

On the geometric densities of random closed sets

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Abstract

In many applications it is of great importance to handle evolution equations about random closed sets of different (even though integer) Hausdorff dimensions, including local information about initial conditions and growth parameters. Following a standard approach in geometric measure theory such sets may be described in terms of suitable measures. For a random closed set of lower dimension with respect to the environment space, the relevant measures induced by its realizations are singular with respect to the Lebesgue measure, and so their usual Radon-Nikodym derivatives are zero almost everywhere. In this paper we suggest to cope with these difficulties by introducing random generalized densities (distributions) à la Dirac-Schwarz, for both the deterministic case and the stochastic case. In this last one we analyze mean generalized densities, and relate them to densities of the expected values of the relevant measures. Many models of interest in material science and in biomedicine are based on time dependent random closed sets, as the ones describing the evolution of (possibly space and time inhomogeneous) growth processes; in such a situation, the Delta formalism provides a natural framework for deriving evolution equations for mean densities at all (integer) Hausdorff dimensions, in terms of the local relevant kinetic parameters of birth and growth. In this context connections with the concepts of hazard function, and spherical contact function are offered.

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1 Preliminaries and notations

We remind here concepts and results of current literature which are relevant for our analysis.

Let us consider the space \mathbb{R}^d and denote by ν^d the usual d -dimensional Lebesgue measure, and by $\mathcal{B}_{\mathbb{R}^d}$ the Borel σ -algebra of \mathbb{R}^d .

We know that every positive Radon measure μ on \mathbb{R}^d can be represented in the form

$$\mu = \mu_{\ll} + \mu_{\perp},$$

where μ_{\ll} and μ_{\perp} are the absolutely continuous part with respect to ν^d , and the singular part of μ , respectively. Denoted by $B_r(x)$ the d -dimensional closed ball centered in x with radius r ,

it is possible to define the following quantities:

$$(\overline{D\mu})(x) := \limsup_{r \rightarrow 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}, \quad (\underline{D\mu})(x) := \liminf_{r \rightarrow 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}.$$

Definition 1 If $(\overline{D\mu})(x) = (\underline{D\mu})(x) < +\infty$, then their common value is called the symmetric derivative of μ at x and is denoted by $(D\mu)(x)$.

$(\overline{D\mu})(x)$ and $(\underline{D\mu})(x)$ are also called upper and lower densities of μ at x .

As a consequence of the Besicovitch Derivation Theorem (see [3], p.54), we have that $(D\mu)(x)$ exists for ν^d -a.e. $x \in \mathbb{R}^d$, and it is the Radon-Nikodym derivative of μ_{\ll} ; while μ_{\perp} is the restriction of μ to the ν^d -negligible set $\{x \in \mathbb{R}^d : \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))} = \infty\}$.

Let us denote by \mathcal{H}^s the s -dimensional Hausdorff measure, and recall the following definition about the dimensional density of a set.

Definition 2 Let A be a subset of \mathbb{R}^d , \mathcal{H}^s -measurable, with $0 < \mathcal{H}^s(A) < \infty$ ($0 \leq s < \infty$). The upper and lower s -dimensional densities of A at a point $x \in \mathbb{R}^d$ are defined as

$$\overline{D}^s(A, x) := \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s)r^s}$$

and

$$\underline{D}^s(A, x) := \liminf_{r \rightarrow 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s)r^s},$$

respectively, where $b(s) \equiv \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$. If $\overline{D}^s(A, x) = \underline{D}^s(A, x)$ we say that the s -dimensional density of A at x exists and we write $D^s(A, x)$ for the common value.

Note that when s is integer, say $s = n$, then $b(n) = b_n$, the volume of the unit ball in \mathbb{R}^n .

We are going to consider a class of subsets of \mathbb{R}^d with integer dimension.

Definition 3 Given an integer $n \in [0, d]$, we say that a closed subset A of \mathbb{R}^d is n -regular, if it satisfies the following conditions:

- (i) $\mathcal{H}^n(A \cap B_R(0)) < \infty$ for any $R > 0$;
- (ii) $\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A \cap B_r(x))}{b_n r^n} = 1$ for \mathcal{H}^n -a.e. $x \in A$.

Note that condition (ii) is related to a characterization of the \mathcal{H}^n -rectifiability of the set A ([15], p.256, 267, [3], p.83).

Remark 4 We may observe that if Θ_n is an n -regular closed set in \mathbb{R}^d , we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_n r^n} = \begin{cases} 1 & \mathcal{H}^n\text{-a.e. } x \in \Theta_n, \\ 0 & \forall x \notin \Theta_n. \end{cases} \quad (1)$$

In fact, since Θ_n^C is open, $\forall x \notin \Theta_n \exists r_0 > 0$ such that $\forall r \leq r_0 \ B_r(x) \subset \Theta_n^C$, that is $\mathcal{H}^n(\Theta_n \cap B_r(x)) = 0$ for all $r \leq r_0$; thus the limit equals 0, $\forall x \in \Theta_n^C$.

For a general set A , problems about “ \mathcal{H}^n -a.e.” and “ \forall ” arise when we consider a point $x \in \partial A$ or singular. For example, if A is a closed square in \mathbb{R}^2 , for all point x on the edges

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^2(A \cap B_r(x))}{b_2 r^2} = \frac{1}{2},$$

while for each of the four vertices the limit equals $1/4$.

Observe that in both of cases the set of such points has \mathcal{H}^2 -measure 0.

From now on we shall consider n -regular closed sets Θ_n in \mathbb{R}^d , with $0 \leq n \leq d$.

As a consequence, for $n < d$, (by assuming $0 \cdot \infty = 0$), by (1) we also have:

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_n r^n} \frac{b_n r^n}{b_d r^d} = \begin{cases} \infty & \mathcal{H}^n\text{-a.e. } x \in \Theta_n, \\ 0 & \forall x \notin \Theta_n. \end{cases}$$

Note that in the particular case $n = 0$, with $\Theta_0 = X_0$ point in \mathbb{R}^d (X_0 is indeed a 0-regular closed set),

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^0(X_0 \cap B_r(x))}{b_d r^d} = \begin{cases} \infty & \text{if } x = X_0, \\ 0 & \text{if } x \neq X_0; \end{cases}$$

Note that, if Θ_n is an n -regular closed set in \mathbb{R}^d with $n < d$, then the Radon measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot)$$

is a singular measure with respect to ν^d , and so $(D\mu_{\Theta_n})(x) = 0$ ν^d -a.e. $x \in \mathbb{R}^d$.

But, in analogy with the Dirac delta function $\delta_{X_0}(x)$ associated with a point $X_0 \in \mathbb{R}^d$, we may introduce the following definition

Definition 5 We call δ_{Θ_n} , the generalized density (or, briefly, the density) associated with Θ_n , the quantity

$$\delta_{\Theta_n}(x) := \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d},$$

finite or not.

In this way $\delta_{\Theta_n}(x)$ can be considered as the *generalized density* (or the *generalized Radon-Nikodym derivative*) of the measure μ_{Θ_n} with respect to ν_d . We may notice that in the case $\Theta_0 = X_0$, $\delta_{X_0}(x)$ coincides with the well known delta function at a point X_0 , that is the (generalized) density of the singular Dirac measure ε_{X_0} [20].

