := \mathbb{R}^d \setminus (A^0 \cup A^1)$  where the density of  $A$  is neither 0 nor 1 is called essential boundary of  $A$ .*

It is proved (e.g., see (Ambrosio *et al.*, 2000)) that all the sets  $A^i$  are Borel sets, and that

$$\mathcal{H}^{d-1}(\partial^* A \cap B) = \mathcal{H}^{d-1}(A^{1/2} \cap B) = P(A, B)$$

for all  $B \in \mathcal{B}_{\mathbb{R}^d}$ , where  $P(A, B) := |D\chi_A|(B)$ , the total variation  $|D\chi_A|$  of the characteristic function  $\chi_A$  of  $A$  in  $B$  (with  $B$  contained in an open set), is called *perimeter of  $A$  in  $B$* . In the sequel we shall write  $P(A)$  instead of  $P(A, \mathbb{R}^d)$ .

The following class of sets has been introduced in (Villa, 2008a):

**Definition 3 (The class  $\mathcal{O}$ )** Let  $\mathcal{O}$  be the class of Borel sets  $A$  of  $\mathbb{R}^d$  with countably  $\mathcal{H}^{d-1}$ -rectifiable and bounded topological boundary, such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial A, \forall r \in (0, 1) \quad (4)$$

holds for some  $\gamma > 0$  and some probability measure  $\eta$  in  $\mathbb{R}^d$  absolutely continuous with respect to  $\mathcal{H}^{d-1}$ .

**Theorem 4** (Villa, 2008a) The class  $\mathcal{O}$  is stable under finite unions and any  $A \in \mathcal{O}$  admits outer Minkowski content, given by

$$\mathcal{M}(A) = P(A) + 2\mathcal{H}^{d-1}(\partial A \cap A^0).$$

**Remark 5** Condition (4) is a kind of quantitative non-degeneracy condition which prevents  $\partial A$  from being too sparse; simple examples (see, e.g., (Ambrosio *et al.*, 2008, Example 3)) show that  $\mathcal{M}(A)$  can be infinite, and  $\mathcal{H}^{d-1}(\partial A)$  arbitrarily small, when this condition fails.

In (Villa, 2008a) it is also proved that the same conclusions of the above theorem hold for a class of Borel subsets of  $\mathbb{R}^d$ , defined similarly to  $\mathcal{O}$  by replacing the condition of absolute continuity of  $\eta$  with the assumption that  $\mathcal{M}^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A)$ ; then it follows that this class of sets contains all Borel sets with  $(d-1)$ -rectifiable boundary (and so finite unions of sets with positive reach or with Lipschitz boundary, in particular).

By (3) and the definition of  $\sigma(x)$  it is intuitive that the following theorem, which may be seen as a generalization of Theorem 4 since the classical (outer) Minkowski content follows by choosing  $f \equiv 1$ , plays an important role in the determination of an explicit formula for  $\sigma(x)$ .

**Theorem 6** Let  $\mu$  be a positive measure in  $\mathbb{R}^d$  absolutely continuous with respect to  $\mathcal{H}^d$  with locally

bounded density  $f$ . Let  $A$  belong to  $\mathcal{O}$ . If the set of all the discontinuity points of  $f$  is  $\mathcal{H}^{d-1}$ -negligible, then

$$\begin{aligned} & \lim_{r \downarrow 0} \frac{\mu(A \oplus r \setminus A)}{r} \\ &= \int_{\partial^* A} f(x) \mathcal{H}^{d-1}(dx) + 2 \int_{\partial A \cap A^0} f(x) \mathcal{H}^{d-1}(dx). \end{aligned}$$

## MEAN BOUNDARY DENSITIES

We give now general regularity conditions on the typical grain  $Z_0$  and on the intensity  $f$  of an inhomogeneous Boolean model  $\Xi$  in  $\mathbb{R}^d$  such that  $\sigma(x)$  exists and is finite for all  $x \in \mathbb{R}^d$ .

