MEAN DENSITY OF INHOMOGENEOUS BOOLEAN MODELS WITH LOWER DIMENSIONAL TYPICAL GRAIN

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The Problem

A random closed set $\Theta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathcal{F}, \sigma_{\mathcal{F}})^1$ in $\mathbb{R}^d$ with integer Hausorff dimension $n$ may induce a random Radon measure $\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$ on $\mathbb{R}^d$, and, as a consequence, an expected measure

$$\mathbb{E}[\mu_\Theta](B) := \mathbb{E}[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Questions

1. Is $\mathbb{E}[\mu_\Theta]$ absolutely continuous w.r.t. $\mathcal{H}^d$?
2. If so, which is its density?

Notation

If $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$ we denote by $\lambda_\Theta$ its density and we call $\lambda_\Theta(x)$ the mean density of $\Theta$ at point $x \in \mathbb{R}^d$.

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$^1\mathcal{F}$ = closed subsets in $\mathbb{R}^d$; $\sigma_{\mathcal{F}} = \sigma$-algebra generated by the hit-or-miss topology
The Problem

A random closed set Θ : (Ω, ℱ, ℙ) → (ℱ, σℱ)₁ in ℝᵈ with integer Hausorff dimension $n$ may induce a random Radon measure $μ_Θ(·) := ℋⁿ(Θ ∩ ·)$ on ℝᵈ, and, as a consequence, an expected measure

$$
E[μ_Θ](B) := E[ℋⁿ(Θ ∩ B)] \quad ∀B ∈ ℬ_{ℝᵈ}.
$$

Questions

1. Is $E[μ_Θ]$ absolutely continuous w.r.t. $ℋᵈ$?

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$₁ℱ$ = closed subsets in ℝᵈ; $σℱ = σ$-algebra generated by the hit-or-miss topology

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The Problem

A random closed set $\Theta : (\Omega, \mathcal{F}, P) \longrightarrow (\mathcal{F}, \sigma_\mathcal{F})$ in $\mathbb{R}^d$ with integer Hausorff dimension $n$ may induce a random Radon measure $\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$ on $\mathbb{R}^d$, and, as a consequence, an expected measure

$$E[\mu_\Theta](B) := E[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$ 

Questions

1. Is $E[\mu_\Theta]$ absolutely continuous w.r.t. $\mathcal{H}^d$?
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A random closed set $\Theta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathcal{F}, \sigma_{\mathcal{F}})^1$ in $\mathbb{R}^d$ with integer Hausdorff dimension $n$ may induce a random Radon measure $\mu_\Theta(\cdot) : = \mathcal{H}^n(\Theta \cap \cdot)$ on $\mathbb{R}^d$, and, as a consequence, an expected measure

$$E[\mu_\Theta](B) : = E[\mathcal{H}^n(\Theta \cap B)] \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$ 

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1. Is $E[\mu_\Theta]$ absolutely continuous w.r.t. $\mathcal{H}^d$?
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If $E[\mu_\Theta] \ll \mathcal{H}^d$ we denote by $\lambda_\Theta$ its density and we call $\lambda_\Theta(x)$ the mean density of $\Theta$ at point $x \in \mathbb{R}^d$.

$^1$\(\mathcal{F}\) = closed subsets in $\mathbb{R}^d$; $\sigma_{\mathcal{F}}$ = $\sigma$-algebra generated by the hit-or-miss topology
What we know

If \( n = d \), then \( E[\mu_\Theta] \ll H^d \) with density \( \lambda_\Theta(x) = P(x \in \Theta) \) for \( H^d \)-a.e. \( x \in \mathbb{R}^d \).


If \( \Theta = X \) random point (\( n = 0 \)), then \( E[\mu_X](\cdot) = P(X \in \cdot) \ll H^d \) iff \( X \) admits pdf \( f \), and \( \lambda_X = f \).

