METHODS OF GEOMETRIC MEASURE THEORY IN STOCHASTIC GEOMETRY

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Introduction

Geometric measure theory concerns the geometric structure of Borel sets and measures in euclidean spaces in order to study their regularities (i.e. if they possess some smoothness) or irregularities, with the aim to find criteria which guarantee rectifiability (see, e.g., [4, 30, 32, 50]). Every set or measure can be decomposed in its rectifiable and its non-rectifiable part and these parts behave entirely different. From geometric measure theory follows connections with the decomposition of a measure into absolutely continuous part, jump part and Cantor part and so the notion of dimensional density of a set in terms of the density of its induced measure. Roughly speaking, rectifiable sets and measures are the largest class of sets on which an analogue of the classical calculus on manifolds (approximation by tangent spaces, etc.) can be performed. Thus Hausdorff measures, Hausdorff dimensions, rectifiable sets, area and coarea formulas, Minkowski content are the basic ingredients of geometric measure theory, and they turn to be essential tools whenever one has to deal with sets at different Hausdorff dimensions and in several problems raised by image analysis, solid and liquid crystals theory, continuous mechanics...

In many real applications, the need of dealing with the same kind of problems in a stochastic context emerges. Application areas include crystallization processes (see [47, 21], and references therein; see also [66] for the crystallization processes on sea shells); tumor growth [6] and angiogenesis [27]; spread of fires in the woods, spread of a pollutant in the environment; etc. All quoted processes may be described by time dependent random closed sets at different Hausdorff dimensions (for instance, crystallization processes are modelled in general by full dimensional growing sets, and lower dimensional interfaces, while angiogenesis by systems of random curves).

Typical problems concern the characterization of such random objects in terms of relevant quantities (mean densities, volume and surface expected measures,...) characterizing the geometric process. Therefore, in a stochastic setting, we have to deal with tools of stochastic geometry, together with concepts of geometric measure theory.
It is well known that if a result holds "almost surely", in general it does not hold "in mean", and viceversa. Thus the application, in a stochastic context, of methods and results which are proper of a deterministic setting (in our case, methods and results form geometric measure theory), requires additional regularity conditions on the referring random set.

In dependence of its regularity, a random closed set Θ_n with Hausdorff dimension n (i.e. $\dim_{\mathcal{H}}\Theta_n(\omega) = n$ for a.e. $\omega \in \Omega$), may induce a random Radon measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot\,)$$

on $\mathbb{R}^d$ ($\mathcal{H}^n$ is the n-dimensional Hausdorff measure), and, as a consequence, an *expected measure*

$$\mathbb{E}[\mu_{\Theta_n}(\cdot)] := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot\,)].$$

In several real applications it is of interest to study the density (said *mean density*) of the measure $\mathbb{E}[\mu_{\Theta_n}]$ [13], and, in the dynamical case, its evolution in time [54, 55].

The principal aim of the present thesis is to provide a framework for dealing with random closed sets at different Hausdorff dimensions in order to describe them in terms of their mean densities, based, whenever necessary, on concepts and results from geometric measure theory.

It is clear that, if $n < d$, for a.e. $\omega \in \Omega$, the measure $\mu_{\Theta_n}(\omega)$ is singular, while its expected measure $\mathbb{E}[\mu_{\Theta_n}]$ may be absolutely continuous with respect to the usual d-dimensional Lebesgue measure ν^d. Thus, a first problem, is to introduce a suitable formalism which apply both to random closed sets with singular expected measure, and not. To this end, rectifiability conditions on the referring random set, together with basic ideas of the theory of generalized functions are needed. So we introduce a *Delta formalism*, á la Dirac, for the description of random measures associated with random closed sets of lower dimensions, such that the well known usual Dirac delta at a point follows as a particular case (see, for instance, [40, 45, 67]). In this way, in the context of time dependent growing random sets, we provide a natural framework for deriving evolution equations for mean densities at all (integer) Hausdorff dimensions, in terms of the relevant kinetic parameters associated with a given growth process [26, 24].

In dealing with mean densities, a concept of *absolutely continuous random closed set* arises in a natural way in terms of the expected measure; indeed, an interesting property of a random set in $\mathbb{R}^d$ is whether the expected measure induced by the random set is absolutely continuous or not with respect to the d-dimensional Lebesgue measure ν^d. In classical literature on stochastic geometry (see, e.g., [49]) the concept of continuity of a random closed set has not been sufficiently analyzed, and does not seem to provide sufficient insight about the structure of the relevant random closed set, so that we have introduced definitions of *discrete, continuous* and *absolutely continuous* random closed set, which extend the standard definition of discrete, continuous and absolutely continuous random variable, respectively [23, 25]. The definition of absolute continuity is given in terms of the absolute continuity of the expected measure associated with the random set with respect to ν^d, so that geometrical stochastic properties of the random set may be related to its mean density.

In many real applications such as fibre processes, n-facets of random tessellations of dimension $n \leq d$ in spaces of dimension $d \geq 1$, several problems are related to the estimation of such mean densities (see e.g. [13, 55, 64]). For instance, in $\mathbb{R}^2$, since points and lines are ν^2-negligible, it is natural to make use of their 2-D box approximation. As a matter of fact, a computer graphics representation of them is anyway provided in terms of pixels, which can only offer a 2-D box approximation of points in $\mathbb{R}^2$. By geometric measure theory we know that the Hausdorff measure of a sufficiently regular set can be obtained as limit of the Lebesgue measure (much more robust and computable) of its *enlargement* by Minkowski addition. Thus, we consider a particular approximation of the mean densities for sufficiently regular random closed sets based on a stochastic version of the existence of the *Minkowski content* [2]. This procedure suggests unbiased density estimators and it is consistent with the usual histogram estimation of probability densities of a random variable, known in literature [58].

The "enlargement" of a growing set in time may be regarded as a suitable Minkowski enlargement; since *first order Steiner formulas* propose a relation between the Hausdorff measure of the boundary of a set and the derivative in $r = 0$ of the volume of the enlarged set by Minkowski addition with a ball of radius r, we ask if a *mean first order Steiner formula* holds for some classes of random closed sets. To treat directly mean first order Steiner formulas is still an open problem. Here we start by almost sure convergence (i.e. the random closed set satisfies a first order Steiner formula $\mathbb{P}$-a.s.), and we obtain the L^1-convergence (i.e. the mean first order Steiner formula) by means of uniform integrability conditions. The most general result known in current literature is that a first order Steiner formula holds (in the deterministic case) for unions of sets with positive reach (see [59, 42, 37]), and it is proved by tools of integral and convex geometry. Here we offer a different proof of this for finite unions of sets with positive reach, which seems to be more tractable in the stochastic case, based on elementary tools of measure theory (in particular on the inclusion-exclusion theorem), togheter with a basic Federer's result on sets with positive reach. Further, through a different approach, by considering sets with finite perimeter, we prove that a first order Steiner formula holds for sets with Lipschitz boundary, and we give also a *local* version of it. Then, by applying the quoted result on the existence of the Minkowski content in a stochastic setting (which guarantees the exchange between limit and expectation), we obtain that a *local mean* first order Steiner formula holds, under suitable regularity assumptions on the random set [1]. This result plays a fundamental role in deriving evolution equations for the mean densities of growing random closed sets.

Since many real phenomena may be modelled as *dynamic germ-grain models*, we consider, as working example, geometric processes associated with *birth-and-growth processes* driven by a marked point process $N = \{(T_i, X_i)\}_{i\in\mathbb{N}}$ on $\mathbb{R}_+$ with marks in $\mathbb{R}^d$, modelling births, at random times $T_i \in \mathbb{R}_+$, and related random spatial locations $X_i \in \mathbb{R}^d$ [39, 64]. Once born, each germ generates a grain, which is supposed to grow according with a deterministic space-time dependent given field.

Denoted by Θ^t the evolving random closed set at time t, we provide a relation between the time derivative of the mean densities of Θ^t and the probability density function of the random time of capture $T(x)$ of a point $x \in \mathbb{R}^d$ by the growing random set. On one hand, by survival analysis [5], $T(x)$ is strictly

related to the hazard and survival functions of x; on the other hand, we prove a relation between the hazard function and the spherical contact distribution function, a well studied tool in stochastic geometry [36], which is in connection with the mean first order Steiner formula associated with Θ^t, and so with the mean density of $\partial\Theta^t$.
Thus, under suitable general condition on the birth-and-growth process, we may write evolution equations of the *mean volume density* associated to Θ^t in terms of the growing rate and of the *mean surface density* (i.e. the mean density associated to $\partial\Theta^t$). Such equations turn to be the stochastic analogous of evolution equations known in literature for deterministic growing sets [12, 17, 19, 65]; indeed they may be formally obtained taking the expected value in the deterministic equation.

In **Chapter 1** we introduce some basic definitions and results from geometric measure theory and from stochastic geometry, which will be useful in the sequel. In **Chapter 2** we give suitable regularity conditions on random closed sets such that it is possible to deal with *densities*, to be meant in a *generalized sense*, in order to treat also singular measures. According to Riesz Representation Theorem, we may regard such densities as (random) linear functionals and interpret them as *generalized Radon-Nikodym derivatives* (in a distributional sense), of the associated expected measured. We do this first in the deterministic case, then in the stochastic one. In **Chapter 3** we introduce definitions of discrete, continuous and absolutely continuous random closed set. In particular we distinguish between absolute continuous *in mean* and *strong*; this last one requires additional regularity conditions on the random set, in order to exclude some "pathological" cases, like sets with Hausdorff dimension n, but n-dimensional Hausdorff measure zero. For absolutely continuous random sets it follows that the associated mean generalized density is a classical real function. Clearly, random variables may be regarded as particular 0-dimensional random closed set; the definitions given extend the standard definition of discrete, continuous and absolutely continuous random variable, respectively. Further, we observe that the well known relations between discrete, continuous and absolutely continuous part of a Radon measure hold also for random closed sets, although in a different context. With the aim to face real problems related to the estimation of mean densities in the general setting of spatially inhomogeneous processes, in **Chapter 4** we suggest and analyze an approximation of mean densities for sufficiently regular random closed sets, based on d-dimensional enlargements by Minkowski addition. We show how some known results in literature follow as particular cases, and we provide a series of examples to exemplify various relevant situations. In **Chapter 5** we consider first order Steiner formulas for closed sets in $\mathbb{R}^d$. Roughly speaking, we may say that a first order Steiner formula holds for a deterministic d-dimensional set $A \subset \mathbb{R}^d$ if there exists the derivative in $r = 0$ of the volume of the enlarged set $A_{\oplus r}$ by Minkowski addition with a ball of radius r. When A is stochastic and we consider its expected volume, we speak of *mean* first order Steiner formula. We prove that under regularity assumptions on the random set, a local mean first order Steienr formula holds. In **Chapter 6** we apply all previous results in the context of birth-and-growth processes. The problem of the absolute continuity of $T(x)$ in terms of quantities characterizing the process is taken into account, and connections with the concepts of hazard functions and spherical contact distribution functions, together with mean local Steiner formulas at first order, are studied. As a result it follows that an evolution equation (to be taken in a weak form) holds for the mean density of the growing set. The particular case of a Poissonian nucleation process, often taken as a model in several real applications, is also considered and compared with other different nucleation processes in order to make clearer the crucial property of independence of increment, typical of the Poisson process.

Chapter 1

Preliminaries and notations

In this chapter we recall concepts and results of current literature which are relevant for our analysis.

1.1 Measures and derivatives

We remember some basic definitions (e.g. see [4, 29, 30, 60, 61]).
A positive *measure* μ over a set X is a function defined on some σ-algebra $\mathcal{S}$ of subsets of X and taking values in the range $[0, \infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$
for every countable sequence of disjoint sets $\{E_j\} \subset \mathcal{S}$.

If the property (ii) is weakened to subadditivity, μ is called *outer measure*; more precisely, an outer measure μ on a set X is a function defined on *all* subsets of X such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(E) \leq \mu(E')$ if $E \subset E'$,

(iii) $\mu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j) \quad$ for any $\{E_j\} \subset X$.

Outer measure are useful since there is always a σ-algebra of subsets on which they behave as measures.
A subset E of X is called *μ-measurable*, or *measurable with respect to the outer measure μ* if

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E) \qquad \forall A \subset X.$$

Theorem 1.1 *([30], p. 3) Let μ be an outer measure. The collection $\mathcal{S}$ of μ-measurable sets forms a σ-algebra, and the restriction of μ to $\mathcal{S}$ is a measure.*

Besides, all Borel sets are μ-measurable if and only if *Caratheodory's criterion* holds:
Let μ be an outer measure on the metric space X such that

$$\operatorname{dist}(E, F) > 0 \implies \mu(E \cup F) = \mu(E) + \mu(F) \qquad \text{for any } E,\ F \subset X;$$

then the restriction of μ to the Borel sets of X is a positive measure.

Let us consider the space $\mathbb{R}^d$ and denote by ν^d the usual d-dimensional Lebesgue measure. Note that ν^d is an outer measure and, according to Caratheodory criterion, its restriction to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ of $\mathbb{R}^d$ is a Radon measure.
We recall now some basic definitions:

- A measure μ on $\mathbb{R}^d$ is called *Borel* if every Borel set is μ-measurable.
- A measure μ on $\mathbb{R}^d$ is *Borel regular* if μ is Borel and for each $A \subset \mathbb{R}^d$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.
- A measure μ on $\mathbb{R}^d$ is a *Radon* measure if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^d$.

Note that, given a Borel regular measure μ on $\mathbb{R}^d$, we can generate a Radon measure by restricting μ to a measurable set of finite measure.
Let us consider a positive Radon measure μ on $\mathbb{R}^d$.

- μ is said to be *concentrated* on a set E_0 if it is defined and equal to zero on every Borel set $E \subset \mathbb{R}^d \setminus E_0$.
- μ is said to be *continuous* if it is defined and equal to zero on every set of measure zero containing a single point (i.e. $\mu(\{x\}) = 0 \ \forall \ x \in \mathbb{R}^d$).
- μ is said to be *absolutely continuous* if it is defined and equal to zero on every set of measure zero.
- μ is said to be *singular* if it is concentrated on a set E_0 of measure zero.
- A singular measure μ is said to be *discrete* if it is concentrated on a set E_0 of measure zero containing no more than countably many points.

Beside, we remind that every measure μ can be represented in the form

$$\mu(E) = A(E) + S(E) + D(E), \tag{1.1}$$

where $A(E)$ is absolutely continuous, $S(E)$ is continuous and singular, and $D(E)$ is discrete.
We denote by $\mu_{\ll}$ and $\mu_{\perp}$ the absolutely continuous part and the singular part of μ, respectively. So, an equivalent form of (1.1) is

$$\mu = \mu_{\ll} + \mu_{\perp}.$$

Remark 1.2

$$\begin{aligned} \mu \text{ absolutely continuous} \quad &\Rightarrow \quad \mu \text{ continuous, but not the reverse;} \\ \mu \text{ discrete} \quad &\Rightarrow \quad \mu \text{ singular, but not the reverse.} \end{aligned}$$

Note that not all the finite measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ that are singular, are also discrete. An example is the measure induced by the Cantor (singular) function:

it is singular with respect to the Lebesgue measure, but it assigns measure zero to every point of $\mathbb{R}$, and so it is continuous and singular. The Cantor set is the subset of $[0, 1]$ constructed as follows:
the "middle third" of the interval $[0, 1]$ is removed, i.e. the open interval $(\frac{1}{3}, \frac{2}{3})$ of length $\frac{1}{3}$. Next, the middle thirds of the two remaining intervals are removed, i.e. the interval $(\frac{1}{9}, \frac{2}{9})$ is removed from $[0, \frac{1}{3}]$ and $(\frac{7}{9}, \frac{8}{9})$ is removed from $[\frac{1}{3}, 1]$. Then the middle thirds of each of the four intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$ are removed, and so on. The remaining closed set C is called the *Cantor set*.
Equivalently, if we denote by

$$\begin{aligned} E_1 &= (\frac{1}{3}, \frac{2}{3}) \\ E_2 &= (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \\ &\vdots \end{aligned}$$

we may represent the Cantor set by

$$C := [0, 1] - \bigcup_{n=1}^{\infty} E_n.$$

It follows that C is closed, it has no interior points, it has the power of continuum and it has Lebesgue measure zero.
Now, let $A_1, A_2, \ldots, A_{2^n-1}$ be the subintervals of $\bigcup_{i=1}^{n} E_i$, ordered with increasing order; for example, if $n = 3$:

$$\begin{aligned} &E_1 \cup E_2 \cup E_3 \\ &= (\frac{1}{27}, \frac{2}{27}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{7}{9}, \frac{8}{9}) \cup (\frac{25}{27}, \frac{26}{27}) \\ &= A_1 \cup A_2 \cup \ldots \cup A_7 \end{aligned}$$

We set

$$\begin{aligned} &F_n(0) = 0 \\ &F_n(x) = \frac{k}{2^n} \qquad \text{if } x \in A_k,\ k = 1, 2, \ldots, 2^n - 1 \\ &F_n(1) = 1, \end{aligned}$$

completing by linear interpolation; for example, if $n = 2$ (see Fig.1.1):

$$\begin{aligned} E_1 \cup E_2 &= (\frac{1}{9}, \frac{2}{9}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{7}{9}, \frac{8}{9}) = A_1 \cup A_2 \cup\ A_3 \\ F_2(x) &= \frac{1}{4} \qquad \text{if } x \in A_1 \\ &= \frac{1}{2} \qquad \text{if } x \in A_2 \\ &= \frac{3}{4} \qquad \text{if } x \in A_3 \end{aligned}$$

It can be shown that

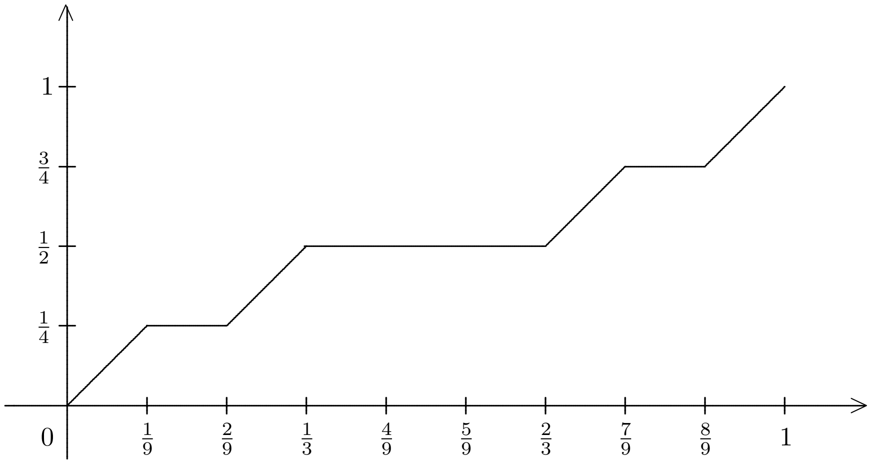

Figure 1.1: The function $F_n(x)$ when $n = 2$.

$$\begin{aligned} F : [0,1] &\longrightarrow [0,1] \\ x &\longmapsto F(x) := \lim_{n\to\infty} F_n(x) \end{aligned}$$

is increasing and continuous. Let μ be the measure on $\mathbb{R}$ so defined:

$\mu((-\infty, 0) = 0,$
$\mu([0, x]) = F(x) \quad \forall x \in [0, 1],$
$\mu((1, +\infty)) = 0.$

Then, $\mu(C) = 1$ and $\mu(\{x\}) = 0 \ \forall x \in \mathbb{R}$; thus it is continuous and singular, but it is not discrete since it is concentrated on a set which is not countable. In other words, it can not be written as $\sum_i \delta_{x_i}$, where $x_i \in [0, 1]$ and δ_{x_i} is the Dirac delta at point x_i.

From now on, r is a positive quantity (i.e. $r \in \mathbb{R}_+$), and so $r \to 0$ has to be intended $r \to 0^+$.

We know that, denoted by $B_r(x)$ the d-dimensional closed ball centered in x with radius r, it is possible to define the following quantities:

$$(\overline{D\mu})(x) := \limsup_{r\to 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}, \qquad (\underline{D\mu})(x) := \liminf_{r\to 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}.$$

Definition 1.3 *If* $(\overline{D\mu})(x) = (\underline{D\mu})(x) < +\infty$, *then their common value is called the* symmetric derivative *of* μ *at* x *and is denoted by* $(D\mu)(x)$.
$(\overline{D\mu})(x)$ *and* $(\underline{D\mu})(x)$ *are also called* upper *and* lower densities *of* μ *at* x.

The following is an application of the Besicovitch derivation theorem ([4], p. 54):

Theorem 1.4 *Let* μ *be a positive Radon measure on* $\mathbb{R}^d$. *Then, for* ν^d*-a.e.* x, *the limit*

$$f(x) := \lim_{r\to 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}$$

exists in $\mathbb{R}$; besides, the Radon-Nikodym decomposition of μ is given by

$$\mu = f\nu^d + \mu_\perp,$$

where $\mu_\perp(\cdot) = \mu(E \cap \cdot)$, with E the ν^d-negligible set

$$E = \left\{ x \in \mathbb{R}^d : \lim_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} = \infty \right\}.$$

Thus, the above theorem provides a characterization of the support of the singular part. In particular, as a consequence of the theorem, we have that

(C1) $\mu \perp \nu^d \iff (D\mu)(x) = 0 \quad \nu^d$-a.e.;

(C2) $\mu \ll \nu^d \iff \mu(B) = \int_B (D\mu)(x)\nu^d(\mathrm{d}x)$ for all Borel set $B \subset \mathbb{R}^d$.

Even if, usually, by "density of μ" it is understood the Radon-Nikodym derivative of μ with respect to ν^d, and so it is meant that μ is absolutely continuous, actually, it is well known in literature the Delta "function" at a point X_0, as the *(generalized) density* of the singular Dirac measure [45].

Remark 1.5 Consider the Dirac measure ε_{X_0} associated with a point $X_0 \in \mathbb{R}^d$, defined as

$$\varepsilon_{X_0}(A) := \begin{cases} 1 & \text{if } X_0 \in A, \\ 0 & \text{if } X_0 \notin A. \end{cases}$$

Obviously ε_{X_0} is singular with respect to ν^d, and by (C1) we have that $D\varepsilon_{X_0} = 0$ ν^d-a.e.; in fact, in this case, in accordance with Theorem 1.4 we have that:

$$\lim_{r\to 0} \frac{\varepsilon_{X_0}(B_r(x))}{b_d r^d} = \begin{cases} 0 & \text{if } x \neq X_0, \\ \lim_{r\to 0} \dfrac{1}{r^d} = +\infty & \text{if } x = X_0, \end{cases} \tag{1.2}$$

where b_j ($j = 0, \ldots, d$) is the j-dimensional measure of the unit ball in $\mathbb{R}^j$; note that $\nu^d(B_r(x)) = b_d r^d$. We introduce the *Dirac delta function* δ_{X_0} as the "generalized density" (distribution) of ε_{X_0} such that (see Section 1.3)

$$\int_A \delta_{X_0}(x)\nu^d(\mathrm{d}x) := \varepsilon_{X_0}(A). \tag{1.3}$$

Due to (1.2) we may claim in a "generalized" sense that

$$\delta_{X_0}(x) = \begin{cases} 0 & \text{if } x \neq X_0, \\ +\infty & \text{if } x = X_0, \end{cases}$$

and

$$\varepsilon_{X_0}(A) = \int_A \delta_{X_0}(x)\nu^d(\mathrm{d}x) := \begin{cases} 1 & \text{if } X_0 \in A, \\ 0 & \text{if } X_0 \notin A; \end{cases}$$

so, formally, $\delta_{X_0}(x)$ is a "fictitious" function which equal zero everywhere except at $x = X_0$ and has an integral equal to 1.

By analogy with the Delta function, we want to introduce a concept of "density" for particular singular measures on $\mathbb{R}^d$ (for example, the measure of the intersection of a given surface with a d-dimensional subset of $\mathbb{R}^d$).
Note that, if we consider the restriction of the Lebesgue measure ν^d to a measurable subset A of $\mathbb{R}^d$, then the density of the absolutely continuous measure $\nu^d(A \cap \cdot\,)$ is also called the *Lebesgue density of A*.
We have the following theorem.

Theorem 1.6 (Lebesgue density theorem) *([30], p. 14) Let A be a measurable subset of $\mathbb{R}^d$. Then the Lebesgue density of A at point x*

$$\lim_{r\to 0} \frac{\nu^d(A \cap B_r(x))}{b_d r^d} \tag{1.4}$$

exists and equals 1 if $x \in A$ and 0 if $x \notin A$ for ν^d-a.e. x.

For lower dimensional sets, the natural analogues of Lebesgue densities are the socalled *dimensional densities*, given in terms of the Hausdorff measure.

1.2 Hausdorff measures and related concepts

Definition 1.7 *Let $A \subset \mathbb{R}^d$, $0 \le s < \infty$, $0 < \delta \le \infty$. Define*

$$\mathcal{H}^s_\delta(A) := \inf\left\{ \sum_{j=1}^{\infty} b(s)\left(\frac{\mathrm{diam}C_j}{2}\right)^s \,:\, A \subset \bigcup_{j=1}^{\infty} C_j,\, \mathrm{diam}C_j < \delta \right\},$$

where $b(s) \equiv \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$, and

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A).$$

*$\mathcal{H}^s$ is called s-*dimensional Hausdorff outer measure *on $\mathbb{R}^d$.*
*The restriction of $\mathcal{H}^s$ to the σ-algebra of $\mathcal{H}^s$-measurable sets is called s-*dimensional Hausdorff measure*.*

Note that when s is integer, say $s = n$, than $b(n) = b_n$, the volume of the unit ball in $\mathbb{R}^n$.
Countable subadditivity follows immediately from the definition, so $\mathcal{H}^s$ is an outer measure; further, $\mathcal{H}^s$ is a Borel regular measure, but not a Radon measure if $0 \le s < d$, since $\mathbb{R}^d$ in not σ-finite with respect to $\mathcal{H}^s$. The measurability of Borel sets follows from Caratheodory's criterion. Indeed, we have the following

Proposition 1.8 *[4] The measure $\mathcal{H}^s$ are outer measures in $\mathbb{R}^d$ and, in particular, σ-additive on $\mathcal{B}_{\mathbb{R}^d}$.*

The σ-algebra of $\mathcal{H}^s$-measurable sets includes the Borel sets and the socalled *Souslin sets.* Souslin sets, unlike the Borel sets, are defined explicitly in terms of unions and intersections of closed sets; in particular, they are sets of the form

$$E = \bigcup_{i_1,i_2\ldots} \bigcap_{k=1}^{\infty} E_{i_1 i_2 \ldots i_k},$$

where $E_{1_1 i_2 \dots i_k}$ is a closed set for each finite sequence $\{i_1, i_2, \dots, i_k\}$ of positive integers.
It may be shown that every Borel set is a Souslin set and that, if the underline metric space is complete, then any continuous image of a Souslin set is Souslin. Further, if μ is an outer measure on a metric space, then the Souslin sets are μ-measurable.

Remark 1.9 If $s = k$ is an integer, $\mathcal{H}^k$ agrees with ordinary "k-dimensional surface area" on nice sets (see [29] p. 61), that is, for C^1 k-dimensional submanifolds of $\mathbb{R}^d$, it coincides with the usual k-dimensional area (e.g. the length for $k = 1$ or the surface for $k = 2$) ([64] p. 22; [4] p. 73).

Theorem 1.10 *([29], p. 63) As outer measures,*
$\mathcal{H}^d = \nu^d$ *on* $\mathbb{R}^d$ *(and so* $\mathcal{H}^d(A) = \nu^d(A)$ *for any Borel set* $A \subset \mathbb{R}^d$*);*
$\mathcal{H}^1 = \nu^1$ *on* $\mathbb{R}^1$*;*
$\mathcal{H}^0 = \nu^0$ *is the usual counting measure.*

In particular, by the definition of Hausdorff measure, it follows that ([29], p. 65):
Let $A \subset \mathbb{R}^d$ and $0 \leq s < t < \infty$; then

i) $\mathcal{H}^s(A) < \infty \;\Rightarrow\; \mathcal{H}^t(A) = 0$,

ii) $\mathcal{H}^t(A) > 0 \;\Rightarrow\; \mathcal{H}^s(A) = \infty$.

Definition 1.11 *The* Hausdorff dimension *of a set* $A \subset \mathbb{R}^d$ *is defined to be*

$$\dim_{\mathcal{H}}(A) := \inf\{0 \leq s < \infty \,|\, \mathcal{H}^s(A) = 0\}.$$

Remark 1.12 $\dim_{\mathcal{H}}(A)$ need not to be an integer; for example, the Hausdorff dimension of the Cantor set is $s = \log 2/\log 3 = 0,6309\dots$ (see [30] p. 14).
Besides, $\dim_{\mathcal{H}}(A) = s$ does not imply that $\mathcal{H}^s(A)$ is positive and finite; we may have $\dim_{\mathcal{H}}(A) = s$ and $\mathcal{H}^s(A) = 0$, or $\mathcal{H}^s(A) = \infty$. (See also [57].)

Here and in the following, we denote by A^C, $\mathrm{int}A$, $\mathrm{clos}A$ and ∂A the complementary set, the interior, the closure and the boundary of a set A, respectively.

Theorem 1.13 *([29], p. 72) Assume that* $A \subset \mathbb{R}^d$*,* A *is* $\mathcal{H}^s$*-measurable, and* $\mathcal{H}^s(A) < \infty$*. Then*

i) for $\mathcal{H}^s$*-a.e.* $x \in A^C$

$$\lim_{r \to 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s) r^s} = 0;$$

ii) for $\mathcal{H}^s$*-a.e.* $x \in A$

$$\frac{1}{2^s} \leq \limsup_{r \to 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s) r^s} \leq 1.$$

Definition 1.14 *Let A be a subset of $\mathbb{R}^d$ $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(A) < \infty$ ($0 \le s < \infty$). The* upper *and* lower s-dimensional densities *of A at a point $x \in \mathbb{R}^d$ are defined as*

$$\overline{D}^s(A, x) := \limsup_{r \to 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s) r^s}$$

and

$$\underline{D}^s(A, x) := \liminf_{r \to 0} \frac{\mathcal{H}^s(A \cap B_r(x))}{b(s) r^s},$$

*respectively. If $\overline{D}^s(A, x) = \underline{D}^s(A, x)$ we say that the s-*dimensional density *of A at x exists and we write $D^s(A, x)$ for the common value.*

Theorem 1.15 *([30], p. 63) Let A be a subset of $\mathbb{R}^d$, $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(A) < \infty$. If s in not integer, then, if there exists, $D^s(A, x) \neq 1$ $\mathcal{H}^s$-q.o.*

Roughly speaking, the existence of the dimensional density equal to 1 of a set means that it can be considered, in some sense, "regular".

Similarly, for a measure μ on $\mathbb{R}^m$, $0 \le m \le d$, m integer, we may define the *m-dimensional density* $D^m(\mu, x)$ of μ at x by

$$D^m(\mu, x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{b_m r^m}. \tag{1.5}$$

Compare also with Definition 1.3.
Note that for any subset A of $\mathbb{R}^d$, $D^m(A, x) = D^m(\mathcal{H}^m_{|A}, x)$, where $\mathcal{H}^m_{|A}$ is the measure defined by

$$\mathcal{H}^m_{|A}(E) \equiv \mathcal{H}^m(A \cap E).$$

Hence, the m-dimensional density of measures actually generalizes the notion of m-dimensional density of sets. (See [56] or [32]).

In the following we will introduce a concept of "generalized density" that recalls both the definition of the Lebesgue density (1.4) and the definition of the dimensional density (1.5). In fact, for a set A with Hausdorff dimension m, we will consider the limit

$$\lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B_r(x))}{b_d r^d}.$$

Definition 1.16 *([32], p. 251) Let A be a subset of a metric space X, m a positive integer and ϕ a measure over X:*

1) *A is* m-rectifiable *if and only if there exists a Lipschitzian function mapping a bounded subset of $\mathbb{R}^m$ onto A;*

2) *A is* countably m-rectifiable *if and only if A equals the union of a countable family whose members are m-rectifiable;*

3) *A is* countably (ϕ, m)-rectifiable *if and only if there exists a countably m-rectifiable set containing ϕ-almost all of A;*

4) A is (ϕ,m)-rectifiable *if and only if A is countably (ϕ,m)-rectifiable and $\phi(A)<\infty$.*

When $X=\mathbb{R}^d$ and $\phi=\mathcal{H}^m$, we write $\mathcal{H}^m$-rectifiable, instead of $(\mathcal{H}^m,m)$-rectifiable.
The tangential properties of $\mathcal{H}^m$-rectifiable sets generalize those of m-dimensional submanifolds of class 1; so, from the point of view of measure theory, a rectifiable set behaves like submanifolds that admit a tangent space at almost every point. Roughly, $\mathcal{H}^m$-rectifiable sets are characterized by the equivalent properties of being countable union of measurable pieces of m-dimensional C^1 submanifolds or of possessing "approximate tangent space" $\mathcal{H}^m$-almost everywhere ([34], p. 90; see also [56]).

By Definition 1.16, an $\mathcal{H}^m$-rectifiable set is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^d$, apart for a null $\mathcal{H}^m$ set. It includes countably unions of immersed manifolds.
More precisely ([34], p. 90 and following):
If A is a countably $\mathcal{H}^m$-rectifiable set, then we can write A as a disjoint union

$$A=A_0\cup\bigcup_{k=1}^{\infty}A_k,$$

where $\mathcal{H}^m(A_0)=0$ and each A_k is a Borel subset of an m-dimensional C^1-submanifold, or, equivalently,

$$A=\mathcal{A}_0\cup\bigcup_{k=1}^{\infty}f_k(\mathcal{A}_k),$$

where $\mathcal{H}^m(\mathcal{A}_0)=0$ and $f_k:\mathcal{A}_k\subset\mathbb{R}^m\to\mathbb{R}^d$ are Lipschitz maps.

As a consequence, countably 0-rectifiable and countably $\mathcal{H}^0$-rectifiable sets correspond to finite or countable sets, while $\mathcal{H}^0$-rectifiable sets correspond to finite sets. In particular, the graph of a Lipschitz function of m variables is an example of countably m-rectifiable set.

Notice that the definition of rectifiability of a set is related to the measure defined on the set itself, and as well a notion of rectifiable set and tangent planes is given, similarly a notion of rectifiable and tangent measure may be defined. (As we said at the beginning, we consider positive Radon measure, but the following definitions can be given for more general vector measures; see [4]).
In order to look at the asymptotic behavior of a measure μ in $\mathbb{R}^d$ near a point x of its support, it is convenient to introduce the rescaled measure

$$\mu_{x,r}(B):=\mu(x+rB),\qquad B\in\mathcal{B}_{\mathbb{R}^d}$$

around x, and to analyze the behavior of suitable normalizations of $\mu_{x,r}$ as $r\to 0$.

Definition 1.17 *We denote by $\mathrm{Tan}(\mu,x)$ the set of all finite Radon measures on $B_1(0)$ which are the weak* limits of*

$$\frac{\mu_{x,r_i}}{\mu(B_r(x))}$$

for some infinitesimal sequence $\{r_i\} \subset (0, \infty)$*. The elements of* $\mathrm{Tan}(\mu, x)$ *are called* tangent measures to μ at x.

Remark 1.18 An interesting example of tangent measure comes considering tangent planes to regular embedded manifolds. Let $\Gamma \subset \mathbb{R}^d$ be a m-dimensional C^1 surface and let $\mu = \mathcal{H}^m_{|\Gamma}$. Then $\mathrm{Tan}(\mu, x)$ contains only the measure

$$\frac{1}{b_m}\mathcal{H}^m_{|\pi(x)} \qquad \forall x \in \Gamma,$$

where $\pi(x)$ is the tangent space to Γ at x.

Definition 1.19 *A Radon measure* μ *is said* m-rectifiable *if there exists a countably* $\mathcal{H}^m$*-rectifiable set* S *and a Borel function* $\theta : S \to \mathbb{R}$ *such that*

$$\mu = \theta \mathcal{H}^m_{|S}. \tag{1.6}$$

Remark 1.20 i) In the extreme cases $m = 0$, and $m = d$ we obtain the class of purely atomic measures and the class of all measures absolutely continuous with respect to ν^d.

ii) According to Radon-Nikodym theorem, μ is representable by (1.6) for suitable $S \in \mathcal{B}_{\mathbb{R}^d}$ and $\theta : S \to \mathbb{R}$, if and only if it is absolutely continuous with respect to $\mathcal{H}^m$, and concentrate on a set σ-finite with respect to $\mathcal{H}^k$. It can be proved (see Theorem 1.22) that if μ is concentrated on a countably $\mathcal{H}^m$-rectifiable set, then the function $\theta(x)$ coincides μ-a.e. with $D^m(\mu, x)$.

m-rectifiable measures are, for almost every point x, asymptotically concentrated near to x on an affine m-plane. This allows to define an approximate tangent space to m-rectifiable measures (and to countably $\mathcal{H}^m$-rectifiable sets as well) which plays, in this context, the same role played by the classical tangent space in differential geometry.

Definition 1.21 *We say that a Radon measure* μ *has* approximate tangent space π with multiplicity $\theta \in \mathbb{R}$ at x, *and we write*

$$\mathrm{Tan}^m(\mu, x) = \theta \mathcal{H}^m_{|\pi}$$

if $\dfrac{\mu_{x,r}}{r^m}$ *locally weakly* converges to* $\theta \mathcal{H}^m_{|\pi}$ *in* $\mathbb{R}^d$ *as* $r \to 0$.

In other words, the approximate tangent space to μ is the unique m-plane π on which the measures $\mu_{x,r}/r^m$ are asymptotically concentrated. (See Appendix A for the particular case of rectifiable curves).
Note that, when the following limits make sense,

$$D^m(\mu, x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{b_m r^m} = \frac{1}{b_m} \lim_{r \to 0} \frac{\mu_{x,r}(B_1(0))}{r^m} = \frac{1}{b_m} \theta \mathcal{H}^m_{|\pi}(B_1(0)) = \theta.$$

More precisely, the following theorem holds:

Theorem 1.22 (Rectifiability criterion for measures) *([4], p. 94) Let μ a positive Radon measure on an open set $\Omega \subset \mathbb{R}^d$.*

(i) If $\mu = \theta \mathcal{H}^m_{|S}$ and S is countably $\mathcal{H}^m$-rectifiable, then μ admits an approximate tangent space with multiplicity $\theta(x)$ for $\mathcal{H}^m$-a.e. $x \in S$. In particular $\theta(x) = D^m(\mu, x)$ for $\mathcal{H}^m$-a.e. $x \in S$.

(ii) If μ is concentred on a Borel set S and admits an approximate tangent space with multiplicity $\theta(x) > 0$ for μ-a.e. $x \in S$, then S is countably $\mathcal{H}^m$-rectifiable and $\mu = \theta \mathcal{H}^m_{|S}$. In particular

$$\exists \ \mathrm{Tan}^m(\mu, x) \ \textit{for } \mu - \textit{a.e. } x \in \Omega \implies \mu \textit{ is } k\textit{-rectifiable.}$$

Definition 1.23 *Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^m$-rectifiable set and let $\{S_j\}$ be a partition of $\mathcal{H}^m$-almost all of S into $\mathcal{H}^m$-rectifiable sets; we define* $\mathrm{Tan}^m(S, x)$ *to be the* approximate tangent space *to $\mathcal{H}^m_{|S_j}$ at x for any $x \in S_j$ where the latter is defined.*

Note that if $\mathcal{M}$ is a m-dimensional C^1-manifold, then the approximate tangent space coincide with the classical tangent space.

From all this, it is clear the following theorem:

Theorem 1.24 *([32], p. 256, 267) A subset A of $\mathbb{R}^d$ is countably $\mathcal{H}^m$-rectifiable if and only if $\mathcal{H}^m$-almost all of A is contained in the union of some countable family of m dimensional submanifolds of class 1 of $\mathbb{R}^d$.*
If A is an $\mathcal{H}^m$-rectifiable set and an $\mathcal{H}^m$-measurable subset of $\mathbb{R}^d$, then, for $\mathcal{H}^m$-a.e. $x \in A$,

$$\lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B_r(x))}{b_m r^m} = 1, \tag{1.7}$$

and the approximate tangent space at x is an m-dimensional vectorsubspace of $\mathbb{R}^d$.

In particular, the reverse holds ([4], p. 83):

Theorem 1.25 (Besicovitch-Marstrand-Mattila) *Let $A \in \mathcal{B}_{\mathbb{R}^d}$ with $\mathcal{H}^m(A) < \infty$. Then, A is $\mathcal{H}^m$-rectifiable if and only if*

$$\lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B_r(x))}{b_m r^m} = 1, \qquad \mathcal{H}^m\textit{-a.e. } x \in A.$$

1.3 Basic ideas of the theory of generalized functions

If $S \subset \mathbb{R}^d$ is given by an at most countable union of points in $\mathbb{R}^d$, we know that it is well defined the counting measure $\mathcal{H}^0(S \cap \cdot)$ induced by S, so that $\mathcal{H}^0(S \cap A)$ "counts" the number of points of S which also belong to a Borel set A.
In case S reduces to a single point X_0, we know that the Dirac measure ε_{X_0} coincides with the counting measure $\mathcal{H}^0(X_0 \cap \cdot)$, and in Remark 1.5 we have

introduced the usual Dirac delta function δ_{X_0} associated to X_0. In the classical theory of generalized functions (see e.g. [45]),

$$\int_A \delta_{X_0}(x)\mathrm{d}x := \mathbf{1}_A(X_0) = \mathcal{H}^0(X_0 \cap A) \tag{1.8}$$

(here $\mathbf{1}_A$ stands for the characteristic function of A), and we remind that $\delta_{X_0}(x)$ can be obtained as the limit of a sequence of "classical" integrable functions $\varphi_m(x)$ ($\delta_{X_0}(x) = \lim_{m\to\infty} \varphi_m(x)$) such that

$$\lim_{m\to\infty} \int_A \varphi_m(x)\mathrm{d}x = \mathcal{H}^0(X_0 \cap A).$$

Remark 1.26 The integral in (1.8) does not represent a usual (Lebesgue) integral, but it is only a "symbol" for $\mathcal{H}^0(X_0 \cap A)$; in fact, since $\delta_{X_0}(x) = 0$ for any $x \neq X_0$, its Lebesgue integral should be equal to zero. Thus, we cannot interchange limit and integral, even if it is *formally* allowed by (1.8).

Well known examples of approximating sequences of δ_{X_0} with $X_0 \in \mathbb{R}$ are the Gaussian functions

$$\varphi_m(x) := \frac{m}{\sqrt{\pi}} e^{-m^2(x-X_0)^2},$$

or the simple functions

$$\varphi_m(x) := \frac{m}{2} \mathbf{1}_{[X_0 - \frac{1}{m}, X_0 + \frac{1}{m}]}(x). \tag{1.9}$$

We revisit now the basics of the theory of generalized functions ([45] p. 206 and following).

Definition 1.27 *Let $\mathcal{X}$ be a set and $\mathcal{F}(\mathcal{X},\mathbb{R})$ be the collection of all real-valued functions defined on $\mathcal{X}$.*
A subset $\mathcal{U} \subset \mathcal{F}(\mathcal{X},\mathbb{R})$ of this set is a linear space *if and only if*

(a) $0 \in \mathcal{U}$*;*

(b) if $f, g \in \mathcal{U}$, then $f + g \in \mathcal{U}$;

(c) if $f \in \mathcal{U}$ and $t \in \mathbb{R}$, then $tf \in \mathcal{U}$.

Definition 1.28 *A* linear functional *is a function $\Phi : \mathcal{U} \to \mathbb{R}$, defined on a linear space $\mathcal{U}$, such that*

(a) $\Phi(0) = 0$*;*

(b) $\Phi(f + g) = \Phi(f) + \Phi(g)$ *for all $f, g \in \mathcal{U}$;*

(c) $\Phi(tf) = t\Phi(f)$ *for all $f \in \mathcal{U}$ and $t \in \mathbb{R}$.*

Let φ be a fixed function on the real line, integrable over any finite interval, and let $f \in C_c(\mathbb{R},\mathbb{R})$, where $C_c(\mathbb{R},\mathbb{R})$ denotes the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. Consider the linear functional

$$f \in C_c(\mathbb{R},\mathbb{R}) \longmapsto T_\varphi(f) = (\varphi, f) := \int_{-\infty}^{\infty} \varphi(x) f(x)\mathrm{d}x \in \mathbb{R}. \tag{1.10}$$

In this way we may identify the function φ as a linear functional defined on $C_c(\mathbb{R}, \mathbb{R})$.
The function f is said *test function*. The space of test functions, called the *test space*, may be chosen in various other ways, depending on specific applications. For example, it might consist of all continuous functions with a compact support, as above; however, it makes sense to require the test functions satisfy rather stringent smoothness conditions whenever we want to differentiate them. In the theory of generalized functions, the test space, denoted by K, is usually chosen as the set of all functions f of compact support with continuous derivatives of all orders (equivalently, the set of all infinitely differentiable functions). Clearly K is a linear space.
In particular, the following notion of convergence in K is introduced.

Definition 1.29 *A sequence $\{f_m\}$ of functions in K is said to* converge *to a function $f \in K$ if*

1. *there exists an interval outside which all the functions f_m vanish;*
2. *the sequence $\{f_m^{(k)}\}$ of derivatives of order k converges uniformly on this interval to $f^{(k)}$, for any $k = 0, 1, 2, \ldots$.*

The linear space K equipped with this notion of convergence is called the test space*, and the functions in K are called* test functions*.*

Definition 1.30 *Every continuous linear functional $T(f)$ on the test space K is called a* generalized function *on $(-\infty, \infty)$, where continuity of $T(f)$ means that $f_m \to f$ in K implies $T(f_m) \to T(f)$.*

However, there are many other linear functionals on K besides functionals of the form (1.10); for example, the linear functional assigning to any function f its value at the point $x = 0$. Generalized functions defined by locally integrable functions are called *regular*, and all other generalized functions are called *singular* (for example the Dirac delta function defined before).

Usually, a representation of the form (1.10) is given to all generalized functions, even in the singular case, where such integrals have to be understood as an extension of the Lebesgue integral, that is, not as usual Lebesgue integrals, but as a formal representation of the linear functional T.

Addition of generalized functions and multiplication of generalized functions by real numbers are defined in the same way as for linear functionals in general.

Definition 1.31 *A sequence of generalized functions T_n is said to* converge *to a generalized function T if $T_n(f) \to T(f)$ for every $f \in K$.*

In other words, convergence of generalized functions is just the weak* convergence of continuous linear functionals on K.

Note that generalized functions are particular linear functionals, and it is well known that integration with respect to a fixed measure is "linear". In the following we will consider linear functionals defined by measures, according to the usual identification by Riesz Representation Theorem, so that we will regard them as generalized functions on the usual test space C_c, in order to interpret them as the "generalized density" of the associated measure, with respect to the usual Lebesgue measure (see Chapter 2).

1.4 Random closed sets

Throughout this section we refer to [49, 52, 53].
Roughly speaking, a random closed set is a random element in the space of all closed subsets of a basic setting space $\mathcal{X}$.
Let $\mathcal{X}$ be a locally compact, Hausdorff, and separable space (i.e. each point in $\mathcal{X}$ admits a compact neighborhood, and the topology of $\mathcal{X}$ admits a countable base). We denote by $\mathbb{F}$ the class of the closed subsets in $\mathcal{X}$, and by $\mathbb{F}_B$ the class of the closed sets hitting B, and by $\mathbb{F}^B$ the complementary class:

$$\mathbb{F}_B := \{F \in \mathbb{F} \,:\, F \cap B \neq \emptyset\},$$

$$\mathbb{F}^B := \{F \in \mathbb{F} \,:\, F \cap B = \emptyset\}.$$

The space $\mathbb{F}$ is topologized by the topology $\sigma_{\mathbb{F}}$, generated by the two families $\mathbb{F}^K$, with K compact subset of $\mathcal{X}$, and $\mathbb{F}_G$, with G open subset of $\mathcal{X}$.
It can be proved (see [49], p. 3) that the space $\mathbb{F}$ is compact, Hausdorff, and separable.

Definition 1.32 *A* random closed set *(RACS) Θ is a measurable map*

$$\Theta : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}).$$

The measurability of such a map Θ is guaranteed if

$$\{\omega \,:\, \Theta(\omega) \cap K \neq \emptyset\} \in \mathcal{F} \tag{1.11}$$

for all compact set K (see [49]). This means that, observing a realization of Θ, we can always say if Θ hits or misses a given compact set K.
Note that Θ is a function whose values are closed sets. Such functions are a usual object of set-valued analysis. For instance, the measurability condition (1.11) coincides with the condition of measurability of set-valued functions, that we remind briefly (see [9], p. 307):

Definition 1.33 *Consider a measurable space $(X, \mathcal{X})$, a complete separable metric space Y, and a set-valued map $F : X \longrightarrow Y$ with closed images.*
The map F is called measurable *if the inverse image of each open set is a measurable set: for every open subset $\mathcal{O} \subset Y$, we have*

$$F^{-1}(\mathcal{O}) := \{x \in X \,:\, F(x) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{X}.$$

We will deal with random closed set in the Euclidean space $\mathbb{R}^d$. Here are several examples of random closed sets: random points and point processes, random spheres and balls, etc.

The definition ensures that many functional $f(\Theta)$ are measurable, i.e. they become random variables. For instance, the d-dimensional Lebesgue measure of Θ, $\nu^d(\Theta)$, is a measurable functional. The same is true for the surface area (if it is well defined), and many other functionals known for convex geometry (see [62]).

Measurability of some set-theoretic operations can be deduced from topological properties (the so called semicontinuity) of related maps.
For instance, if Θ and Ξ are random closed sets, then $\Theta \cup \Xi$, $\Theta \cap \Xi$ and $\partial\Theta$ are random closed sets.

Definition 1.34 *We say that a random closed set* Θ has Hausdorff dimension s *if* $\dim_{\mathcal{H}}\Theta(\omega) = s$ *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$.

Note that a random variable is a particular case of a random closed set with 0 Hausdorff dimension.

1.4.1 Capacity functionals and Choquet Theorem

The distribution of a random closed set Θ is described by the corresponding probability measure $\mathbf{P}$ on $\sigma_{\mathbb{F}}$. Fortunately, $\mathbf{P}$ is determined by its values on $\mathbb{F}_K$ for K running through the class $\mathcal{K}^d$ of compacts in $\mathbb{R}^d$, only.
It well known that the probability law of an ordinary random variable is entirely determined if the corresponding distribution function is given. In the case of a random closed set, there exists a very similar notion.
Let Θ be a random closed set associated with a probability law on $\sigma_{\mathbb{F}}$. For any $K \in \mathcal{K}^d$ let $T_\Theta(K)$ be equal to $\mathbf{P}(\mathbb{K}_K)$, i.e.

$$T_\Theta(K) = \mathbf{P}(\mathbb{K}_K) = \mathbb{P}(\{\omega : \Theta(\omega) \cap K \neq \emptyset\}), \quad K \in \mathcal{K}^d.$$

Definition 1.35 *The functional* T_Θ *is said to be the* capacity (or hitting) functional *of* Θ.

Considered as a function on $\mathcal{K}^d$, the capacity functional T_Θ is an *alternating Choquet capacity* of infinite order, i.e. it satisfies the following properties:

(T1) $T_\Theta(\emptyset) = 0$;

(T2) $T_\Theta(K_1) \leq T_\Theta(K_2)$ if $K_1 \subseteq K_2$, so that T_Θ is a monotone functional;

(T3) T_Θ is upper semi-continuous on $\mathcal{K}^d$, i.e. $T_\Theta(K_n) \downarrow T_\Theta(K)$ as $K_n \downarrow K$;

(T4) the following functionals recurrently defined by

$$\begin{aligned}
S_1(K_0; K) &= T_\Theta(K_0 \cup K) - T_\Theta(K_0) \\
\dots \quad & \quad \dots \\
S_n(K_0; K_1, \dots, K_n) &= S_{n-1}(K_0; K_1, \dots, K_{n-1}) \\
&\qquad\qquad -S_{n-1}(K_0 \cup K_n; K_1, \dots, K_{n-1})
\end{aligned}$$

are non-negative for all $n \geq 0$ and $K_0, K_1, \dots, K_n \in \mathcal{K}^d$.

Note that the value of $S_n(K_0; K_1, \ldots, K_n)$ is equal to the probability that Θ misses K_0, but hits $K_1, \ldots, K_n$
The properties of T_Θ resemble those of distribution function. Property (T3) is the same as right-continuity, and (T2) is the extension of the notion of monotonicity. However, in contrast to measures, the functional T_Θ is not additive, but only *subadditive*, i.e.

$$T_\Theta(K_1 \cup K_2) \leq T_\Theta(K_1) + T_\Theta(K_2)$$

for all compact sets K_1 and K_2.

Remark 1.36 If $\Theta = X$ is a random point in $\mathbb{R}^d$, then $T_X(K) = \mathbb{P}(X \in K)$ is the probability distribution of X. Besides, it can be proven that the capacity functional T_Θ is additive if and only if Θ is a random singleton.

The following theorem establishes one-to-one correspondence between Choquet capacities and distributions of random closed sets.

Theorem 1.37 (Choquet) *Let T be a functional on $\mathcal{K}^d$. Then there exists a (necessarily unique) probability $\mathbf{P}$ on $\sigma_{\mathbb{F}}$ satisfying*

$$\mathbf{P}(\mathbb{F}_K) = T(K), \quad K \in \mathcal{K}^d,$$

if and only if T is an alternating Choquet capacity of infinite order such that $0 \leq T(K) \leq 1$ and $T(\emptyset) = 0$.

Thus, the capacity functional determines uniquely the distribution of a random closed set. It plays in the theory of random sets the same role as the distribution function in classical probability theory.

Remark 1.38 Generally speaking, the family of all compact sets is too large, which makes it difficult to define $T_\Theta(K)$ for *all* $K \in \mathcal{K}^d$. In this connection, an important problem arises to reduce the class of test sets needed. That is to say, is the distribution of a random closed set determined by the values $T(K)$, $K \in \mathcal{M}$, for a certain class $\mathcal{M} \subset \mathcal{K}^d$? In general, it can be proved that the distribution of a random closed set is determined by the values of its capacity functional on the class of all finite unions of balls of positive radii, or the class of all finite unions of parallelepipeds.

1.4.2 Stationary and isotropic random closed sets

Definition 1.39 *A random closed set Θ is said to be* stationary *if Θ has the same distribution as $\Theta + a$ for all $a \in \mathbb{R}^d$.*

By Choquet Theorem, Θ is stationary if and only if $T_\Theta(K)$ is translation-invariant.

Definition 1.40 *A random closed set Θ is said to be* isotropic *if Θ has the same distribution as $r\Theta$ for all rotation r.*

Let Θ be a random closed set in $\mathbb{R}^d$. Fubini's theorem implies that, for a general σ-finite measure μ (for example the d-dimensional Lebesgue measure), $\mu(\Theta)$ is a random variable, and

$$\mathbb{E}[\mu(\Theta)] = \int_{\mathbb{R}^d} \mathbb{P}(x \in \Theta)\mu(\underset{\cdot}{\mathrm{x}}). \tag{1.12}$$

Note that, if Θ is stationary, then $\mathbb{P}(x \in \Theta)$ does not depend on x and is equal to $\mathbb{P}(0 \in \Theta)$. By (1.12, for any $W \in \mathcal{K}^d$,

$$\mathbb{E}[\mu(\Theta \cap W)] = \mathbb{P}(0 \in \Theta)\mu(W).$$

The constant $\mathbb{E}[\mu(\Theta \cap W)]/\mu(W) = \mathbb{P}(0 \in \Theta)$ is called the *volume fraction* of Θ; it characterizes the part of volume covered by Θ.

1.4.3 Weak convergence of random closed sets

Weak convergence of random closed sets is a particular case of weak convergence of probability measures, since a random closed set is associated with a certain probability measure on $\sigma_{\mathbb{F}}$. (For further details see e.g. [52].)

Definition 1.41 *A sequence of random closed sets Θ_n, $n \geq 1$, is said to* converge weakly *if the corresponding probability measures $\mathbf{P}_n$, $n \geq 1$, converge weakly in the usual sense; namely,*

$$\lim_{n\to\infty} \mathbf{P}(\mathcal{U}) = \mathbf{P}(\mathcal{U}) \tag{1.13}$$

for each $\mathcal{U} \in \sigma_{\mathbb{F}}$ such that $\mathbb{P}(\partial\mathcal{U}) = 0$, for the boundary of $\mathcal{U}$ with respect to the hit-or-miss topology (i.e. $\mathcal{U}$ is a continuity set for the limiting measure).

To check (1.13) for all $\mathcal{U} \in \sigma_{\mathbb{F}}$ is rather difficult, but it has been proved (see [52] and references therein) that a reduction is possible, by letting $\mathcal{U}$ to be equal to $\mathbb{F}_K$ for K running through $\mathcal{K}^d$. In particular, the class $\mathbb{F}_K$ is a continuity set for the limiting measure if

$$\mathbf{P}(\mathbb{F}_K) = \mathbf{P}(\mathbb{F}_{\mathrm{int}K}).$$

In other words,

$$\mathbb{P}(\Theta \cap K \neq \emptyset\,,\, \Theta \cap \mathrm{int}K = \emptyset) = 0, \tag{1.14}$$

for the corresponding limiting random set Θ.
In terms of the limiting capacity functional T_Θ, denoting by

$$T_\Theta(\mathrm{int}K) := \sup\{T_\Theta(K') \,:\, K' \in \mathcal{K}^d, K' \subset \mathrm{int}K\},$$

the class of compact sets satisfying (1.14) is denoted by $\mathcal{S}_T$, and it is easy to see that

$$\mathcal{S}_T\{K \in \mathcal{K}^d \,:\, T_\Theta(K) = T_\Theta(\mathrm{int}K)\}.$$

It follows that the pointwise convergence of capacity functionals on $\mathcal{S}_T$ implies the weak convergence of the corresponding probability measures on $\sigma_{\mathbb{F}}$:

Proposition 1.42 *The sequence of random closed sets Θ_n converges to Θ if*

$$T_{\Theta_n}(K) = T_\Theta(K)$$

for each K belonging to $\mathcal{S}_T$.

1.5 Point processes

1.5.1 General definitions

Throughout this Section we refer to [28, 41, 46].
Roughly speaking, a point process in a space $\mathbb{R}^d$ is a random distribution of points in $\mathbb{R}^d$. A point process can be described in two equivalent ways: as random counting measure, and as random closed set.

Point processes as random measures

Let $\mathbf{M}$ be the set of all *locally finite* measures μ on $(\mathbb{R}^d,\ \mathcal{B}_{\mathbb{R}^d})$. Denote by $\mathcal{M}$ the smallest σ-algebra on $\mathbf{M}$ such that all the maps

$$\mu \in \mathbf{M} \longrightarrow \mu(A) \in \mathbb{R} \quad \forall A \in \mathcal{B}_{\mathbb{R}^d}$$

are measurable; i.e. it is the σ-algebra containing all sets of the type

$$\{\mu \in \mathbf{M} \,:\, \mu(A) \in B \text{ with } A \in \mathcal{B}_{\mathbb{R}^d},\ B \in \mathcal{B}_{\mathbb{R}}\}. \tag{1.15}$$

If $\mu(A) \in \mathbb{N}$ for any A, then μ is said *counting measure*; further, it is said *simple* if $\mu(\{x\}) \leq 1 \quad \forall x \in \mathbb{R}^d$.
Let $\mathbf{N}$ be the set of all counting measures, and $\mathcal{N}$ be the associated σ-algebra as in (1.15).

Definition 1.43 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*

- *A* random measure *on $\mathbb{R}^d$ is a measurable map $M : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbf{M}, \mathcal{M})$.*
- *A* point process *in $\mathbb{R}^d$ is a measurable map $\Phi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbf{N}, \mathcal{N})$.*

Thus, point processes are particular random measures, and they may be written in the form:

$$\Phi = \sum_{i=1}^{+\infty} \varepsilon_{X_i},$$

(where ε is the Dirac measure, $X_i : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ is a random vector, $i \in \mathbb{N}$), or, explicitly:

$$\Phi(A, \omega) = \sum_{i=1}^{+\infty} \varepsilon_{X_i(\omega)}(A) \qquad \forall A \in \mathcal{B}_{\mathbb{R}^d}.$$

Note that
- for any fixed $A \in \mathcal{B}_{\mathbb{R}^d}$, $\Phi(A, \cdot)$ is a discrete random variable;
- for any fixed $\omega \in \Omega$, $\Phi(\cdot, \omega)$ is a measure on $\mathcal{B}_{\mathbb{R}^d}$.

Notation: in the following we will write $\Phi(A)$ instead of $\Phi(A, \omega)$, and we will consider only simple processes.

Point processes as random closed sets

Let S be the set of all sequences $\varphi = \{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that:

(i) φ is *locally finite*, i.e. any bounded subset of $\mathbb{R}^d$ contains finitely many points of φ;

(ii) φ is *simple*, i.e. $x_i \neq x_j \ \forall i \neq j$.

Denote by $\mathcal{S}$ the smallest σ-algebra containing the family

$$\tilde{S} := \{S_{B,s} \mid B \in \mathcal{B}_{\mathbb{R}^d}, s \in \{1, 2, ...\}\}$$

where $S_{B,s} := \{\varphi \in S \mid card(\varphi \cap B) = s\}$ ($card$ stands for cardinality).

Definition 1.44 *A* point process *on* $\mathbb{R}^d$ *is a measurable map*

$$\Phi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (S, \mathcal{S}).$$

Thus, a point process can be seen as a random sequence of point in $\mathbb{R}^d$, and so as a random closed set of null dimension.

Equivalence of the two definitions

There exists an one-to-one correspondence between $\mathbf{N}$ and S, which allows to identify counting measures and sequences of point in $\mathbb{R}^d$.
The bijection is given by the function $card$ as follows:

$$card(\varphi \cap A) = card(\Phi(A, \omega)) = \mu(A)$$

where $\varphi \in S$ and $\mu \in \mathbf{N}$ are the realizations in ω of Φ, as sequence of points in $\mathbb{R}^d$, and as counting measure, respectively.
It will be clear by the context when Φ is regarded as counting measure or random closed set.

1.5.2 Distribution of a point process

As random measure, a point process $\Phi : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbf{N}, \mathcal{N})$ induces a distribution $\tilde{P}$ on $(\mathbf{N}, \mathcal{N})$ such that

$$\tilde{P}(Y) = \mathbb{P}(\Phi \in Y) = \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \in Y\}), \qquad Y \in \mathcal{N}.$$

As random closed set, a point process $\Phi : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ induces a distribution P on $(S, \mathcal{S})$ such that

$$P(Y) = \mathbb{P}(\Phi \in Y) = \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \in