MEAN DENSITY OF INHOMOGENEOUS BOOLEAN MODELS WITH LOWER DIMENSIONAL TYPICAL GRAIN

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The Problem

A random closed set $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$[1] in $\mathbb{R}^d$ with integer Hausorff dimension n may induce a random Radon measure $\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$ on $\mathbb{R}^d$, and, as a consequence, an *expected measure*

$$\mathbb{E}[\mu_\Theta](B) := \mathbb{E}[\mathcal{H}^n(\Theta \cap B)] \qquad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

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2. If so, which is its density?

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2. If so, which is its density?

Notation

If $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$ we denote by λ_Θ its density and we call $\lambda_\Theta(x)$ the **mean density of Θ at point $x \in \mathbb{R}^d$**.

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- If $\Theta = X$ random point ($n = 0$), then $\mathbb{E}[\mu_X](\cdot) = \mathbb{P}(X \in \cdot) \ll \mathcal{H}^d$ iff X admits pdf f, and $\lambda_X = f$.
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Problem

What if $0 < n < d$ and Θ is NOT stationary?

Boolean model in $\mathbb{R}^d$

Definition

- $\Psi = \{x_i\}_{i\in\mathbb{N}}$: Poisson point process in $\mathbb{R}^d$ with intensity f;
- $\{Z_i\}_{i\in\mathbb{N}}$: sequence of IID random compact sets in $\mathbb{R}^d$, which are also independent of the Poisson process Ψ;
- Z_0: random compact set of the same distribution as the Z_i's.

The random closed set

$$\boldsymbol{\Theta} := \bigcup_i (x_i + Z_i)$$

is said **(inhomogeneous) Boolean model with intensity f and typical grain Z_0**.

It is usually assumed that

$$\mathbb{E}[\mathrm{card}\{i \,:\, (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \qquad \forall \text{ compact } K \subset \mathbb{R}^d.$$

We assume that the typical grain Z_0 is a lower dimensional random closed set in $\mathbb{R}^d$, uniquely determined by a random quantity in a suitable mark space $\mathbf{K}$; i.e. $\forall s \in \mathbf{K}$

$$Z_0(s) = n\text{-dimensional compact subset of } \mathbb{R}^d \text{ containing the origin.}$$

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Let us consider the Boolean model

$$\Theta(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \qquad \forall \omega \in \Omega,$$

with Φ Poisson point process in $\mathbb{R}^d \times \mathbf{K}$ with intensity measure

$$\Lambda(\mathrm{d}y \times \mathrm{d}s) = f(y)\mathrm{d}y Q(\mathrm{d}s).$$

The main result

Under general regularity assumptions on Z_0, related to the existence of its *Minkowski content*, and on the intensity f of the underlying Poisson point process, we can prove that

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$\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$ with density

$$\lambda_\Theta(x) = \int_{\mathbf{K}} \int_{Z^{x,s}} f(y)\mathcal{H}^n(\mathrm{d}y)\, Q(\mathrm{d}s) \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

where $Z^{x,s} := x - Z_0(s)$.

Rectifiability and Minkowski content

Notation:

- b_m =volume of the unit ball in $\mathbb{R}^m$;
- $S_{\oplus r} := S \oplus B_r(0)$.

Definition (Minkowski content)

The ***n*-dimensional Minkowski content** $\mathcal{M}^n(S)$ of a closed set $S \subset \mathbb{R}^d$ is defined by

$$\mathcal{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}}$$

whenever the limit exists finite.

Definition

We say that a compact set $S \subset \mathbb{R}^d$ is

- *n-rectifiable*, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;

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$\mathcal{M}^n(S) = \mathcal{H}^n(S)$ for any compact n-rectifiable set $S \subset \mathbb{R}^d$.

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- ***n*-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;
- is **countably $\mathcal{H}^n$-rectifiable** if there exist countably many Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\Big(S \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n)\Big) = 0.$$

Theorem (H.Federer (1969))

$\mathcal{M}^n(S) = \mathcal{H}^n(S)$ *for any compact n-rectifiable set* $S \subset \mathbb{R}^d$.

Theorem (L.Ambrosio-N.Fusco-D.Pallara (2000))

Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \;\; \forall r \in (0,1) \tag{1}$$

holds for some $\gamma > 0$ and some Radon measure η in $\mathbb{R}^d$, $\eta \ll \mathcal{H}^n$. Then

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- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\widetilde{S} \cap \cdot\,)$ for some closed set $\widetilde{S} \supseteq S$;

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- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\widetilde{S} \cap \cdot\,)$ for some closed set $\widetilde{S} \supseteq S$;
- η can be assumed to be a probability measure;
- it can be proved that $\mathcal{H}^n(S) < \infty$ and

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r} \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap A)$$

for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^n(S \cap \partial A) = 0$.

A generalization of $\mathcal{M}^n$

Theorem (EV (2007))

Let μ be a positive measure in $\mathbb{R}^d$ absolutely continuous w.r.t. $\mathcal{H}^d$ with density f such that

i) *f is locally bounded (i.e. $\sup_{x\in K} f(x) < \infty$ for any compact $K \subset \mathbb{R}^d$);*

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This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with

- f = intensity of the underlying Poisson point process in $\mathbb{R}^d$,
- $S = x - Z_0(s), \; s \in \mathbf{K}$.

Assumptions

Let us consider the Boolean model Θ in $\mathbb{R}^d$

$$\Theta(\omega) := \bigcup_{(x_i,s_i)\in\Phi(\omega)} x_i + Z_0(s_i) \qquad \forall \omega \in \Omega,$$

with Φ Poisson point process in $\mathbb{R}^d \times \mathbf{K}$ having intensity measure

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such that

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such that

(IC) $$\int_{\mathbf{K}} \int_{(-Z_0(s))_{\oplus R}} \Lambda(\mathrm{d}y \times \mathrm{d}s) < \infty \qquad \forall R > 0.$$

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$$\mathcal{H}^n(\widetilde{Z}_0(s) \cap B_r(x)) \geq \gamma r^n \qquad \forall x \in Z_0(s),\ \forall r \in (0,1).$$

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(A2) the set of all discontinuity points of f is $\mathcal{H}^n$-negligible and f is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K_{\oplus\delta}} f(y) \leq \xi_K \qquad (\delta := \mathrm{diam} Z_0)$$

holds for some random variable ξ_K with $\mathbb{E}_Q[\mathcal{H}^n(\widetilde{Z}_0)\xi_K] < \infty$.

Main Theorem

Theorem (EV (2007))

For any Boolean model Θ *as in Assumptions*

- $\mathbb{E}[\mu_\Theta]$ *is locally finite and absolutely continuous w.r.t.* $\mathcal{H}^d$*;*
- *the mean density* λ_Θ *is given by*

$$\lambda_\Theta(x) = \int_{\mathbf{K}} \int_{Z^{x,s}} f(y)\mathcal{H}^n(\mathrm{d}y)\, Q(\mathrm{d}s) \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

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Special cases

1. Stationary case. If
   - $f \equiv c > 0$,
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2. Deterministic typical grain. If
   - f is locally bounded and such that the set of all its discontinuity points is $\mathcal{H}^n$-negligible,
   - Z_0 is a countably $\mathcal{H}^n$-rectifiable compact set such that
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$$\lambda_\Theta(x) = \int_{Z_0} f(x-y)\mathcal{H}^n(\mathrm{d}y) \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

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- $\mathbb{E}[\mu_\Theta] = \lambda_\Theta \mathcal{H}^d$ for some integrable function λ_Θ on $\mathbb{R}^d$.
- For any bounded Borel set $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(\partial A) = 0$,

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta \cap A)],$$

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i.e.

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- By (A1) and (A2)

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} \mathrm{d}x = \int_A \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} \mathrm{d}x.$$

- For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

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&= \int_{\mathbf{K}} \int_{Z^{x,s}} f(y) \mathcal{H}^n(\mathrm{d}y) \, Q(\mathrm{d}s).
\end{aligned}
$$

Estimation of the mean density

By the proof of the main theorem we get that, for any Boolean model Θ as in the Assumptions,

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{b_{d-n} r^{d-n}} \in \mathbb{R} \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d$$

This suggest an estimator of $\lambda_\Theta(x)$ in terms of the empirical capacity functional of Θ:

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This suggest an estimator of $\lambda_\Theta(x)$ in terms of the empirical capacity functional of Θ:
Let $\Theta_1, \ldots, \Theta_N$ be a random sample of Θ; we define

$$\widehat{\lambda}^N_\Theta(x) := \frac{\sum_{i=1}^N \mathbf{1}_{\Theta_i \cap B_{R_N}(x) \neq \emptyset}}{N b_{d-n} R_N^{d-n}},$$

with R_N such that $R_N \to 0$ and $N R_N^{d-n} \to \infty$ for $N \to \infty$.
Then

$$\lim_{N \to \infty} \widehat{\lambda}^N_\Theta(x) = \lambda_\Theta(x) \quad \text{in probability}, \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Remark:

Even if the extreme case $n = 0$ can be handle with much more elementary tools, we may notice that if $\Theta = X$ is a random variable with pdf f_X, then

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Even if the extreme case $n = 0$ can be handle with much more elementary tools, we may notice that if $\Theta = X$ is a random variable with pdf f_X, then

- $\lambda_\Theta = f_X$;
- If $X_1, \dots, X_N$ is a random sample of X, $\hat{\lambda}^N_\Theta(x)$ becomes in this case

$$\widehat{f}^N_X(x) = \frac{\sum_{i=1}^N \mathbf{1}_{B_{R_N}(x)}(X_i)}{N b_1 R_N} = \frac{\mathrm{card}\{i : X_i \in \mathcal{I}_x\}}{N|\mathcal{I}_x|},$$

where $\mathcal{I}_x$ is the interval in $\mathbb{R}$ centered in x with length $|\mathcal{I}_x| = 2R_N$ with the usual condition

$$|\mathcal{I}_x| \longrightarrow 0 \ \text{ and } \ N|\mathcal{I}_x| \longrightarrow \infty \qquad \text{as } N \to \infty.$$