If \( 0 < n < d \) and \( \Theta \) is stationary, then \( E[\mu_\Theta] \ll H^d \) with density \( \lambda_\Theta(x) = c > 0 \) \( \forall x \in \mathbb{R}^d \).

Problem

What if \( 0 < n < d \) and \( \Theta \) is NOT stationary?
What we know

- If \( n = d \),

\[
E[\mu_\Theta] \ll \mathcal{H}^d \quad \text{with density} \quad \lambda_\Theta(x) = \mathbb{P}(x \in \Theta) \quad \text{for} \quad \mathcal{H}^d \text{-a.e.} \ x \in \mathbb{R}^d.
\]


If \( \Theta = X \) random point \((n = 0)\),

\[
E[\mu_X](\cdot) = \mathbb{P}(X \in \cdot) \ll \mathcal{H}^d \quad \text{iff} \quad X \text{admits pdf } f \quad \text{and} \quad \lambda_X = f.
\]

If \( 0 < n < d \) and \( \Theta \) is stationary,

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\lambda_\Theta(x) = \mathbb{P}(x \in \Theta) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.
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- If $n = d$, then $E[\mu_{\Theta}] \ll \mathcal{H}^d$ with density
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- If $\Theta = X$ random point ($n = 0$), then $E[\mu_X](\cdot) = P(X \in \cdot) \ll \mathcal{H}^d$ iff
  $X$ admits pdf $f$, and $\lambda_X = f$.

- If $0 < n < d$
What we know

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  $$\lambda_\Theta(x) = P(x \in \Theta) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$ (Robbins H.E. (1944). On the measure of a random set, *Ann.Math.Statistics*, 15, 70–74)

- If $\Theta = X$ random point ($n = 0$), then $E[\mu_X](\cdot) = P(X \in \cdot) \ll \mathcal{H}^d$ iff $X$ admits pdf $f$, and $\lambda_X = f$.

- If $0 < n < d$ and $\Theta$ is stationary, then $E[\mu_\Theta] \ll \mathcal{H}^d$ with density
  $$\lambda_\Theta(x) = c > 0 \quad \forall x \in \mathbb{R}^d.$$ 

Problem

What if $0 < n < d$ and $\Theta$ is NOT stationary?
Boolean model in $\mathbb{R}^d$

**Definition**

- $\Psi = \{x_i\}_{i \in \mathbb{N}}$: Poisson point process in $\mathbb{R}^d$ with intensity $f$;
- $\{Z_i\}_{i \in \mathbb{N}}$: sequence of IID random compact sets in $\mathbb{R}^d$, which are also independent of the Poisson process $\Psi$;
- $Z_0$: random compact set of the same distribution as the $Z_i$’s.

The random closed set

$$\Theta := \bigcup_i (x_i + Z_i)$$

is said **(inhomogeneous) Boolean model with intensity $f$ and typical grain $Z_0$**.

It is usually assumed that

$$\mathbb{E}[\text{card}\{i : (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^d.$$
We assume that the typical grain $Z_0$ is a lower dimensional random closed set in $\mathbb{R}^d$, uniquely determined by a random quantity in a suitable mark space $K$; i.e. $\forall s \in K$

$$Z_0(s) = \text{n-dimensional compact subset of } \mathbb{R}^d \text{ containing the origin.}$$
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Let us consider the Boolean model

$$\Theta(\omega) := \bigcup_{(x_i,s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,$$

with $\Phi$ Poisson point process in $\mathbb{R}^d \times K$ with intensity measure

$$\Lambda(dy \times ds) = f(y)dyQ(ds).$$
The main result

Under general regularity assumptions on $Z_0$, related to the existence of its Minkowski content, and on the intensity $f$ of the underlying Poisson point process, we can prove that

\[ E[\mu_\Theta] \ll H^d \] with density

\[ \lambda_\Theta(x) = \int K \int Z_x, s f(y) H^n(dy) Q(ds) \] for $H^d$-a.e. $x \in \mathbb{R}^d$, where $Z_{x,s} := x - Z_0(s)$. 
The main result

Under general regularity assumptions on $Z_0$, related to the existence of its Minkowski content, and on the intensity $f$ of the underlying Poisson point process, we can prove that

$$E[\mu] \ll \mathcal{H}^d$$

with density

$$\lambda_\Theta(x) = \int_K \int_{Z_{x,s}} f(y) \mathcal{H}^n(dy) Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

where $Z_{x,s} := x - Z_0(s)$. 

