SOME REMARKS ON THE CONTINUITY OF RANDOM CLOSED SETS

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Definition: *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{B}_{\mathbb{R}}$ the Borel σ-algebra on $\mathbb{R}$.*
A real-valued random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is said to be

- discrete, *if its probability law $\mathbb{P}_X$ is concentrated on an at most countable subset D of $\mathbb{R}$; i.e. the set of its realizations is discrete;*

$$\mathbb{P}_X(\{x\}) = \mathbb{P}(X = x) > 0, \text{for } x \in D, \text{ and } \mathbb{P}_X(D) = 1;$$

- continuous *if $\mathbb{P}_X(\{x\}) = \mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$;*

- absolutely continuous *if $\mathbb{P}_X$ is absolutely continuous with respect to the usual Lebesgue measure on $\mathbb{R}$.*

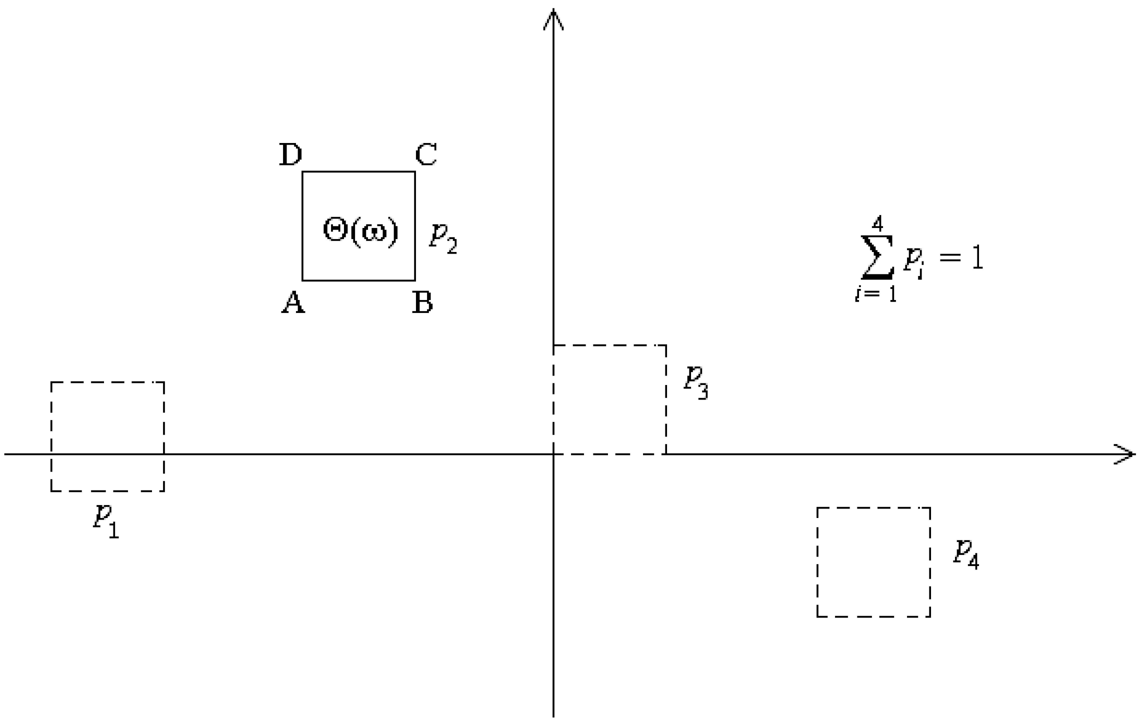
Let us consider a random closed set Θ in $\mathbb{R}^d$:

$$\Theta : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ is the family of closed sets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the sigma algebra generated by the hit-or-miss topology.