**Assumptions:** (A1)  $\partial Z_0$  is countably  $\mathcal{H}^{d-1}$ -rectifiable and compact, and such that there exist  $\gamma > 0$  and a random closed set  $\Theta \supseteq \partial Z_0$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)] < \infty$  such that, for  $Q$ -a.e.  $s \in \mathbf{K}$ ,

$$\mathcal{H}^{d-1}(\Theta(s) \cap B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial Z_0(s), \forall r \in (0, 1). \quad (5)$$

(A2) The set of all the discontinuity points of  $f$  is  $\mathcal{H}^{d-1}$ -negligible and  $f$  is locally bounded such that, denoted by  $\delta$  the diameter of  $Z_0$ , for any compact set  $K \subset \mathbb{R}^d$

$$\sup_{y \in K \oplus \delta} f(y) \leq \xi_K \quad (6)$$

for some random variable  $\xi_K$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)\xi_K] < \infty$ .

**Remark 7** i) condition (6) is trivially satisfied whenever  $f$  is bounded, or  $f$  is locally bounded and  $\delta \leq c \in \mathbb{R}_+$   $Q$ -a.s.;

ii) the assumption (A1) is often fulfilled with  $\Theta = \partial Z_0$  or  $\Theta = \partial Z_0 \cup \tilde{A}$  for some sufficiently regular random closed set  $\tilde{A}$ . As a matter of fact, it can be seen as the stochastic version of (4), which, in many applications, is satisfied with  $\eta(\cdot) = \mathcal{H}^n(\tilde{A} \cap \cdot)$  for some closed set  $\tilde{A} \supseteq A$ , as proved in (Ambrosio *et al.*, 2000, p. 111) (see also (Ambrosio *et al.*, 2008)).

**Proposition 8** (Villa, 2008b) Let  $\Xi$  be a Boolean model as in the Assumptions. Then

$$\begin{aligned} \sigma(x) &= \exp \left\{ - \mathbb{E}_Q \left[ \int_{Z^x} f(y) dy \right] \right\} \\ &\cdot \mathbb{E}_Q \left[ \int_{\partial^* Z^x} f(y) \mathcal{H}^{d-1}(dy) + 2 \int_{\partial Z^x \cap (Z^x)^0} f(y) \mathcal{H}^{d-1}(dy) \right] \end{aligned} \quad (7)$$

for all  $x \in \mathbb{R}^d$ .

It can be proved (Villa, 2008b) that for any Boolean model  $\Xi$  as in the Assumptions satisfying (2),  $\mathbb{E}[\mathcal{H}_{|\partial\Xi}^{d-1}]$  is a Radon measure absolutely continuous with respect to  $\mathcal{H}^d$ . As a consequence,  $\mathbb{E}[\mathcal{H}_{|\partial^*\Xi}^{d-1}]$  and  $E[\mathcal{H}_{|\partial\Xi\cap\Xi^0}^{d-1}]$  are Radon measures with density  $\lambda_{\partial^*\Xi}$  and  $\lambda_{\partial\Xi\cap\Xi^0}$ , respectively.

The next theorem shows that, without any further regularity assumption on  $Z_0$ ,  $\sigma$  differs from the mean surface density  $\lambda_{\partial\Xi}$  of  $\Xi$ , in general.

**Theorem 9** (Villa, 2008b) *If  $\Xi$  is a Boolean model as in the Assumptions satisfying (2), then*

$$\sigma(x) = \lambda_{\partial^*\Xi}(x) + 2\lambda_{\partial\Xi\cap\Xi^0}(x) \quad (8)$$

for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .

The proof of the above theorem is based on the following two main steps:

1. the limit in the left side of (1) gives rise (for  $B$  varying in  $\mathcal{B}_{\mathbb{R}^d}$ ) to a Radon measure in  $\mathbb{R}^d$ , namely  $\mathbb{E}[\mathcal{H}_{|\partial^*\Xi}^{d-1} + 2\mathcal{H}_{|\partial\Xi\cap\Xi^0}^{d-1}]$ , absolutely continuous with respect to  $\mathcal{H}^d$ ;
2. limit and integral can be exchanged in the right side of (1).

**Remark 10** By the proof of Theorem 9 it follows in particular that  $\Xi$  admits *local mean outer Minkowski content*, i.e. for any Borel set  $B \subset \mathbb{R}^d$  such that  $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \partial B)] = 0$  it holds

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Xi_{\oplus r} \setminus \Xi) \cap B)]}{r} = \mathbb{E}[P(\Xi, B)] + 2\mathbb{E}[\mathcal{H}_{|\partial\Xi\cap\Xi^0}^{d-1}(B)]. \quad (9)$$

Taking into account the general inequality  $P(A) \leq \mathcal{H}^{d-1}(\partial A) \forall A \in \mathcal{B}_{\mathbb{R}^d}$ , by assumption (A1) we know that  $Z_0$  has finite perimeter  $Q$ -a.s.; we also remind that a classical result of geometric measure theory states that any set  $A \subset \mathbb{R}^d$  of finite perimeter has density either 0 or 1 or 1/2 at  $\mathcal{H}^{d-1}$ -almost every point of its boundary (e.g., see (Ambrosio et al., 2000, Theorem 3.61)); as a consequence,  $\Xi$  is a random set with locally finite perimeter and

$$\lambda_{\partial\Xi} = \lambda_{\partial\Xi\cap\Xi^0} + \lambda_{\partial^*\Xi} + \lambda_{\partial\Xi\cap\Xi^1}.$$

Then, it follows that  $\sigma(x) = \lambda_{\partial\Xi}(x) = \lambda_{\partial^*\Xi}(x)$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$  if  $\mathbb{E}[\mathcal{H}_{|\partial\Xi\cap(\Xi^0 \cup \Xi^1)}^{d-1}] = 0$ .

The proof of the following proposition, which provides a sufficient regularity condition on the typical grain

ensuring  $\sigma = \lambda_{\partial\Xi}$ , is based on the fact that if  $A_1$  and  $A_2$  are random closed sets of finite perimeter in  $\mathbb{R}^d$  with  $\mathbb{E}[P(A_i)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial A_i)]$ ,  $i = 1, 2$ , such that  $\mathbb{E}[\mathcal{H}^{d-1}(\partial A_1 \cap \partial A_2)] = 0$ , then  $\mathbb{E}[P(A_1 \cup A_2)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial(A_1 \cup A_2))]$ .

**Proposition 11** (Villa, 2008b) *Let  $\Xi$  be a Boolean model as in the Assumptions satisfying (2), such that  $\mathbb{E}_Q[P(Z_0)] = \mathbb{E}_Q[\mathcal{H}^{d-1}(\partial Z_0)]$ . Then*

$$\sigma(x) = \exp \left\{ -\mathbb{E}_Q \left[ \int_{Z^x} f(y) dy \right] \right\} \cdot \mathbb{E}_Q \left[ \int_{\partial Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] = \lambda_{\partial\Xi}(x)$$

for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .

We remind that any compact subset  $A$  of  $\mathbb{R}^d$  with Lipschitz boundary satisfies  $P(A) = \mathcal{H}^{d-1}(\partial A)$ ; the same holds also for a certain class of compact sets with positive reach, containing, in particular, all the  $d$ -dimensional convex bodies. (See (Ambrosio et al., 2008).)

## FURTHER REMARKS AND PARTICULAR CASES

### The $(d-1)$ -dimensional case

Let us notice that  $\sigma(x)$  might be not trivial for  $(d-1)$ -dimensional random closed sets, as well. The  $(d-1)$ -dimensional counterpart of Proposition 8 and Theorem 9 is given by the following theorem (Villa, 2008b)

**Theorem 12** *Let  $\Xi$  be a Boolean model in  $\mathbb{R}^d$  satisfying the two following conditions on the intensity  $f$  and the typical grain  $Z_0$ :*