The usefulness of introducing this *generalized function* will turn to be clear in the following, in particular in the stochastic case, where we shall give an example in which it is natural to deal with this kind of *density* associated to a random lower-dimensional closed set.

For a full comprehension of this, we expose now our definitions and results, and we will summarize in the Conclusions why we find necessary to work directly with these *delta functions* in general situations.

We like to notice that the possible use of random distributions in spatial statistics had already been anticipated by Matheron in [21].

2 Densities as linear functionals

2.1 The deterministic case

We know that the Dirac delta δ_{X_0} at a point $X_0 \in \mathbb{R}^d$ can be defined as a linear functional associated with a finite Borel measure, the well known Dirac measure ε_{X_0} , concentrated at X_0 ; as such it is the (generalized) density of ε_{X_0} .

In fact, we recall that, according to Riesz theorem, Radon measures in \mathbb{R}^d (i.e. nonnegative and σ -additive set functions defined on the Borel σ -algebra $\mathcal{B}_{\mathbb{R}^d}$ which are finite on bounded sets) can be canonically identified with linear and order preserving functionals on $C_c(\mathbb{R}^d, \mathbb{R})$, the space of continuous functions with compact support in \mathbb{R}^d . The identification is provided by the integral operator, i.e.

$$(\mu, f) = \int_{\mathbb{R}^d} f d\mu \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

If $\mu \ll \nu^d$, it admits, as Radon-Nikodym density, a classical function δ_μ defined almost everywhere in \mathbb{R}^d , so that

$$(\mu, f) = \int_{\mathbb{R}^d} f(x) \delta_\mu(x) dx \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R})$$

in the usual sense of Lebesgue integral.

If $\mu \perp \nu^d$, we may speak of a density δ_μ only in the sense of distributions (it is almost everywhere trivial, but it is ∞ on a set of ν^d -measure zero). In this case the symbol

$$\int_{\mathbb{R}^d} f(x) \delta_\mu(x) dx := (\mu, f)$$

can still be adopted, provided the integral on the left hand side is understood in a generalized sense, and not as a Lebesgue integral.

In either cases, from now on, we will denote by (δ_μ, f) the quantity (μ, f) .

Accordingly, we say that a sequence of measures μ_n *weakly* converges* to a Radon measure μ if (δ_{μ_n}, f) converges to (δ_μ, f) for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$. A classical criterion (see for instance [14] or [3]) states that μ_n weakly* converge to μ if and only if $\mu_n(A) \rightarrow \mu(A)$ for any bounded open set A with $\mu(\partial A) = 0$.

Using the common integral representation for generalized functions

$$\int_A \delta_{X_0}(x) dx := \varepsilon_{X_0}(A) = \mathcal{H}^0(X_0 \cap A),$$

we have

$$(\delta_{X_0}, f) = \int_{\mathbb{R}^d} \delta_{X_0}(x) f(x) dx = f(X_0), \quad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

If we define, for $m \in \mathbb{N}$,

$$\varphi_m(x) := \frac{m}{2} \mathbf{1}_{(X_0 - \frac{1}{m}, X_0 + \frac{1}{m})}(x), \quad (2)$$

(here $\mathbf{1}_A$ stands for the characteristic function of A), we have

$$\lim_{m \rightarrow \infty} \int_A \varphi_m(x) dx = \mathcal{H}^0(X_0 \cap A),$$

for any measurable set A such that $\mathcal{H}^0(X_0 \cap \partial A) = 0$; in other words, the associated measures $\mu_m = \varphi_m \nu^d$ weakly* converge to the measure ε_{X_0} (equivalently, the linear functionals φ_m weakly* converge to the linear functional δ_{X_0}), as $m \rightarrow \infty$.

Now we are ready to introduce the delta function of an n -regular set Θ_n as the linear functional (the generalized function) $\delta_{\Theta_n}(x)$ in a similar way.

Consider the measure defined on the Borel σ -algebra of \mathbb{R}^d , as follows

$$\mu_{\Theta_n}(A) := \mathcal{H}^n(\Theta_n \cap A), \quad A \in \mathcal{B}_{\mathbb{R}^d}. \quad (3)$$

Define now the function

$$\delta_{\Theta_n}^{(r)}(x) := \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d},$$

and correspondingly the associated measure $\mu_{\Theta_n}^{(r)} = \delta_{\Theta_n}^{(r)} \nu^d$:

$$\mu_{\Theta_n}^{(r)}(A) := \int_A \delta_{\Theta_n}^{(r)}(x) dx, \quad A \in \mathcal{B}_{\mathbb{R}^d}.$$

In accordance with the the functional notation we have introduced in terms of the respective (generalized) densities, we have

$$\begin{aligned} (\delta_{\Theta_n}^{(r)}, f) &:= \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n}^{(r)} dx, \\ (\delta_{\Theta_n}, f) &:= \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n} dx, \end{aligned} \quad (4)$$

for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

We may prove the following result.

Proposition 6 *For all $f \in C_c(\mathbb{R}^d, \mathbb{R})$, it holds*

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n}^{(r)} dx = \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n} dx.$$

Proof. Thanks to the quoted criterion on weak* convergence of measures on metric spaces, we may limit ourselves to prove that for any bounded Borel A of \mathbb{R}^d such that $\mu_{\Theta_n}(\partial A) = 0$, the following holds

$$\lim_{r \rightarrow 0} \mu_{\Theta_n}^{(r)}(A) = \mu_{\Theta_n}(A).$$

It is clear that, for any fixed $r > 0$ and for any bounded fixed set A , there exists a compact set K containing A such that $\mathcal{H}^n(\Theta_n \cap B_r(x)) = \mathcal{H}^n(\Theta_n \cap K \cap B_r(x))$ for all $x \in A$. Thus, we have

$$\begin{aligned} \lim_{r \rightarrow 0} \mu_{\Theta_n}^{(r)}(A) &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} dx \\ &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x)}{b_d r^d} \left(\int_{\Theta_n \cap K} \mathbf{1}_{B_r(x)}(y) \mathcal{H}^n(dy) \right) dx \\ &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \left(\int_{\Theta_n \cap K} \frac{\mathbf{1}_A(x) \mathbf{1}_{B_r(x)}(y)}{b_d r^d} \mathcal{H}^n(dy) \right) dx; \end{aligned}$$

by exchanging the integrals and using the identity $\mathbf{1}_{B_r(x)}(y) = \mathbf{1}_{B_r(y)}(x)$,

$$\begin{aligned} &= \lim_{r \rightarrow 0} \int_{\Theta_n \cap K} \left(\int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x) \mathbf{1}_{B_r(y)}(x)}{b_d r^d} dx \right) \mathcal{H}^n(dy) \\ &= \lim_{r \rightarrow 0} \int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy); \end{aligned}$$

since $\frac{\nu^d(A \cap B_r(y))}{b_d r^d} \leq 1$, and by hypothesis we know that $\mathcal{H}^n(\Theta_n \cap K) < \infty$,

$$= \int_{\Theta_n \cap K} \lim_{r \rightarrow 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy);$$

by $\mathcal{H}^n(\Theta_n \cap \partial A) = 0$, and $\lim_{r \rightarrow 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} = 0$ for all $y \in (\text{clos} A)^C$,

$$= \int_{\Theta_n \cap \text{int} A} \lim_{r \rightarrow 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) = \mathcal{H}^n(\Theta_n \cap \text{int} A),$$

since, by the Lebesgue density theorem ([15], p.14), $\lim_{r \rightarrow 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} = 1$ for every $y \in \text{int} A$.

So, by the condition $\mathcal{H}^n(\Theta_n \cap \partial A) = 0$, we conclude that

$$\lim_{r \rightarrow 0} \mu_{\Theta_n}^{(r)}(A) = \mu_{\Theta_n}(A). \quad (5)$$

□

By the above proposition we may claim that the sequence of measures $\mu_{\Theta_n}^{(r)}$ weakly* converges to the measure μ_{Θ_n} ; in other words, the sequence of linear functionals $\delta_{\Theta_n}^{(r)}$ weakly* converges to the linear functional δ_{Θ_n} , i.e.

$$(\delta_{\Theta}, f) = \lim_{r \rightarrow 0} (\delta_{\Theta_n}^{(r)}, f) \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}). \quad (6)$$

We may like to point out that the role of the sequence $\{\varphi_m(x)\}$ for $n = 0$ in (2), is played here, for any $n \in \{0, 1, \dots, d\}$, by $\left\{ \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} \right\}$, by taking $r = 1/m$. We notice that if $n = 0$ and $\Theta_0 = X_0$, then

$$\frac{\mathcal{H}^0(X_0 \cap B_r(x))}{b_d r^d} = \frac{\mathbf{1}_{B_r(X_0)}(x)}{b_d r^d} = \frac{\mathbf{1}_{X_0 \oplus_r}(x)}{b_d r^d},$$

which is the usual “enlargement” of the point X_0 ($X_{0 \oplus_r}$ is the Minkowski sum $X_0 \oplus B_r(0)$); in the case $d = 1$ we have in particular that

$$\delta_{X_0}^{(r)}(x) = \frac{1}{2r} \mathbf{1}_{(X_0-r, X_0+r)}(x),$$

in accordance with (2).

Remark 7 This convergence result can also be understood noticing that $\delta_{\Theta_n}^{(r)}(x)$ is the convolution (e.g. [3]) of the measure μ_{Θ_n} with the kernel

$$\rho_r(y) := \frac{1}{b_d r^d} \mathbf{1}_{B_r(0)}(y).$$

In analogy with the classical Dirac delta, we may regard the continuous linear functional δ_{Θ_n} as a generalized function on the usual test space $C_c(\mathbb{R}^d, \mathbb{R})$, and, in accordance with the usual representation of distributions in the theory of generalized functions, we formally write

$$\int_{\mathbb{R}^d} f(x) \delta_{\Theta_n}(x) dx := (\delta_{\Theta_n}, f). \quad (7)$$

If we rewrite (6) with the notation in (7), we have a formal exchange of limit and integral

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(x) \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} dx = \int_{\mathbb{R}^d} f(x) \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} dx.$$

Further, we notice that the classical Dirac delta $\delta_{X_0}(x)$ associated to a point X_0 now follows as a particular case.

Remark 8 *If Θ is a piecewise smooth surface S in \mathbb{R}^n (and so n -regular), then, by the definition in (4), it follows that, for any test function f ,*

$$(\delta_S, f) = \int_S f(x) dS,$$

which is the definition of δ_S in [24] on page 33.

In terms of the above arguments, we may state that $\delta_{\Theta_n}(x)$ is the (*generalized*) *density* of the measure μ_{Θ_n} , defined by (3), with respect to the usual Lebesgue measure ν^d on \mathbb{R}^d and, formally, we may define

$$\frac{d\mu_{\Theta_n}}{d\nu^d}(x) := \delta_{\Theta_n}(x). \quad (8)$$

Note that if $n = d$, then μ_{Θ_d} is absolutely continuous with respect to ν^d , and $\frac{d\mu_{\Theta_d}}{d\nu^d}(x)$ is the classical Radon-Nikodym derivative.

2.2 The stochastic case

We recall that a *random closed set* Ξ in \mathbb{R}^d is a measurable map

$$\Xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where \mathbb{F} denotes the class of the closed subsets in \mathbb{R}^d , and $\sigma_{\mathbb{F}}$ is the so called *hit-or-miss topology* (see [22]).

Definition 9 *Given an integer n , with $0 \leq n \leq d$, we say that a random closed set Θ_n in \mathbb{R}^d is n -regular, if it satisfies the following conditions:*

- (i) *for almost all $\omega \in \Omega$, $\Theta_n(\omega)$ is an n -regular closed set in \mathbb{R}^d ;*
- (ii) *$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_R(0))] < \infty$ for any $R > 0$.*

(For a discussion about measurability of $\mathcal{H}^n(\Theta_n)$ we refer to [5, 21, 25]).

Suppose now that Θ_n is a random n -regular closed set in \mathbb{R}^d . By condition (ii) the random measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot)$$

is almost surely a Radon measure, and we may consider the corresponding *expected measure*

$$\mathbb{E}[\mu_{\Theta_n}](\cdot) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot)].$$

In this case $\delta_{\Theta_n}(x)$ is a random quantity, and δ_{Θ_n} is a *random linear functional* in the following sense:

Definition 10 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $T(\omega)$ be a linear functional on a suitable test space \mathcal{S} for any $\omega \in \Omega$.*

We say that T is a random linear functional on \mathcal{S} if and only if (T, s) is a real random variable $\forall s \in \mathcal{S}$; i.e.

$$\forall s \in \mathcal{S}, \forall V \in \mathcal{B}_{\mathbb{R}} \quad \{\omega \in \Omega : (T(\omega), s) \in V\} \subset \mathcal{F}.$$

Remark 11 *The definition above is the analogous of the well known definition for Banach valued random variables (see, e.g., [4, 7, 8]).*

Now, if T is a random linear functional on \mathcal{S} , then it makes sense to compute the expected value of the random variable (T, s) for any $s \in \mathcal{S}$:

$$\mathbb{E}[(T, s)] = \int_{\Omega} (T(\omega), s) \, d\mathbb{P}(\omega).$$

If for any $s \in \mathcal{S}$ the random variable (T, s) is integrable, then the map

$$s \in \mathcal{S} \longmapsto \mathbb{E}[(T, s)] \in \mathbb{R}$$

is well defined.

Hence, by extending the definition of expected value of a random operator à la Pettis (or Gelfand-Pettis, [4, 7, 8]), we may define the *expected linear functional* associated with T as follows (see [21]) .

Definition 12 *Let T be a random linear functional T on \mathcal{S} .*

If for any $s \in \mathcal{S}$ the random variable (T, s) is integrable, then we define the expected linear functional of T as the linear functional $\mathbb{E}[T]$ such that

$$(\mathbb{E}[T], s) = \mathbb{E}[(T, s)] \quad \forall s \in \mathcal{S};$$

i.e.

$$\mathbb{E}[T] : s \in \mathcal{S} \longmapsto (\mathbb{E}[T], s) := \int_{\Omega} (T(\omega), s) \, d\mathbb{P}(\omega) \in \mathbb{R}.$$

Note that $\mathbb{E}[T]$ is well defined since $(\mathbb{E}[T], s) < \infty$ for all $s \in \mathcal{S}$, and if $s = r$, then $(\mathbb{E}[T], s) = (\mathbb{E}[T], r)$. Besides, it is easy to check the linearity of $\mathbb{E}[T]$:

$$(\mathbb{E}[T], \alpha s + \beta r) = \alpha(\mathbb{E}[T], s) + \beta(\mathbb{E}[T], r).$$

for any $\alpha, \beta \in \mathbb{R}$ and $s, r \in \mathcal{S}$.

Let us now come back to consider the random linear functional δ_{Θ_n} , associated with an n -regular random closed set Θ_n . We have

$$\omega \in (\Omega, \mathcal{F}, \mathbb{P}) \longmapsto \delta_{\Theta_n}(\omega) \equiv \delta_{\Theta_n(\omega)},$$

and, for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$, (δ_{Θ_n}, f) is an integrable random variable, since certainly an $M \in \mathbb{R}$ exists such that $|f(x)| \leq M$ for any x in the support E of f , and by hypothesis we know that $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap E)] < \infty$.

As before, for the measurability of (δ_{Θ_n}, f) we refer to [5, 25].

Thus, we may define the expected linear functional $\mathbb{E}[\delta_{\Theta_n}]$ on $C_c(\mathbb{R}^d, \mathbb{R})$ by

$$(\mathbb{E}[\delta_{\Theta_n}], f) := \mathbb{E}[(\delta_{\Theta_n}, f)]. \quad (9)$$

Remark 13 By condition (ii) in Definition 9, the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is a Radon measure in \mathbb{R}^d ; as usual, we may consider the associated linear functional as follows:

$$(\tilde{\delta}_{\Theta_n}, f) := \int_{\mathbb{R}^d} f(x) \mathbb{E}[\mu_{\Theta_n}](dx), \quad f \in C_c(\mathbb{R}^d, \mathbb{R}). \quad (10)$$

We show that $\mathbb{E}[\delta_{\Theta_n}] = \tilde{\delta}_{\Theta_n}$.

Proposition 14 The linear functionals $\mathbb{E}[\delta_{\Theta_n}]$ and $\tilde{\delta}_{\Theta_n}$ defined in (9) and (10), respectively, are equivalent.

Proof. Let us consider a function $f \in C_c(\mathbb{R}^d, \mathbb{R})$. By definition (10) we have

$$(\tilde{\delta}_{\Theta_n}, f) := \lim_{k \rightarrow \infty} \sum_{j=1}^k a_j \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A_j)],$$

where $f_k = \sum_{j=1}^k a_j \mathbf{1}_{A_j}$, $k=1, 2, \dots$, is, as usual, a sequence of simple functions converging to f . (Note that the limit does not depend on the chosen approximating sequence of simple functions, and the convergence is uniform.)

For any k ,

$$F_k := \sum_{j=1}^k a_j \mathcal{H}^n(\Theta_n \cap A_j)$$

is a random variable, and $\lim_{k \rightarrow \infty} F_k = (\delta_{\Theta_n}, f)$.

Consider the sequence $\{F_k\}$. We know that an $M \in \mathbb{R}$ exists such that $|f| \leq M$, and so $|a_j| \leq M \forall j$; besides, since A_j is a partition of the support E of f , it follows that $\forall k$

$$F_k \leq M \sum_{j=1}^k \mathcal{H}^n(\Theta_n \cap A_j) = M \mathcal{H}^n(\Theta_n \cap E).$$

By hypothesis $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap E)] < \infty$, so that the Dominated Convergence Theorem implies the following chain of equalities:

$$\begin{aligned} (\tilde{\delta}_{\Theta_n}, f) &= \int_{\mathbb{R}^d} f(x) \mathbb{E}[\mu_{\Theta_n}](dx) = \lim_{k \rightarrow \infty} \sum_{j=1}^k a_j \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A_j)] = \lim_{k \rightarrow \infty} \mathbb{E} \left[\sum_{j=1}^k a_j \mathcal{H}^n(\Theta_n \cap A_j) \right] \\ &= \mathbb{E} \left[\lim_{k \rightarrow \infty} \sum_{j=1}^k a_j \mathcal{H}^n(\Theta_n \cap A_j) \right] = \mathbb{E}[(\delta_{\Theta_n}, f)] = (\mathbb{E}[\delta_{\Theta_n}], f). \end{aligned}$$

□

As in the deterministic case, we may define the *mean generalized density* $\mathbb{E}[\delta_{\Theta_n}](x)$ of $\mathbb{E}[\mu_{\Theta_n}]$ by the following formal integral representation:

$$\int_A \mathbb{E}[\delta_{\Theta_n}](x) dx := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)],$$

with

$$\mathbb{E}[\delta_{\Theta_n}](x) := \lim_{r \rightarrow 0} \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d}.$$

Further, we may represent $\mathbb{E}[\delta_{\Theta_n}]$ as the limit of a sequence of functionals defined by suitable measures, in a similar way as in the previous section.

Let us define

$$\mathbb{E}[\delta_{\Theta_n}^{(r)}](x) := \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d},$$

and denote by $\mathbb{E}[\mu_{\Theta_n}^{(r)}]$ the measure with density the function $\mathbb{E}[\delta_{\Theta_n}^{(r)}](x)$, with respect to the Lebesgue measure ν^d .

Let us introduce the linear functional $\mathbb{E}[\delta_{\Theta_n}^{(r)}]$ associated with the measure $\mathbb{E}[\mu_{\Theta_n}^{(r)}]$, as follows:

$$(\mathbb{E}[\delta_{\Theta_n}^{(r)}], f) := \int_{\mathbb{R}^d} f(x) \mathbb{E}[\mu_{\Theta_n}^{(r)}](dx), \quad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

By the same arguments as in the deterministic case, we now show that the measures $\mathbb{E}[\mu_{\Theta_n}^{(r)}]$ weakly* converge to the measure $\mathbb{E}[\mu_{\Theta_n}]$. In fact, the following result, which may be regarded as the stochastic analogue of Proposition 6, holds.

Proposition 15 *For any bounded Borel set A of \mathbb{R}^d such that $\mathbb{E}[\mu_{\Theta_n}(\partial A)] = 0$ we have*

$$\lim_{r \rightarrow 0} \mathbb{E}[\mu_{\Theta_n}^{(r)}(A)] = \mathbb{E}[\mu_{\Theta_n}(A)].$$

Proof. It is clear that, for any fixed $r > 0$ and for any bounded fixed set A , there exists a compact set K containing A such that $\mathcal{H}^n(\Theta_n(\omega) \cap B_r(x)) = \mathcal{H}^n(\Theta_n(\omega) \cap K \cap B_r(x))$ for all $x \in A$, $\omega \in \Omega$; further, the condition $\mathbb{E}[\mu_{\Theta_n}(\partial A)] = 0$ implies

$$\mathbb{P}(\mathcal{H}^n(\Theta_n \cap \partial A) > 0) = 0; \tag{11}$$

Thus we have that

$$\begin{aligned}\lim_{r \rightarrow 0} \mathbb{E}[\mu_{\Theta_n}^{(r)}(A)] &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d} dx \\ &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x)}{b_d r^d} \int_{\Omega} \int_{\Theta_n(\omega) \cap K} \mathbf{1}_{B_r(x)}(y) \mathcal{H}^n(dy) d\mathbb{P}(\omega) dx;\end{aligned}$$

by exchanging the integrals and using the identity $\mathbf{1}_{B_r(x)}(y) = \mathbf{1}_{B_r(y)}(x)$,

$$\begin{aligned}&= \lim_{r \rightarrow 0} \int_{\Omega} \int_{\Theta_n(\omega) \cap K} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x) \mathbf{1}_{B_r(y)}(x)}{b_d r^d} dx \mathcal{H}^n(dy) d\mathbb{P}(\omega) \\ &= \lim_{r \rightarrow 0} \mathbb{E} \left[\int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \right].\end{aligned}$$

Note that, by (11),

$$\mathbb{E} \left[\int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \right] = \mathbb{E} \left[\int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \mid \mathcal{H}^n(\Theta_n \cap \partial A) = 0 \right],$$

and that

$$(i) \quad \int_{\Theta_n(\omega) \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \leq \int_{\Theta_n(\omega) \cap K} \mathcal{H}^n(dy) = \mathcal{H}^n(\Theta_n(\omega) \cap K);$$

$$(ii) \quad \mathbb{E}[\mathcal{H}^n(\Theta_n \cap K)] < \infty \text{ by hypothesis};$$

$$(iii) \quad \text{by (5), for any } A \text{ as in (11),}$$

$$\lim_{r \rightarrow 0} \int_{\Theta_n(\omega) \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) = \mathcal{H}^n(\Theta_n(\omega) \cap K \cap A) = \mathcal{H}^n(\Theta_n(\omega) \cap A);$$

thus, by the Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{r \rightarrow 0} \mathbb{E} \left[\int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \right] &= \lim_{r \rightarrow 0} \mathbb{E} \left[\int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \mid \mathcal{H}^n(\Theta_n \cap \partial A) = 0 \right] \\ &= \mathbb{E} \left[\lim_{r \rightarrow 0} \int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(dy) \mid \mathcal{H}^n(\Theta_n \cap \partial A) = 0 \right] = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)].\end{aligned}$$

□

The quoted criterion on the characterization of weak convergence of sequences of measures implies that the sequence of measures $\mathbb{E}[\mu_{\Theta_n}^{(r)}]$ weakly* converges to the measure $\mathbb{E}[\mu_{\Theta_n}]$, i.e.

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(x) \mathbb{E}[\mu_{\Theta_n}^{(r)}](dx) = \int_{\mathbb{R}^d} f(x) \mathbb{E}[\mu_{\Theta_n}](dx) \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R});$$

or, in other words, the sequence of linear functionals $\mathbb{E}[\delta_{\Theta_n}^{(r)}]$ converges weakly* to the linear functional $\mathbb{E}[\delta_{\Theta_n}]$, i.e.

$$(\mathbb{E}[\delta_{\Theta_n}], f) = \lim_{r \rightarrow 0} (\mathbb{E}[\delta_{\Theta_n}^{(r)}], f) \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}). \quad (12)$$

By using the integral representation of (δ_{Θ_n}, f) and $(\mathbb{E}[\delta_{\Theta_n}], f)$, Eq. (9) becomes

$$\int_{\mathbb{R}^d} f(x) \mathbb{E}[\delta_{\Theta_n}](x) dx = \mathbb{E} \left[\int_{\mathbb{R}^d} f(x) \delta_{\Theta_n}(x) dx \right]; \quad (13)$$

so that, formally, we may exchange integral and expectation.

Further, by (12), as for the deterministic case, we have the formal exchange of limit and integral

$$\lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(x) \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d} dx = \int_{\mathbb{R}^d} f(x) \lim_{r \rightarrow 0} \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d} dx.$$

Remark 16 When $n = d$, integral and expectation in (13) can be really exchanged by Fubini's theorem, since in this case both μ_{Θ_d} and $\mathbb{E}[\mu_{\Theta_d}]$ are absolutely continuous with respect to ν^d and $\delta_{\Theta_d}(x) = \mathbf{1}_{\Theta_d}(x)$, ν^d -a.s.

In particular $\delta_{\Theta_d}(x) = \mathbf{1}_{\Theta_d}(x)$, ν^d -a.s. implies that

$$\mathbb{E}[\delta_{\Theta_d}](x) = \mathbb{P}(x \in \Theta_d), \quad \nu^d\text{-a.s.},$$

and it is well known the following chain of equalities according with our definition of $\mathbb{E}[\delta_{\Theta_n}]$ ([19], p.46):

$$\mathbb{E}[\nu^d(\Theta_d \cap A)] = \mathbb{E} \left(\int_{\mathbb{R}^d} \mathbf{1}_{\Theta_d \cap A}(x) dx \right) = \mathbb{E} \left(\int_A \mathbf{1}_{\Theta_d}(x) dx \right) = \int_A \mathbb{E}(\mathbf{1}_{\Theta_d}(x)) dx = \int_A \mathbb{P}(x \in \Theta_d) dx.$$

In material science, the density

$$\rho(x) := \mathbb{E}[\delta_{\Theta_d}](x) = \mathbb{P}(x \in \Theta_d)$$

is known as the (degree of) crystallinity.

Again, we may formally state that (see (8))

$$\frac{d\mathbb{E}[\mu_{\Theta_n}]}{d\nu^d}(x) := \mathbb{E}[\delta_{\Theta_n}](x).$$

We know that $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot)]$ is singular with respect to ν^d if and only if its density equals zero almost everywhere, i.e., by our notations, if and only if $\frac{d\mathbb{E}[\mu_{\Theta_n}]}{d\nu^d}(x) = 0$ ν^d -a.e. In this case $\mathbb{E}[\delta_{\Theta_n}](x)$ has the same role of a Dirac delta, so, as in the deterministic case, we may interpret $\mathbb{E}[\delta_{\Theta_n}]$ as a generalized function on the usual test space $C_c(\mathbb{R}^d, \mathbb{R})$, the mean Delta function of the random closed set Θ_n , or, in term of the measure $\mathbb{E}[\mu_{\Theta_n}]$, as its generalized density.

On the other hand, if Θ_n is not a pathological set, i.e. $\mathcal{H}^n(\Theta_n)(\omega) > 0$ for \mathbb{P} -a.e. $\omega \in \Omega$ ($n < d$), we may notice that, even though for a.e. realization θ_n of Θ_n the measure μ_{θ_n} is positive and singular (and so it is not absolutely continuous), the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ may be absolutely continuous with respect to ν^d .

Example: Consider the case $n = 0$. Let $\Theta_0 = X_0$ be a random point in \mathbb{R}^d ; then, in this case, $\mathcal{H}^0(X_0 \cap A) = \mathbf{1}_A(X_0)$, and so

$$\mathbb{E}[\mathcal{H}^0(X_0 \cap A)] = \mathbb{P}(X_0 \in A).$$

If X_0 is a continuous random point with pdf p_{X_0} , then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)]$ is absolutely continuous and, in this case, $\mathbb{E}[\delta_{X_0}](x)$ is just the probability density function $p_{X_0}(x)$, so $\int_A \mathbb{E}[\delta_{X_0}](x) \nu^d(dx)$ is the usual Lebesgue integral.

Note that we formally have

$$\mathbb{E}[\delta_{X_0}](x) = \int_{\mathbb{R}^d} \delta_y(x) p_{X_0}(y) \nu^d(dy) = \int_{\mathbb{R}^d} \delta_x(y) p_{X_0}(y) \nu^d(dy) = p_{X_0}(x);$$

and, in accordance with (13),

$$\int_A \mathbb{E}[\delta_{X_0}](x) \nu^d(dx) = \int_A p_{X_0}(x) \nu^d(dx) = \mathbb{P}(X_0 \in A) = \mathbb{E}[\mathcal{H}^0(X_0 \cap A)] = \mathbb{E} \left[\int_A \delta_{X_0}(x) \nu^d(dx) \right].$$

If instead X_0 is discrete, i.e. $X_0 = x_i$ with probability p_i , only for an at most countable set of points $x_i \in \mathbb{R}$, then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)]$ is singular and, as in the previous case, we have that $\mathbb{E}[\delta_{X_0}](x)$ coincides with the probability distribution p_{X_0} of X_0 .

In fact, in this case $p_{X_0}(x) = \sum_i p_i \delta_{x_i}(x)$, and by computing the expectation of δ_{X_0} , we formally obtain

$$\mathbb{E}[\delta_{X_0}](x) = \delta_{x_1}(x)p_1 + \delta_{x_2}(x)p_2 + \cdots = \sum_i p_i \delta_{x_i}(x) = p_{X_0}(x).$$

Remark 17 By Remark 16 and the considerations on the above example, we may claim that, in the cases $n = d$ and $n = 0$ with X_0 continuous, the expected linear functionals $\mathbb{E}[\delta_{\Theta_d}]$ and $\mathbb{E}[\delta_{X_0}]$ are defined by the function $\rho(x) := \mathbb{P}(x \in \Theta_d)$ and by the pdf p_{X_0} of X_0 , respectively, in the following way

$$(\mathbb{E}[\delta_{\Theta_d}], f) := \int_{\mathbb{R}^d} f(x) \rho(x) dx$$

and

$$(\mathbb{E}[\delta_{X_0}], f) := \int_{\mathbb{R}^d} f(x) p_{X_0}(x) dx.$$

In fact, let us consider the random point X_0 ; in accordance with the definition in (9):

$$(\mathbb{E}[\delta_{X_0}], f) := \int_{\mathbb{R}^d} f(x) p_{X_0}(x) dx = \mathbb{E}[f(X_0)] = \mathbb{E}[(\delta_{X_0}, f)].$$

For a discussion about continuity and absolute continuity of random closed sets we refer to [12, 13].

3 Space-time dependent linear functionals

In this section we wish to analyze the case in which a random closed set Θ may depend upon time as, for example, in the case in which it models the evolution due to a growth process, so that we have a geometric random process $\{\Theta^t, t \in \mathbb{R}_+\}$, such that for any $t \in \mathbb{R}_+$, the random set Θ^t satisfies all the relevant assumptions required in Section 2.2.

Correspondingly the associated linear functional δ_{Θ^t} will also be a function of time.

In order to provide evolution equations for such space-time dependent linear functionals, we need to define partial derivatives of linear functionals depending on more than one variable.

Consider a linear functional L acting on the test space \mathcal{S}_k of functions s in k variables; we formally represent it as

$$(L, s) =: \int_{\mathbb{R}^k} \phi(x_1, \dots, x_k) s(x_1, \dots, x_k) d(x_1, \dots, x_k).$$

Let us denote by L_i^h the linear functional defined by

$$(L_i^h, s) =: \int_{\mathbb{R}^k} \phi(x_1, \dots, x_i + h, \dots, x_k) s(x_1, \dots, x_k) d(x_1, \dots, x_k).$$

We define the *weak partial derivative* of the functional L with respect to the variable x_i as follows (see also [17], p.20).

Definition 18 *We say that a linear functional L on the space \mathcal{S}_k , admits a weak partial derivative with respect to x_i , denoted by $\frac{\partial}{\partial x_i} L$, if and only if $\frac{\partial}{\partial x_i} L$ is a linear functional on the same space \mathcal{S}_k and $\left\{ \frac{L_i^h - L}{h} \right\}$ weakly* converges to $\frac{\partial}{\partial x_i} L$, i.e.*

$$\lim_{h \rightarrow 0} \left(\frac{L_i^h - L}{h}, s \right) = \left(\frac{\partial}{\partial x_i} L, s \right) \quad \text{for all } s \in \mathcal{S}_k.$$

Consider, as an example, a growth process satisfying the following assumptions [9, 10]:

- (i) for any $t \in \mathbb{R}_+$, and any $s > 0$, $\Theta^t \subset \Theta^{t+s}$;
- (ii) for any $t \in \mathbb{R}_+$, Θ^t is a d -regular random closed set in \mathbb{R}^d , and $\partial\Theta^t$ is a $(d-1)$ -regular random closed set.

For any $x \in \mathbb{R}^d$, we may introduce a *time of capture*, as the random variable $T(x)$ such that

$$x \in \text{int}\Theta^t \text{ if } t > T(x),$$

$$x \notin \Theta^t \text{ if } t < T(x),$$

so that

$$x \in \partial\Theta^{T(x)}.$$

Let us introduce, on the test space $C_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ the following two linear functionals

$$(L_1, f) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} \delta_{\Theta^t}(x) f(t, x) dt dx$$

and

$$(L_2, f) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} H_{T(x)}(t) f(t, x) dt dx.$$

We know that

$$\delta_{\Theta^t}(x) = \begin{cases} 1 & \forall x \in \text{int}\Theta^t \\ 0 & \forall x \notin \Theta^t \end{cases}.$$

As a consequence we may easily check that, for any test function f , $(L_1, f) = (L_2, f)$, so that we may formally write

$$\delta_{\Theta^t}(x) = H_{T(x)}(t).$$

We have denoted by H_s the Heaviside distribution associated with $s \in \mathbb{R}$, such that, for any $g \in C_c(\mathbb{R}, \mathbb{R})$, we have

$$(H_s, g) = \int_s^{+\infty} g(t) dt.$$

We know that the distributional derivative of H_s is the delta function δ_s ; as a consequence the following holds.

Proposition 19 *For any test function $f \in C_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$,*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t, x) \frac{\partial}{\partial t} \delta_{\Theta^t}(x) \, dx dt = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t, x) \delta_{T(x)}(t) \, dx dt = \int_{\mathbb{R}^d} f(T(x), x) \, dx.$$

Formally we may write

$$\frac{\partial}{\partial t} \delta_{\Theta^t}(x) = \delta_{T(x)}(t).$$

Proof. According to the previous definition,

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t, x) [\delta_{\Theta^{t+\Delta t}}(x) - \delta_{\Theta^t}(x)] \, dx dt \\ = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t, x) [H_{T(x)}(t + \Delta t) - H_{T(x)}(t)] \, dx dt \\ = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}_+} dt f(t, x) \frac{\partial H_{T(x)}}{\partial t}(t) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}_+} dt f(t, x) \delta_{T(x)}(t). \end{aligned}$$

□

Consider the case in which $T(x)$ is a continuous random variable, and denote by $p_{T(x)}(t)$ its probability density function. Then, by Remark 17, we may claim that, in a distributional sense,

$$\mathbb{E} \left[\frac{\partial}{\partial t} \delta_{\Theta^t} \right] (x) = p_{T(x)}(t).$$

In fact, coherently with the definition of expected linear functional, and by Proposition 19 we have

$$\mathbb{E} \left[\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t, x) \frac{\partial}{\partial t} \delta_{\Theta^t}(x) \, dx dt \right] = \mathbb{E} \left[\int dx f(T(x), x) \right] = \int dx \int dt f(t, x) p_{T(x)}(t).$$

We may observe that, in this case, even if for any realization $\Theta^t(\omega)$ of Θ^t , $\frac{\partial}{\partial t} \delta_{\Theta^t(\omega)}$ is a singular generalized function, when we consider the expectation we obtain a regular generalized function, i.e. a real integrable function. In particular the derivative is the usual derivative of

functions. Thus, by observing that, since $T(x)$ is the random time of capture of x , $\mathbb{P}(x \in \Theta^t) = \mathbb{P}(T(x) < t)$, and $\mathbb{E}[\delta_{\Theta^t}](x) = \mathbb{P}(x \in \Theta^t)$ (see Remark 17), the following holds too:

$$\mathbb{E} \left[\frac{\partial}{\partial t} \delta_{\Theta^t} \right] (x) = p_{T(x)}(t) = \frac{\partial}{\partial t} \mathbb{P}(x \in \Theta^t) = \frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}](x). \quad (14)$$

Hence, $\mathbb{E} \left[\frac{\partial}{\partial t} \delta_{\Theta^t} \right] (x)$ and $\frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}](x)$ coincide as functions, and by the equation above, we have the formal exchange between derivative and expectation.

3.1 An evolution equation for growth processes

Definition 20 [2] *We say that a compact subset Ξ of \mathbb{R}^d admits one-sided Minkowski content if the limit*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^d(\Xi_{\oplus r} \setminus \Xi)}{r}$$

exists finite.

In [2] conditions are provided on the set Ξ in order to satisfy

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^d(\Xi_{\oplus r} \setminus \Xi \cap A)}{r} = \mathcal{H}^{d-1}(\partial \Xi \cap A) \quad (15)$$

for any open set $A \subset \mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial \Xi \cap \partial A) = 0$.

We shall see that an analogous result for random closed sets plays here a crucial role in deriving an evolution equation for the mean density of an evolving random closed set.

With reference to the growth process considered above, an *hazard function* can be defined as the rate of capture of point $x \in \mathbb{R}^d$ by the growth process, at time t .

Definition 21 *The function*

$$h(t, x) := \lim_{\Delta t \rightarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} | x \notin \Theta^t)}{\Delta t}. \quad (16)$$

is called the hazard function associated to a point $x \in \mathbb{R}^d$ at time t .

When the time of capture of point x , $T(x)$ is a continuous random variable, as in our hypotheses, the following holds (see [11]):

$$h(t, x) = \frac{p_{T(x)}(t)}{\mathbb{P}(x \notin \Theta^t)}; \quad (17)$$

further, since

$$\mathbb{P}(x \notin \Theta^t) = \mathbb{P}(x \notin \text{int} \Theta^t),$$

by (16) we have that

$$\begin{aligned}
h(t, x) &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t}) - \mathbb{P}(x \in \Theta^t)}{\mathbb{P}(x \notin \text{int}\Theta^t)} \\
&= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} \mid x \notin \text{int}\Theta^t) - \mathbb{P}(x \in \Theta^t \mid x \notin \text{int}\Theta^t)}{\Delta t} \\
&= \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}[\delta_{\Theta^{t+\Delta t}}(x) \mid x \notin \text{int}\Theta^t] - \mathbb{E}[\delta_{\Theta^t}(x) \mid x \notin \text{int}\Theta^t]}{\Delta t} \\
&= \frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}(x) \mid x \notin \text{int}\Theta^t]
\end{aligned} \tag{18}$$

Without loss of generalization, we may assume that, for any fixed $t \in \mathbb{R}_+$, $\partial\Theta^t$ is a compact random set. Then we have the following general result [2].

Theorem 22 *Let Θ^t be a random closed set in \mathbb{R}^d satisfying (15) for a.e. $\omega \in \Omega$, with boundary $\partial\Theta^t$ countably \mathcal{H}^{d-1} -rectifiable and compact. Let $\Gamma : \Omega \rightarrow \mathbb{R}$ be the function so defined:*

$$\Gamma(\omega) := \max\{\gamma \geq 0 : \exists \text{ a probability measure } \eta \text{ such that} \\
\eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial\Theta^t(\omega), r \in (0, 1)\}.$$

If there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $1/\Gamma(\omega) \leq Y(\omega)$ for \mathbb{P} -a.e. $\omega \in \Omega$, then

$$\lim_{r \rightarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r}^t \setminus \Theta^t \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)], \tag{19}$$

for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap \partial A)] = 0$.

Proof. We note that $\partial\Theta^t$ satisfies the hypotheses of Theorem 17 in [1]; from its proof we know that for \mathbb{P} -a.e. $\omega \in \Omega$

$$\frac{\mathcal{H}^d(\partial\Theta_{\oplus r}^t(\omega))}{2r} \leq Y(\omega) 2^{d-1} 4^d \frac{b_d}{2}. \tag{20}$$

Since

$$\frac{\mathcal{H}^d((\Theta_{\oplus r}^t(\omega) \setminus \Theta^t(\omega)) \cap A)}{r} \leq \frac{\mathcal{H}^d(\Theta_{\oplus r}^t(\omega) \setminus \Theta^t(\omega))}{r} \leq \frac{\mathcal{H}^d(\partial\Theta_{\oplus r}^t(\omega))}{2r},$$

by (20) and the Dominated Convergence Theorem the thesis follows. \square

We may like to notice that a wide class of random closed sets satisfy the above theorem; in [1] several relevant examples are provided.

Let us consider a normal growth process induced by a space and time dependent growth rate $G(t, x)$; i.e. we assume that almost every point of $\partial\Theta^t$ admits a unit outer normal, and that the growth of Θ^t occurs with nonnegative normal velocity $G(t, x)$.

In particular G has to be continuous and such that

$$0 < g_0 \leq G(t, x) \leq G_0 < \infty \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

for some $g_0, G_0 \in \mathbb{R}$. (E.g. see [9].)

Remark 23 In the particular case of spherical growth, i.e. when G is constant, it is clear that

$$\lim_{\Delta t \rightarrow 0} \frac{\mathcal{H}^d(\Theta^{t+\Delta t}(\omega) \setminus \Theta^t(\omega))}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathcal{H}^d(\Theta_{\oplus G\Delta t}^t(\omega) \setminus \Theta^t(\omega))}{\Delta t} = G \lim_{r \rightarrow 0^+} \frac{\mathcal{H}^d(\Theta_{\oplus r}^t(\omega) \setminus \Theta^t(\omega))}{r},$$

and, for any $t \in \mathbb{R}_+$, $\Theta^t(\omega)$ admits one-sided Minkowski content, so that we have

$$\lim_{\Delta t \rightarrow 0} \frac{\mathcal{H}^d(\Theta^{t+\Delta t}(\omega)) - \mathcal{H}^d(\Theta^t(\omega))}{\Delta t} = G\mathcal{H}^{d-1}(\partial\Theta^t(\omega)).$$

In particular Theorem 22 applies (see also [1]) and so we get

$$\lim_{\Delta t \rightarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta^{t+\Delta t})] - \mathbb{E}[\mathcal{H}^d(\Theta^t)]}{\Delta t} = G\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t)]. \quad (21)$$

We recall now the definition of the spherical contact distribution function associated to a random closed set, which will play a crucial role in the sequel.

Definition 24 The local spherical contact distribution function $H_{S,\Xi}$ of an inhomogeneous random set Ξ is defined as

$$H_{S,\Xi}(r, x) := \mathbb{P}(x \in \Xi_{\oplus r} \mid x \notin \Xi).$$

Proposition 25 Under the above assumption on the growth model, let G be sufficiently regular so that Theorem 22 applies. In the hypothesis that the time of capture $T(x)$ is a continuous random variable with density, we have that the following evolution equation holds for the mean density $\mathbb{E}[\delta_{\Theta^t}](x)$:

$$\frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}](x) = G(t, x) \mathbb{E}[\delta_{\partial\Theta^t}](x), \quad (22)$$

to be taken, as usual, in weak form.

Proof. By hypothesis Θ^t satisfies (19), which, in terms of the linear functionals previously introduced, is equivalent to write

$$\lim_{r \rightarrow 0} \frac{\mathbb{E}[\delta_{\Theta_{\oplus r}^t}](x) - \mathbb{E}[\delta_{\Theta^t}](x)}{r} = \mathbb{E}[\delta_{\partial\Theta^t}](x). \quad (23)$$

For any fixed t , let us consider the spherical contact distribution $H_{S,\Theta^t}(\cdot, x)$ of the crystallized region Θ^t associated to a point x .

Under the regularity assumptions on $G(t, x)$, it follows that (see also Proposition 2 in [11])

$$h(t, x) = G(t, x) \frac{\partial}{\partial r} H_{S,\Theta^t}(r, x)|_{r=0}. \quad (24)$$

Note that, by Definition 24,

$$H_{S,\Theta^t}(r, x) = \frac{\mathbb{P}(x \in (\Theta_{\oplus r}^t \setminus \Theta^t))}{\mathbb{P}(x \notin \Theta^t)},$$

and so, by (24),

$$h(t, x) = \frac{G(t, x)}{\mathbb{P}(x \notin \Theta^t)} \frac{\partial}{\partial r} \mathbb{P}(x \in (\Theta_{\oplus r}^t \setminus \Theta^t))|_{r=0}. \quad (25)$$

Thus, by (14), (17) and (25) we obtain that

$$\mathbb{E} \left[\frac{\partial}{\partial t} \delta_{\Theta^t}(x) \right] = \frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}](x) = G(t, x) \frac{\partial}{\partial r} \mathbb{P}(x \in (\Theta_{\oplus r}^t \setminus \Theta^t))|_{r=0}. \quad (26)$$

Now, we may notice that

$$\begin{aligned} \frac{\partial}{\partial r} \mathbb{P}(x \in (\Theta_{\oplus r}^t \setminus \Theta^t))|_{r=0} &= \lim_{h \rightarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus h}^t) - \mathbb{P}(x \in \Theta^t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus h}^t) - \mathbb{P}(x \in \Theta^t)}{h} = \lim_{h \rightarrow 0} \frac{\mathbb{E}[\delta_{\Theta_{\oplus h}^t}](x) - \mathbb{E}[\delta_{\Theta^t}](x)}{h} \stackrel{(23)}{=} \mathbb{E}[\delta_{\partial\Theta^t}](x), \end{aligned}$$

so that the thesis follows. \square

Remark 26 Since for any fixed $x \in \mathbb{R}^d$ we said that the time of capture $T(x)$ is a continuous random variable with probability density function $p_{T(x)}(t)$, it is clear by (26) that $\frac{\partial}{\partial t} \mathbb{E}[\delta_{\Theta^t}]$ is a classical real function. It follows that $\mathbb{E}[\delta_{\partial\Theta^t}]$ is a classical real function as well. As a consequence, $\mathbb{E}[\delta_{\partial\Theta^t}]$ is a version of the usual Radon-Nikodym derivative of the measure $\mathbb{E}[\mu_{\partial\Theta^t}]$ with respect to ν^d , and so we may claim that it is absolutely continuous.

4 Conclusions

For the growth process Θ^t introduced in the previous sections, we may notice that the evolution of the realization $\Theta^t(\omega)$ may be described for a.e. $\omega \in \Omega$, by the following (weak) equation (e.g. [6, 9]):

$$\frac{\partial}{\partial t} \delta_{\Theta^t}(x) = G(t, x) \delta_{\partial\Theta^t}(x). \quad (27)$$

The advantage of this expression, even though to be understood in a weak sense in terms of viscosity solutions, is in the fact that it makes explicit the local dependence (both in time and space) upon the growth field G by means of the (geometric) Dirac delta at a point $x \in \partial\Theta^t$. In this way equation (22) can be formally obtained by taking the expected value in (27), thanks to the linearity of expectation, since we have assumed that G is a deterministic function. (Obviously, it involves exchanges between limit and expectation, as in (19) for example). In this paper we have shown that indeed, under suitable regularity assumptions on the process Θ^t , we may obtain (22) from (27) in a rigorous way, thus making effective our motivation to introduce (mean) generalized geometric densities δ_{Θ} ($\mathbb{E}[\delta_{\Theta}]$) associated to a (random) closed set Θ .

We know that if Θ_n is a lower dimensional random closed set in \mathbb{R}^d with Hausdorff dimension $n < d$, then the Radon measures $\mu_{\Theta_n(\omega)}$ induced by its realizations are singular with respect to the d -dimensional Lebesgue measure ν^d , and so their usual Radon-Nikodym derivatives are zero almost everywhere. On the other hand, depending upon the specific probability law of Θ_n , the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ may still be singular, but it may also happen that it is absolutely continuous with respect to ν^d ; consequently in this case its Radon-Nikodym

derivative happens to be a classical non trivial function.

Hence, by introducing the generalized density $\mathbb{E}[\delta_{\Theta_n}]$ for any n -regular random closed set Θ_n , we may formally deal with it as a classical function; it will be clear by the context, i.e. by the specific hypotheses, whether Θ_n is an absolutely continuous random closed set or not. In [12] we have defined Θ_n as an absolutely continuous random closed set if $\mathbb{E}[\mu_{\Theta_n}]$ is an absolute continuous measure with respect to ν^d , in which case $\mathbb{E}[\delta_{\Theta_n}](x)$ is taken as the usual Radon-Nykodym derivative associated with the measure $\mathbb{E}[\mu_{\Theta_n}]$; otherwise it is taken as a generalized function, the generalized Radon-Nykodym derivative. As a consequence, in all situations in which no distinction is required between absolutely continuous random closed sets or not, it is clear how much convenient is to work directly with these generalized densities.

Further, we have shown how to approximate δ_{Θ_n} (respectively $\mathbb{E}[\delta_{\Theta_n}]$) by sequences of classical functions $\{\delta_{\Theta_n}^{(r)}\}$ ($\{\mathbb{E}[\delta_{\Theta_n}^{(r)}]\}$, respectively).

This turns to be useful in several real applications in which one needs to estimate the density of the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ as, for example, when $n = 1$ in the case of fibre processes, or line processes, or when $n = d - 1$ in the case of surface processes (see [1]).

By comparing (27) with (18), we may claim that

$$h(t, x) = G(t, x) \mathbb{E}[\delta_{\partial\Theta^t}(x) \mid x \notin \text{int}\Theta^t],$$

which leads to the interesting interpretation of

$$\frac{\partial}{\partial r} H_{S, \Theta^t}(r, x)|_{r=0} = \mathbb{E}[\delta_{\partial\Theta^t}(x) \mid x \notin \text{int}\Theta^t].$$

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