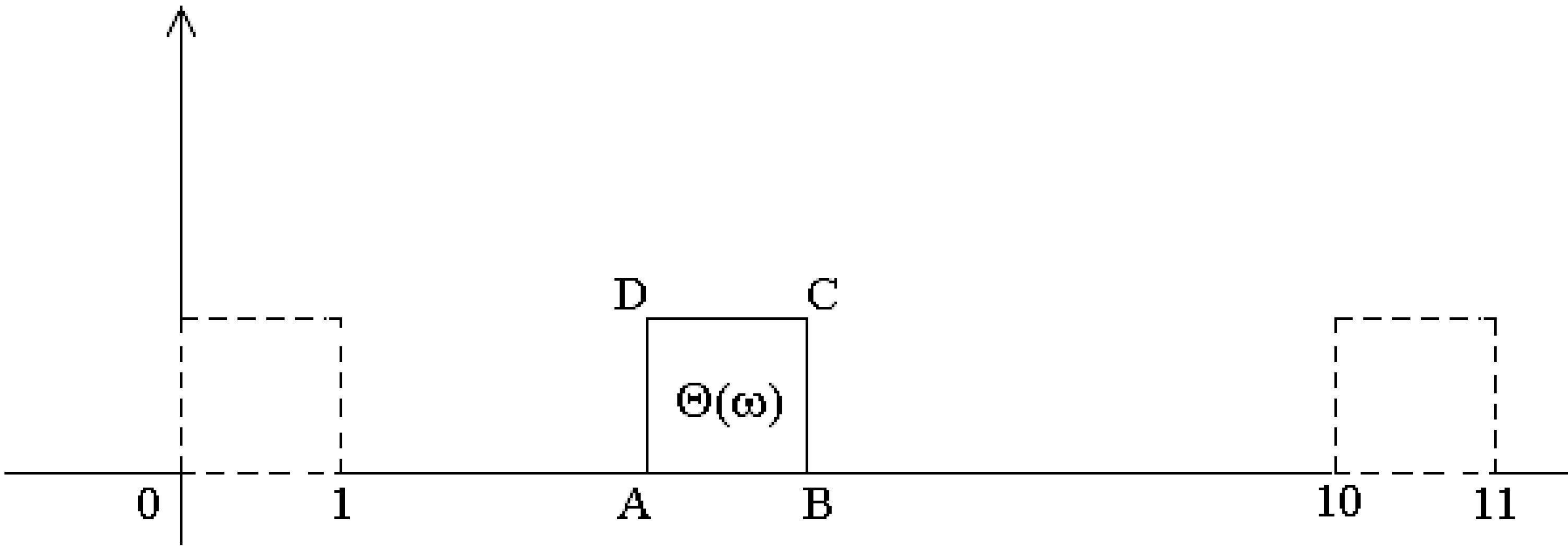
Definition: *We say that Θ is*

• discrete *if its probability law $\mathbb{P}_\Theta$ is concentrated on an at most countable subset of $\mathbb{F}$;*
i.e there exists a family $\theta_1, \theta_2, \ldots$ of closed subsets of $\mathbb{R}^d$, and a family of real numbers $p_1, p_2, \ldots \in [0,1]$ such that $\mathbb{P}(\Theta = \theta_i) = p_i$ and $\sum_i p_i = 1$;



• <u>continuous</u> *if*

$$\mathbb{P}(\Theta = \theta) = 0, \qquad \forall \theta \subset \mathbb{R}^d$$



$$A = (a, 0), \quad a \sim U[0, 10]$$

Note that:

1. The definitions given are consistent with the particular case $\Theta = X$, random point:
   since the possible realizations of X are points in $\mathbb{R}^d$, then $\mathbb{P}(X = \theta) = 0$ for every subset θ of $\mathbb{R}^d$ that is not a point, and so we say that X is continuous if and only if $\mathbb{P}(X = x) = 0$ for any $x \in \mathbb{R}^d$ (that is the usual definition).