(A1')  $Z_0$  is countably  $\mathcal{H}^{d-1}$ -rectifiable and compact, and such that there exist  $\gamma > 0$  and a random closed set  $\Theta \supseteq Z_0$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)] < \infty$  such that, for  $Q$ -a.e.  $s \in \mathbf{K}$ ,

$$\mathcal{H}^{d-1}(\Theta(s) \cap B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in Z_0(s), \forall r \in (0, 1).$$

(A2') The set of all the discontinuity points of  $f$  is  $\mathcal{H}^{d-1}$ -negligible and  $f$  is locally bounded such that, denoted by  $\delta$  the diameter of  $Z_0$ , for any compact set  $K \subset \mathbb{R}^d$ ,  $\sup_{y \in K_{\oplus \delta}} f(y) \leq \xi_K$  for some random variable  $\xi_K$  with  $\mathbb{E}_Q[\mathcal{H}^{d-1}(\Theta)\xi_K] < \infty$ .

Then  $\mathbb{E}[\mathcal{H}_{|\Xi}^{d-1}]$  is a Radon measure in  $\mathbb{R}^d$  absolutely continuous with respect to  $\mathcal{H}^d$ , whose

density is given by

$$\lambda_{\Xi}(x) = \mathbb{E}_Q \left[ \int_{Z^x} f(y) \mathcal{H}^{d-1}(dy) \right] \\ = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r})}{2r} = \frac{\sigma(x)}{2} \quad (10)$$

for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .

**Remark 13** It can be shown (Villa, 2008b) that, contrary to the  $d$  dimensional case, we do not have to assume also the usual condition (2) on Boolean models, being this implied now by assumptions (A1') and (A2'). As a simple counterexample let us consider a stationary Boolean model  $\Xi$  of balls; namely, let  $f \equiv c > 0$  and  $Z_0 = B_\rho(0)$  with  $\rho$  random variable greater than 1. If  $\rho$  is such that  $\mathbb{E}[\rho^{d-1}] < \infty$ , but  $\mathbb{E}[\rho^d] = \infty$ , then it is easy to check that  $\Xi$  satisfies the Assumptions with  $\Xi = \partial Z_0$  and  $\xi_K = c$  for all compacts  $K$  in  $\mathbb{R}^d$ , but

$$\int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(dy \times ds) = cb_d \mathbb{E}[(R + \rho)^d] = \infty.$$

Clearly, the equalities in (10) are in accordance with (7) and (8): it is sufficient to notice that  $Z_0 = \partial Z_0$  (since  $Z_0$  has empty interior) and that  $Z_0$  has null density at any point of  $\mathbb{R}^d$ , and so  $\Xi$  as well.

### The stationary case

The previous results simplify in the stationary case. Let us notice that if  $\Xi$  is stationary with  $f \equiv c > 0$ , then only the regularity assumption (A1) on the typical grain  $Z_0$  and the usual condition (2) are required (only (A1') in the lower dimensional case, by Remark 13). Then  $\sigma$  is now independent of  $x$ , as expected, given by

$$\sigma = e^{-c\mathbb{E}_Q[\mathcal{H}^d(Z_0)]} c \mathbb{E}_Q[\mathcal{S}\mathcal{M}(Z_0)],$$

where  $\mathcal{S}\mathcal{M}(Z_0)$  exists finite equal to  $P(Z_0) + 2\mathcal{H}^{d-1}(Z_0^0 \cap \partial Z_0)$  as a consequence of (A1).

### Deterministic typical grain

Whenever the typical grain  $Z_0$  is deterministic, all the above results still hold (clearly removing the expected value with respect to  $Q$ ) under weaker corresponding assumptions, analogous to those ones of Theorem 6. Namely, it is sufficient to assume that  $Z_0$  is a compact set in  $\mathcal{O}$ , and that the intensity  $f$  is locally bounded such that the set of all its discontinuity points is  $\mathcal{H}^{d-1}$ -negligible.

**Remark 14** By the definition of the class  $\mathcal{O}$ , there exists a probability measure  $\eta$  in  $\mathbb{R}^d$  such that

$\eta(B_r(x)) \geq \gamma r^{d-1}$  for all  $x \in Z_0$  and  $r \in (0, 1)$ . The role played by  $\mathcal{H}^{d-1}(\Xi)$  in the assumption (A1) is played here by  $\eta$ . We also recall that, by geometric measure theory arguments, condition (4) implies that  $\mathcal{H}^{d-1}(\partial Z_0) < \infty$ . Further, since  $Z_0$  is compact then  $\text{diam} Z_0 = \delta < \infty$  and so the assumption (A2) and the condition (2) are easily checked.

### A statistical application

Let  $\Xi$  be a Boolean model as in the assumptions of Proposition 11; then

$$\lambda_{\partial \Xi}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Xi_{\oplus r} \setminus \Xi)}{r} \quad (11)$$

for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .

Therefore, given an i.i.d. random sample  $\Xi_1, \dots, \Xi_N$  of  $\Xi$ , by repeating the same argument in (Villa, 2008b), a natural estimator  $\hat{\lambda}_{\partial \Xi}^N(x)$  of  $\lambda_{\partial \Xi}(x)$  can be defined as follows:

$$\hat{\lambda}_{\partial \Xi}^N(x) := \frac{\sum_{i=1}^N \mathbf{1}_{\Xi_i \oplus R_N \setminus \Xi_i}(x)}{NR_N} \\ = \frac{\sum_{i=1}^N (\mathbf{1}_{\Xi_i \cap B_{R_N}(x) \neq \emptyset} - \mathbf{1}_{\Xi_i \cap \{x\} \neq \emptyset})}{NR_N}, \quad (12)$$

with  $R_N$  such that

$$\lim_{N \rightarrow \infty} R_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} NR_N = \infty. \quad (13)$$

It can be easily seen that  $\hat{\lambda}_{\partial \Xi}^N(x)$  is an asymptotically unbiased and weakly consistent estimator of  $\lambda_{\partial \Xi}(x)$  for  $\mathcal{H}^d$ -a.e.  $x \in \mathbb{R}^d$ .

Then, a problem of statistical interest could be to find the optimal *width*  $R_N$  satisfying condition (13) which minimizes the mean squared error of  $\hat{\lambda}_{\partial \Xi}^N(x)$  (i.e.  $\mathbb{E}[(\hat{\lambda}_{\partial \Xi}^N(x) - \lambda_{\partial \Xi}(x))^2]$ ).

### REFERENCES

- Ambrosio L, Colesanti A, Villa E (2008). Outer Minkowski content for some classes of closed sets. *Math Ann* 342:727–748.
- Ambrosio L, Fusco N, Pallara D (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford: Clarendon Press.
- Baddeley A, Barany I, Schneider R, Weil W (2007). *Stochastic Geometry*, Lecture Notes in Mathematics 1982. Berlin: Springer.
- Falconer KJ (1985). *The Geometry of Fractal Sets*, Cambridge: Cambridge University Press.
- Federer H (1969). *Geometric Measure Theory*. Berlin: Spriger.

- Hug D (2000). Contact distributions of Boolean models. *Rend Circ Mat Palermo* (2 Suppl. No. 65, part I:137–181.
- Hug D, Last G, Weil W (2006). Polynomial parallel volume, convexity and contact distributions of random sets. *Probab Theory Relat Fields* 135: 169–200.
- Matheron G (1975). *Random Sets and Integral Geometry*. New York: John Wiley & Sons.
- Stoyan D, Kendall WS, Mecke J (1995). *Stochastic Geometry and its Applications*. Chichester: Wiley.
- Villa E (2008a). On the outer Minkowski content of sets. *Ann. Mat. Pura Appl. To appear*.
- Villa E (2008b) Mean densities and spherical contact distribution function of inhomogeneous Boolean models. *Submitted*.
- Weil W. (2001). Densities of mixed volumes for Boolean models. *Adv Appl Prob* 33:39–60.
- Zähle M (1982) Random processes of Hausdorff rectifiable closed sets. *Math Nachr* 108:49–72.