On the outer Minkowski content of sets

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Abstract

We provide general conditions, stable under finite unions, ensuring the existence of the outer Minkowski content of Borel subsets of $\mathbb{R}^d$. Such conditions turn out to be the same which guarantee the existence of the $(d-1)$-dimensional Minkowski content of the boundary of the involved sets. Moreover, our results also apply to the study of the differentiability of the volume function of bounded sets, extending some known results in literature.

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1 Introduction and main result

Denoted by $E_{\ominus r}$ the parallel set of a subset E of $\mathbb{R}^d$ at distance r and by $\mathcal{H}^n$ the n-dimensional Hausdorff measure in $\mathbb{R}^d$, the *outer Minkowski content* $\mathcal{SM}(E)$ of E is the quantity so defined

$$\mathcal{SM}(E) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(E_{\oplus r} \setminus E)}{r},$$

provided that the limit exists finite, and it is of interest in many problems arising from real applications (see [1] and reference therein). While quite general results on the existence of the n-dimensional Minkowski content of compact subsets of $\mathbb{R}^d$ are available in literature, only partial results on the outer Minkowski content are known. The most recent paper on this subject until now [1] provides a class of sets stable under finite unions for which the outer Minkowski content exists and equals the perimeter (in the sense of geometric measure theory) of the involved sets, containing, for instance, all sets with Lipschitz boundary and a type of sets with positive reach. Simple examples show that the outer Minkowski content of a set can be greater than its perimeter, but general results about its value are not available in the literature yet. This is the main goal of the present paper. Improving some techniques in [1], we prove here that the existence of the outer Minkowski content of a subset E of $\mathbb{R}^d$ is ensured by the same well-known conditions which guarantee the existence of the $(d-1)$-dimensional Minkowski content of its boundary. Namely, referring to the next section for formal definitions and notation, our main theorem (see Section 3) states that

$$\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0)$$

(here $P(E)$ denotes the perimeter of E, and E^0 is the set of points where E has null density) for any subset E of $\mathbb{R}^d$ which belongs to the following class of sets, stable under finite unions.

Definition 1.1 (The class $\mathcal{O}$) *Let $\mathcal{O}$ be the class of Borel sets E of $\mathbb{R}^d$ such that* *(i)* ∂E *is a countably* $\mathcal{H}^{d-1}$*-rectifiable bounded set;*

(ii) there exist $\gamma > 0$ *and a probability measure* η *in* $\mathbb{R}^d$ *absolutely continuous with respect to* $\mathcal{H}^{d-1}$ *such that*

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial E,\ \forall r \in (0,1).$$

As it will emerge in the sequel, the density of E at its boundary points plays a central role in the determination of the value of $\mathcal{SM}(E)$.

In the last section a series of further results, which follow as applications of the main theorem, are provided. For instance, we observe that the proof of the above formula for $\mathcal{SM}(E)$ also applies to Borel sets with $(d-1)$-rectifiable boundary, and so to unions of compact sets with positive reach. In particular, we show that the same conclusions stated for the class $\mathcal{O}$, still hold for another class of sets, defined similarly to the class $\mathcal{O}$, by replacing the condition of absolute continuity of η with the assumption that ∂E admits $(d-1)$-dimensional Minkowski content.
Finally, we study the differentiability of the so-called *volume function*

$$V_E(r) := \mathcal{H}^d(E_{\oplus r}), \qquad r \geq 0 \tag{1.1}$$

of a given bounded subset E of $\mathbb{R}^d$, at $r > 0$ (clearly, the existence of the right derivative in $r = 0$ corresponds to the existence of $\mathcal{SM}(E)$), improving, in particular, a recent result of Hug et al. in [5].

2 Basic notation and preliminaries

Throughout the paper $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, $\mathcal{B}_{\mathbb{R}^d}$ is the Borel σ-algebra of $\mathbb{R}^d$ and $\mathcal{H}^n_{|A}$ denotes the restriction of $\mathcal{H}^n$ to a $\mathcal{H}^n$-measurable set $A \subset \mathbb{R}^d$ (i.e. $\mathcal{H}^n_{|A}(E) = \mathcal{H}^n(A \cap E)$ for all $E \in \mathcal{B}_{\mathbb{R}^d}$). We shall mainly follow the notation used in [1], to which we refer for a more detailed presentation of some common definitions and results introduced in the present section. For $r \geq 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ is the closed ball with center x and radius r, while for every integer n we denote by b_n the volume of the unit ball in $\mathbb{R}^n$ (for $n = 0$, we set $b_0 := 1$).
Given a subset E of $\mathbb{R}^d$, ∂E will be its (topological) boundary, E^c the complement set of E, $\mathrm{int}E$ and $\mathrm{cl}E$ the interior and the closure of E, respectively. We denote by $E_{\oplus r} := \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) \leq r\}$ (where "dist" is the usual distance function) the parallel set of E at distance r, and by $d_E : \mathbb{R}^d \to \mathbb{R}$ the *signed distance function* from E, defined as follows

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, E^c).$$

The *upper* and *lower* outer Minkowski content of E are defined, respectively, as

$$\mathcal{SM}^*(E) := \limsup_{r\downarrow 0} \frac{\mathcal{H}^d(E_r \setminus E)}{r}, \qquad \mathcal{SM}_*(E) := \liminf_{r\downarrow 0} \frac{\mathcal{H}^d(E_r \setminus E)}{r}.$$

We say that a compact set $S \subset \mathbb{R}^d$ is n-rectifiable if it is representable as the image of a compact set $K \subset \mathbb{R}^n$, with $f : \mathbb{R}^n \to \mathbb{R}^d$ Lipschitz, and we recall that, given a subset S of $\mathbb{R}^d$ and an integer n with $0 \leq n \leq d$, the *n-dimensional Minkowski content* of S is defined by

$$\mathcal{M}^n(S) := \lim_{r\downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}},$$

whenever the limit exists finite. The following theorem is proved in [4, p. 275].