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Notation:
- \( b_m \) = volume of the unit ball in \( \mathbb{R}^m \);
- \( S \oplus r := S \oplus B_r(0) \).

**Definition (Minkowski content)**

The **\( n \)-dimensional Minkowski content** \( \mathcal{M}^n(S) \) of a closed set \( S \subset \mathbb{R}^d \) is defined by

\[
\mathcal{M}^n(S) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(S \oplus r)}{b_{d-n} r^{d-n}}
\]

whenever the limit exists finite.
Definition

We say that a compact set $S \subset \mathbb{R}^d$ is

- **$n$-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;
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Theorem (H. Federer (1969))

$\mathcal{M}^n(S) = \mathcal{H}^n(S)$ for any compact $n$-rectifiable set $S \subset \mathbb{R}^d.$
**Definition**

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- **$n$-rectifiable**, if there exist a compact $K \subset \mathbb{R}^n$ and a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $S = g(K)$;

- is **countably $\mathcal{H}^n$-rectifiable** if there exist countably many Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\left(S \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n)\right) = 0.$$

**Theorem (H.Federer (1969))**

$\mathcal{M}^n(S) = \mathcal{H}^n(S)$ for any compact $n$-rectifiable set $S \subset \mathbb{R}^d$. 
Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)$$

(1)

holds for some $\gamma > 0$ and some Radon measure $\eta$ in $\mathbb{R}^d$, $\eta \ll \mathcal{H}^n$. Then

$$\mathcal{M}^n(S) = \mathcal{H}^n(S).$$
Theorem (L.Ambrosio-N.Fusco-D.Pallara (2000))

Let \( S \subset \mathbb{R}^d \) be a countably \( H^n \)-rectifiable compact set and assume that

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holds for some \( \gamma > 0 \) and some Radon measure \( \eta \) in \( \mathbb{R}^d \), \( \eta \ll H^n \). Then

\[
\mathcal{M}^n(S) = H^n(S).
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Remarks:

• in many applications condition (1) is satisfied with \( \eta(\cdot) = H^n(\widetilde{S} \cap \cdot) \) for some closed set \( \widetilde{S} \supseteq S \);
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- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\tilde{S} \cap \cdot)$ for some closed set $\tilde{S} \supseteq S$;
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Remarks:

- in many applications condition (1) is satisfied with $\eta(\cdot) = \mathcal{H}^n(\tilde{S} \cap \cdot)$ for some closed set $\tilde{S} \supseteq S$;
- $\eta$ can be assumed to be a probability measure;
- it can be proved that $\mathcal{H}^n(S) < \infty$ and

$$
\lim_{r \downarrow 0} \frac{\mathcal{H}^d(S \oplus r \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap A)
$$

for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^n(S \cap \partial A) = 0$. 

A generalization of $M^n$

**Theorem (EV (2007))**

Let $\mu$ be a positive measure in $\mathbb{R}^d$ absolutely continuous w.r.t. $\mathcal{H}^d$ with density $f$ such that

i) $f$ is locally bounded (i.e. $\sup_{x \in K} f(x) < \infty$ for any compact $K \subset \mathbb{R}^d$);

ii) the set of all discontinuity points of $f$ is $\mathcal{H}^n$-negligible.

Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set as in the previous theorem. Then

$$\lim_{r \downarrow 0} \mu(S \oplus r) = \int_S f(x) \mathcal{H}^n(dx).$$

This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with $f$ = intensity of the underlying Poisson point process in $\mathbb{R}^d$, $S = x - Z_0(s)$, $s \in K$. 

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$$\lim_{r \to 0} \mu(S \oplus r) = \int_S f(x) \, d\mathcal{H}^n(x).$$

This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with $f$ the intensity of the underlying Poisson point process in $\mathbb{R}^d$, $S = x - Z_0(s)$, $s \in K$.
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This result applies in the proof of the formula of the mean density $\lambda_\Theta(x)$, with

- $f =$ intensity of the underlying Poisson point process in $\mathbb{R}^d$,
- $S = x - Z_0(s), \ s \in K$. 

Assumptions

Let us consider the Boolean model $\Theta$ in $\mathbb{R}^d$

$$\Theta(\omega) := \bigcup_{(x_i,s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,$$

with $\Phi$ Poisson point process in $\mathbb{R}^d \times K$ having intensity measure

$$\Lambda(dy \times ds) = f(y)dyQ(ds)$$

such that

$$\mathbb{E}[\text{card}\{i : (x_i + Z_i) \cap K \neq \emptyset\}] < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^d.$$
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$$\Theta(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + Z_0(s_i) \quad \forall \omega \in \Omega,$$

with $\Phi$ Poisson point process in $\mathbb{R}^d \times K$ having intensity measure

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such that

$$(IC) \quad \int_K \int_{(-Z_0(s)) \oplus R} \Lambda(dy \times ds) < \infty \quad \forall R > 0.$$
Let us assume that the following conditions on $Z_0$ and $f$ are fulfilled:

(A1) $Z_0(s)$ is a countably $H^n$-rectifiable compact set for $Q$-a.e. $s \in K$. Further there exist $\gamma > 0$ and a random closed set $\tilde{Z}_0 \supseteq Z_0$ with $E_Q[H^n(\tilde{Z}_0)] < \infty$ such that, for $Q$-a.e. $s \in K$,

$$H^n(\tilde{Z}_0(s) \cap B_r(x)) \geq \gamma r^n \forall x \in Z_0(s), \forall r \in (0,1).$$

(A2) the set of all discontinuity points of $f$ is $H^n$-negligible and $f$ is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K} \|\delta f(y)\| \leq \xi_K(\delta := \text{diam } Z_0)$$

holds for some random variable $\xi_K$ with $E_Q[H^n(\tilde{Z}_0) \xi_K] < \infty$. 
Let us assume that the following conditions on $Z_0$ and $f$ are fulfilled:

(A1) $Z_0(s)$ is a countably $\mathcal{H}^n$-rectifiable compact set for $Q$-a.e. $s \in K$. Further there exist $\gamma > 0$ and a random closed set $\tilde{Z}_0 \supseteq Z_0$ with $E_Q[\mathcal{H}^n(\tilde{Z}_0)] < \infty$ such that, for $Q$-a.e. $s \in K$,

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(A2) the set of all discontinuity points of $f$ is $\mathcal{H}^n$-negligible and $f$ is locally bounded such that for any compact set $K \subset \mathbb{R}^d$

$$\sup_{y \in K \oplus \delta} f(y) \leq \xi_K \quad (\delta := \text{diam}Z_0)$$

holds for some random variable $\xi_K$ with $\mathbb{E}_Q[\mathcal{H}^n(\tilde{Z}_0)\xi_K] < \infty$. 


Main Theorem

**Theorem (EV (2007))**

For any Boolean model \(\Theta\) as in Assumptions

- \(\mathbb{E}[\mu_\Theta]\) is locally finite and absolutely continuous w.r.t. \(\mathcal{H}^d\);
- the mean density \(\lambda_\Theta\) is given by

\[
\lambda_\Theta(x) = \int_K \int_{Z^{x,s}_0} f(y) \mathcal{H}^n(dy) Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,
\]

where \(Z^{x,s}_0 := x - Z_0(s)\).
**Special cases**

1. **Stationary case.** If
   - $f \equiv c > 0$,
   - $Z_0$ satisfies the assumption (A1),
then the integrability condition (IC) on $\Lambda$ and the assumption (A2) are satisfied.
Special cases

1. **Stationary case.** If
   - $f \equiv c > 0$,
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then $E[\mu_\Theta] \ll \mathcal{H}^d$ with density

$$
\lambda_\Theta(x) = c E_Q[\mathcal{H}^n(Z_0)] \quad \forall x \in \mathbb{R}^d.
$$
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   then $\mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d$ with density
     \[
     \lambda_\Theta(x) = c\mathbb{E}_Q[\mathcal{H}^n(Z_0)] \quad \forall x \in \mathbb{R}^d.
     \]

2. **Deterministic typical grain.** If
   - $f$ is locally bounded and such that the set of all its discontinuity points is $\mathcal{H}^n$-negligible,
   - $Z_0$ is a countably $\mathcal{H}^n$-rectifiable compact set such that
     \[
     \eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)
     \]
   holds for some $\gamma > 0$ and some probability measure $\eta \ll \mathcal{H}^n$. 
Special cases

1. **Stationary case.** If
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   \]

2. **Deterministic typical grain.** If
   - \( f \) is locally bounded and such that the set of all its discontinuity points is \( \mathcal{H}^n \)-negligible,
   - \( Z_0 \) is a countably \( \mathcal{H}^n \)-rectifiable compact set such that
     \[
     \eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S \quad \forall r \in (0, 1)
     \]
holds for some \( \gamma > 0 \) and some probability measure \( \eta \ll \mathcal{H}^n \),
then \( \mathbb{E}[\mu_\Theta] \ll \mathcal{H}^d \) with density
   \[
   \lambda_\Theta(x) = \int_{Z_0} f(x - y)\mathcal{H}^n(dy) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.
   \]
Main steps of the proof

\[ E[\| \Theta \|] \text{ is locally finite.} \]

\[ E[\| \Theta \|] = \lambda \Theta \text{ for some integrable function } \lambda \Theta \text{ on } \mathbb{R}^d \]

For any bounded Borel set \( A \subset \mathbb{R}^d \) with \( \text{H}^d(\partial A) = 0 \),

\[ \lim_{r \downarrow 0} E[\text{H}^d(\Theta \oplus r \cap A)] - n r d = E[\text{H}^n(\Theta \cap A)] , \]

i.e.

\[ \lim_{r \downarrow 0} \int_A P(x \in \Theta \oplus r) - n r d = \int_A \lim_{r \downarrow 0} P(x \in \Theta \oplus r) - n r d \]

By (A1) and (A2)
Main steps of the proof

- (IC) and (A1) implies that $E[\mu_\Theta]$ is locally finite.
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- (IC) and (A1) implies that $E[\mu_\Theta]$ is locally finite.
- $E[\mu_\Theta] = \lambda_\Theta \mathcal{H}^d$ for some integrable function $\lambda_\Theta$ on $\mathbb{R}^d$. 
Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_{\Theta}]$ is locally finite.
- $\mathbb{E}[\mu_{\Theta}] = \lambda_{\Theta} \mathcal{H}^d$ for some integrable function $\lambda_{\Theta}$ on $\mathbb{R}^d$.
- For any bounded Borel set $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(\partial A) = 0$,
  $$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta \oplus r \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta \cap A)],$$
Main steps of the proof

- (IC) and (A1) implies that $\mathbb{E}[\mu_{\Theta}]$ is locally finite.
- $\mathbb{E}[\mu_{\Theta}] = \lambda_{\Theta} H^d$ for some integrable function $\lambda_{\Theta}$ on $\mathbb{R}^d$.
- For any bounded Borel set $A \subset \mathbb{R}^d$ with $H^d(\partial A) = 0$,

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[H^d(\Theta \oplus r \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[H^n(\Theta \cap A)],$$

i.e.