2. In a large number of cases the conditions

$$\mathbb{P}(\Theta = \theta) = 0, \qquad \forall \theta \subset \mathbb{R}^d$$

   and

$$\mathbb{P}(\partial\Theta = \partial\theta) = 0, \qquad \forall \theta \subset \mathbb{R}^d$$

   are equivalent [1] ;
   e.g.: if $\dim(\Theta) < d$, or $\Theta(\omega) = \mathrm{clos}(\mathrm{int}\Theta(\omega))$.

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[1] Capasso V., Villa E.: On the continuity and absolute continuity of random closed sets. *Stoch. An. Appl.*, **24**, 381–397 (2006).

A Comparison with Current Literature

Definition [*M-continuity*] (see Matheron G. (1975). *Random sets and integral geometry*)
A random closed set Θ *in* $\mathbb{R}^d$ *is* a.s. continuous *if*

$$\mathbb{P}(x \in \partial\Theta) = 0 \qquad \forall x \in \mathbb{R}^d.$$

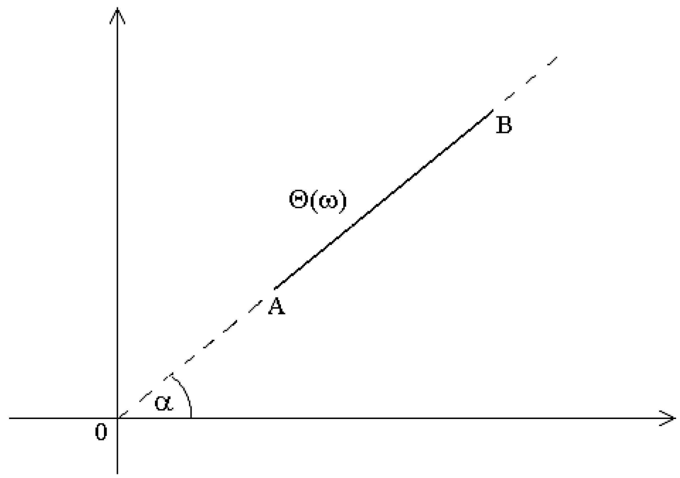
If Θ is a.s. continuous with capacity functional T_Θ, then

$$x = \lim_{n\to\infty} x_n \text{ in } \mathbb{R}^d \;\Rightarrow\; T_\Theta(\{x\}) = \lim_{n\to\infty} T_\Theta(\{x_n\})$$

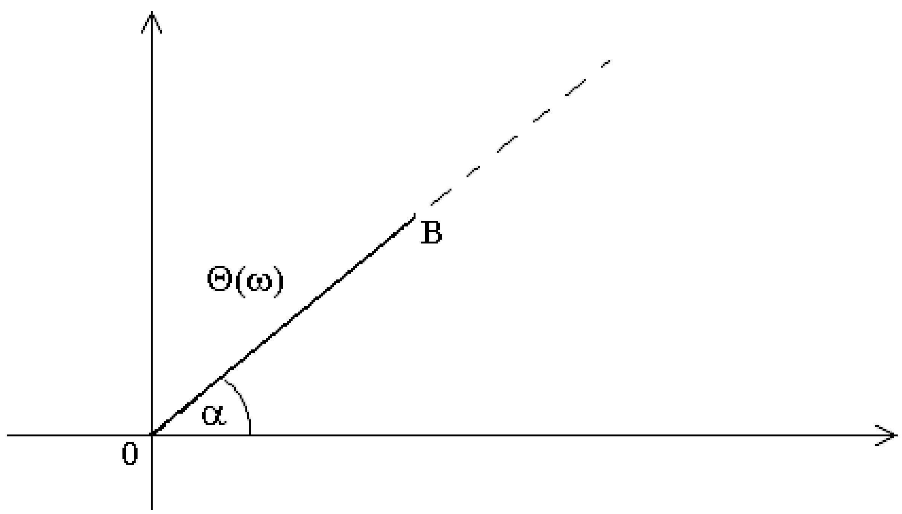
It is easy to prove that M-continuity $\begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array}$ continuity.

MAIN DIFFERENCE between M-continuity and continuity:

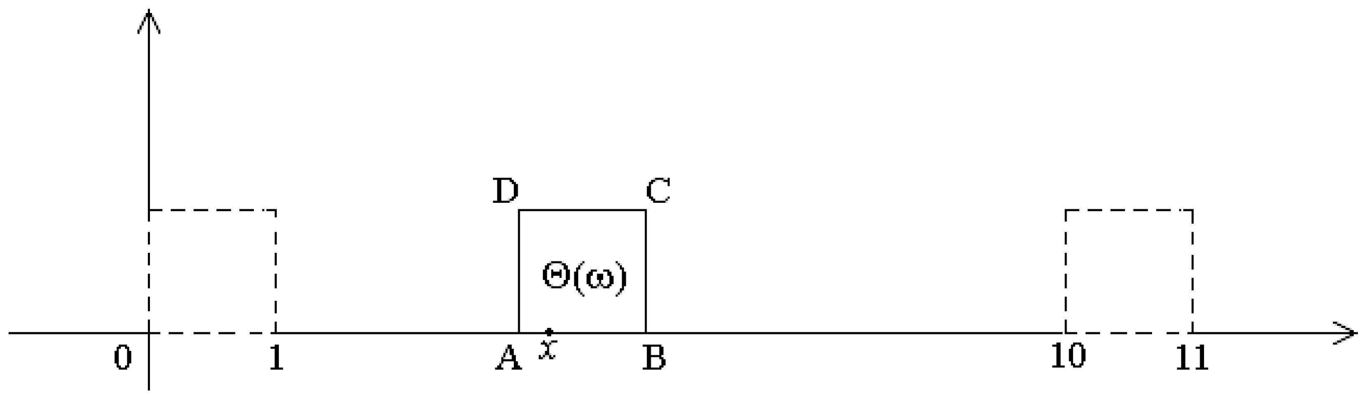
to know that the random set Θ is not continuous by our definition implies that it may assume some configuration with probability bigger than 0;
to know that the random set Θ is not M-continuous does not give this kind of information.



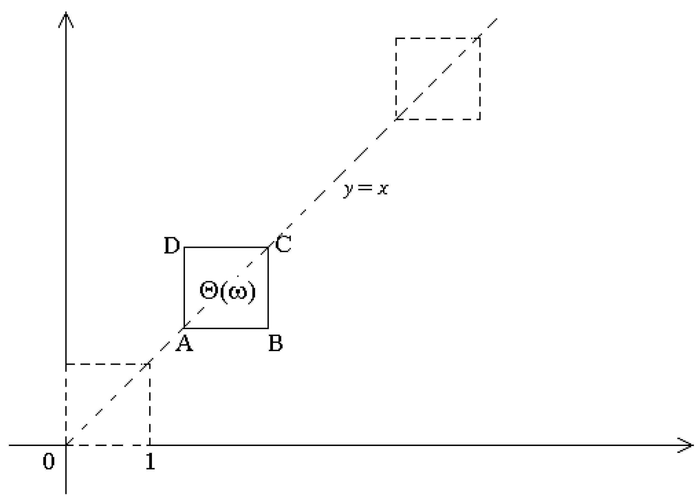
$\alpha \sim U[0, 2\pi]$, $A, B \neq 0 \implies$ Θ is M-continuous, Θ is continuous.



$\alpha \sim U[0, 2\pi]$, $A = 0 \implies$ Θ is <u>not</u> M-continuous, Θ is continuous.



$a \sim U[0,10],\ A = (a,0) \Longrightarrow \mathbb{P}(x \in \partial\Theta) = 1/10$
Θ is not M-continuous, Θ is continuous



$A = (a,a)$, Θ is M-continuous, Θ is continuous

Absolutely Continuous RACS

Definition [*Absolute continuity in mean*]
Let Θ be a random closed set in $\mathbb{R}^d$ with Hausdorff dimension n, such that its associated expected measure

$$\mathbb{E}[\mu_\Theta] := \mathbb{E}[\mathcal{H}^n(\Theta \cap \cdot\,)]$$

is a Radon measure.
We say that Θ is absolutely continuous in mean *if*

$$\mathbb{E}[\mu_\Theta] \ll \nu^d.$$

Even if this definition is consistent with the case $\Theta = X$ random point, it may give no information on the geometric stochastic properties of Θ in $\mathbb{R}^d$.
For instance:

- dim$\Theta = d$;
- if dim$\Theta = n < d$ and $\mathcal{H}^n(\Theta(\omega)) = 0$ $\mathbb{P}$-a.s.

It is well known that for a r.v. X

$$X \text{ absolutely continuous} \quad \Rightarrow \quad X \text{ continuous, but not the reverse;}$$
$$X \text{ discrete} \quad \Rightarrow \quad X \text{ singular, but not the reverse.}$$

Definition [$\mathcal{R}$ *class*] *We say that a random closed set* Θ *in* $\mathbb{R}^d$ belongs to the class $\mathcal{R}$ *if*

$$\dim(\partial\Theta) < d \quad \textit{and} \quad \mathbb{P}[\mathcal{H}^{\dim(\partial\Theta)}(\partial\Theta) > 0] = 1.$$

Definition [*Strong absolute continuity*]
We say that a random closed set Θ *is* (strongly) absolutely continuous *if* $\Theta \in \mathcal{R}$ *and*

$$\mathbb{E}[\mu_{\partial\Theta}] \ll \nu^d. \tag{1}$$

Remarks: Let $\Theta \in \mathcal{R}$.

1. If $\dim\Theta < d$, than $\partial\Theta = \Theta$ and so there is no distinction between absolute continuity *strong* and *in mean*.

2. if $\dim\Theta = d$ and $\dim\partial\Theta = d-1$, than Θ is absolutely continuous if

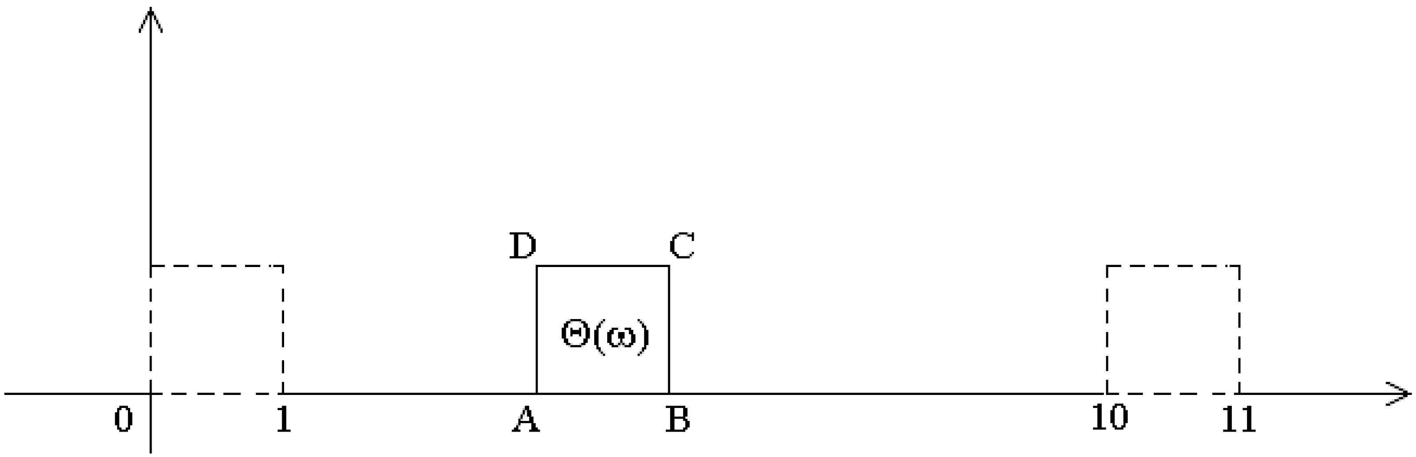
$$\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \cdot\,)] \ll \nu^d(\cdot).$$

3. If $\Theta = X$ r.v., then

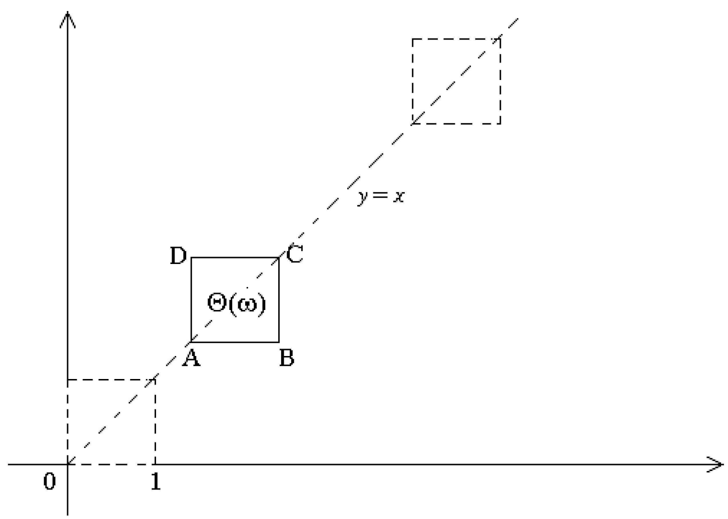
$$\mathrm{dim}X = 0,\ \partial X = X \text{ and } \mathbb{E}[\mathcal{H}^0(X)] = \mathbb{P}(X \in \mathbb{R}^d) = 1,$$

thus $X \in \mathcal{R}$ and condition (1) is equivalent to say

$$\mathbb{E}[\mathcal{H}^0(X \cap \cdot\,)] = \mathbb{P}(X \in \cdot\,) \ll \nu^d.$$



Θ is not absolutely continuous ($B = [0, 11]$, $\nu^2(B) = 0$, $\mathbb{E}[\mathcal{H}^1(\partial\Theta \cap B)] = 1$)



Θ is absolutely continuous

Question: M-continuity is equivalent to absolute continuity?

Answer: No. For instance:

$$X \text{ r.v. M-continuous} \iff \mathbb{P}(X = x) = 0 \quad \forall x \in \mathbb{R}^d$$

But

$$\mathbb{P}(X = x) = 0 \quad \forall x \in \mathbb{R}^d \not\Longrightarrow X \text{ absolutely continuous.}$$

Definition: *We say that $\Theta \in \mathcal{R}$ is* singular *if and only if it is not absolutely continuous.*

In terms of our definitions we may claim that:

Proposition:

$$\begin{aligned} \Theta \text{ absolutely continuous} \quad &\Rightarrow \quad \Theta \text{ continuous, but not the reverse;} \\ \Theta \text{ discrete} \quad &\Rightarrow \quad \Theta \text{ singular, but not the reverse.} \end{aligned}$$

Definition: *Let Θ and Q be random closed sets in $\mathbb{R}^d$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $Q \subseteq \Theta$ if and only if*

$$\forall \omega \in \Omega \qquad Q(\omega) \subseteq \Theta(\omega).$$

In general there are no relations between Θ and its subsets; for example we may have Θ absolutely continuous and
$Q \subset \Theta$ discrete, or viceversa. But it is easy to prove that

Proposition:

If $\dim_{\mathcal{H}} \Theta < d$ and Θ is discrete

$$\Downarrow$$

any subset $Q \in \mathcal{R}$ of Θ is singular.