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Outer Minkowski content for some classes of closed sets

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Abstract   We find conditions ensuring the existence of the *outer Minkowski content* for d-dimensional closed sets in $\mathbb{R}^d$, in connection with regularity properties of their boundaries. Moreover, we provide a class of sets (including all sufficiently regular sets) stable under finite unions for which the outer Minkowski content exists. It follows, in particular, that finite unions of sets with Lipschitz boundary and a type of sets with positive reach belong to this class.

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1 Introduction

This paper is concerned with the so-called *outer Minkowski content* $\mathcal{SM}(A)$ of a compact set $A \subset \mathbb{R}^d$. It is defined, whenever the limit exists, by

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r}.$$

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Here we denote by A_r the closed r-neighborhood of A, and by $\mathcal{H}^k$ the Hausdorff k-dimensional measure in $\mathbb{R}^d$; to make the presentation of our results lighter, we do not give here formal definitions and we refer for this and other (typically standard) notation to the next section. If for any $x \in \partial A$ the set A is locally representable as the subgraph of a sufficiently smooth function, it is intuitive that this limit gives the surface measure of ∂A, namely $\mathcal{H}^{d-1}(\partial A)$. It is also intuitive that in many situations this coincides with the (two-sided or usual) Minkowski content $\mathcal{M}^{d-1}(\partial A)$, namely

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d((\partial A)_r)}{2r},$$

an object on which many more results are available in the literature.

Our goals, which are motivated by problems in stochastic geometry described in more detail in the second part of this introduction are:

(a) to find general conditions ensuring the existence of the outer Minkowski content;
(b) to find a class of sets $\mathcal{S}$ stable under finite unions for which the outer Minkowski content is defined.

The first goal can be considered as a variation of the more classical theme for the Minkowski content. In this analysis, besides the topological boundary ∂A, also the smaller *essential* boundary $\partial^* A$ (i.e. the set of points where the density is neither 0 nor 1) plays an important role. We can prove for instance in Theorem 5 that, whenever the Minkowski content exists and it coincides with $\mathcal{H}^{d-1}(\partial^* A)$, then the outer Minkowski content exists, and has the same value. This criterion can be applied, for instance, to show that the outer Minkowski content exists for all sets A with a Lipschitz boundary. But, the existence of the content is only loosely related to the regularity of the boundary: for instance we characterize in Theorem 9, among all *sets with positive reach*, those for which the outer Minkowski content coincides with $\mathcal{H}^{d-1}(\partial A)$. Notice that for sets of positive reach a complete polynomial expansion of $r \mapsto \mathcal{H}^d(A_r)$ is available, the so-called Steiner formula: therefore the outer Minkowski content always exists, and corresponds to the first coefficient in this expansion. In connection with the most recent developments on this subject we mention the paper [9], where the authors present a generalization of the Steiner formula to closed sets; nevertheless such general formula is not polynomial in r, and so the existence of the outer Minkowski content cannot be obtained directly, without assuming further regularity conditions. We also mention the paper [12], where existence of the outer Minkowski content is proved for finite unions $\bigcup_i A_i$ of sets A_i with positive reach such that all possible finite intersections of the A_i's have positive reach as well.

The second goal is more demanding, as simple examples show that regularity properties of the boundary are not stable under unions. The same is true (see Example 2) for other typical regularity conditions considered in geometric measure theory, as positive reach, or $\mathcal{H}^{d-1}(\partial A \setminus \partial^* A) = 0$ (which implies, by the theory of sets of finite perimeter, that an approximate normal exists at $\mathcal{H}^{d-1}$-a.e. point of ∂A). Nevertheless, we are able to identify two conditions, both stable under finite unions: the first one, see (1), is a kind of quantitative non-degeneracy condition which prevents ∂A from being too sparse; simple examples (see Example 3) show that $\mathcal{SM}(A)$ can be infinite, and $\mathcal{H}^{d-1}(\partial A)$ arbitrarily small, when this condition fails. The second condition, in

analogy with the above mentioned Theorem 5, is the existence of the outer Minkowski content and its coincidence with $\mathcal{H}^{d-1}(\partial^* A)$. The proof of stability of these two conditions, given in Theorem 6, is one of the main contributions of this paper: it requires a careful measure-theoretic analysis of the regions where the boundaries of the sets intersect, either with same or with opposite normals.

With the role of "surface measure", the outer Minkowski content is important in many problems arising from real applications. Moreover, it may be seen as derivative of the volume of a set with respect to its Minkowski enlargement, so that, for a time-dependent closed set, which can be taken as model for evolution problems, the outer Minkowski content turns to be related to evolution equations for relevant quantities associated to the model (see, e.g., [4, 11, 14]). Clearly, several real situations are studied by stochastic models (see, e.g., [15], and [10] for further applications). In a stochastic setting one deals with random closed sets and their expected volumes; so one introduces the *mean* outer Minkowski content of a random closed set Θ in $\mathbb{R}^d$, i.e. a measurable map from a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ to the space of closed subsets in $\mathbb{R}^d$, as the limit (whenever it exists)

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_r \setminus \Theta)]}{r}.$$

The problem is now to find general conditions for the existence of the mean outer Minkowski content. Starting from the results obtained in the deterministic case, along the same lines of [1] one can obtain sufficient conditions on a random compact set ensuring the existence of the mean outer Minkowski content and its coincidence with $\mathbb{E}[\mathcal{SM}(\Theta)]$. In addition, these conditions are stable under finite unions; this feature is particularly relevant, for instance, in connection with the so-called birth-and-growth stochastic processes (see [5] and references therein).

2 Notation and preliminaries

In this section, after giving some basic notation, we recall some definitions and results, mainly belonging to the area of geometric measure theory, that will be used in the paper.

We work in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, endowed with the usual norm $\| \cdot \|$ and scalar product $(\cdot, \cdot)$. For $s \geq 0$, we denote by $\mathcal{H}^s$ the Hausdorff measure of dimension s; in particular, $\mathcal{H}^d$ is the Lebesgue measure in $\mathbb{R}^d$. $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel σ-algebra of $\mathbb{R}^d$. Given a subset A of $\mathbb{R}^d$, ∂A will be its topological boundary, A^c the complement set of A, $\mathrm{int}A$ and $\mathrm{cl}A$ the interior and the closure of A, respectively. For $r \geq 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ is the closed ball with center x and radius r; finally, for every n we set $b_n = \mathcal{H}^n(B_1(0))$, i.e. the volume of the unit ball in $\mathbb{R}^n$.

A crucial notion in this paper is the *parallel set* of a subset of $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ be closed and let $r \geq 0$; the parallel set of A at distance r, denoted by A_r, is defined by

$$A_r = \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) \leq r\}.$$