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}} dx = \int_A \lambda_{\Theta}(x) dx.$$
Main steps of the proof

- (IC) and (A1) implies that $E[\mu_\Theta]$ is locally finite.
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- By (A1) and (A2)
  $$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}} dx = \int_A \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}} dx.$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_d - n r^{d-n}}$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

$$
\lambda_{\Theta}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}} = \lim_{r \downarrow 0} \int_{\mathbb{K}} \int_{Z_{x,s} \oplus r} f(y)dy \frac{Q(ds)}{b_{d-n}r^{d-n}}.
$$
For $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$,

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n} r^{d-n}}$$

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$$
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$$

$$
= \lim_{r \downarrow 0} \int_{K} \int_{Z_{x,s}^r} f(y) dy \frac{1}{b_{d-n} r^{d-n}} Q(ds)
$$

$$
= \int_{K} \lim_{r \downarrow 0} \int_{Z_{x,s}^r} f(y) dy \frac{1}{b_{d-n} r^{d-n}} Q(ds)
$$

$$
= \int_{K} \int_{Z_{x,s}} f(y) \mathcal{H}^n(dy) Q(ds).
$$
By the proof of the main theorem we get that, for any Boolean model $\Theta$ as in the Assumptions,

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{bd-nr^{d-n}} \in \mathbb{R}$$

for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$.

This suggests an estimator of $\lambda_\Theta(x)$ in terms of the empirical capacity functional of $\Theta$: 

$$\hat{\lambda}_N^\Theta(x) := \sum_{i=1}^N 1_{\Theta_i \cap B_{RN}(x) \neq \emptyset}$$

with $N$ such that $N \rightarrow 0$ and $NR^d \rightarrow \infty$ for $N \rightarrow \infty$.

Then

$$\lim_{N \rightarrow \infty} \hat{\lambda}_N^\Theta(x) = \lambda_\Theta(x)$$

in probability, for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$. 
By the proof of the main theorem we get that, for any Boolean model $\Theta$ as in the Assumptions,

$$\lambda_\Theta(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta \oplus r)}{b_{d-n}r^{d-n}} \in \mathbb{R} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d$$

This suggests an estimator of $\lambda_\Theta(x)$ in terms of the empirical capacity functional of $\Theta$:
Let $\Theta_1, \ldots, \Theta_N$ be a random sample of $\Theta$; we define

$$\hat{\lambda}_\Theta^N(x) := \frac{\sum_{i=1}^N 1_{\Theta_i \cap B_{R_N}(x) \neq \emptyset}}{N b_{d-n} R_N^{d-n}},$$

with $R_N$ such that $R_N \to 0$ and $NR_N^{d-n} \to \infty$ for $N \to \infty$. Then

$$\lim_{N \to \infty} \hat{\lambda}_\Theta^N(x) = \lambda_\Theta(x) \quad \text{in probability, for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$
Remark:
Even if the extreme case $n = 0$ can be handle with much more elementary tools, we may notice that if $\Theta = X$ is a random variable with pdf $f_X$, then

\[ \lambda_\Theta = f_X; \]
Remark:
Even if the extreme case $n = 0$ can be handle with much more elementary tools, we may notice that if $\Theta = X$ is a random variable with pdf $f_X$, then

- $\lambda_\Theta = f_X$;
- If $X_1, \ldots, X_N$ is a random sample of $X$, $\hat{\lambda}_\Theta^N(x)$ becomes in this case

$$
\hat{f}_X^N(x) = \frac{\sum_{i=1}^N 1_{B_{RN}(x)}(X_i)}{Nb_1 R_N} = \frac{\text{card}\{i : X_i \in I_x\}}{N|I_x|},
$$

where $I_x$ is the interval in $\mathbb{R}$ centered in $x$ with length $|I_x| = 2R_N$ with the usual condition

$$
|I_x| \longrightarrow 0 \quad \text{and} \quad N|I_x| \longrightarrow \infty \quad \text{as} \quad N \longrightarrow \infty.
$$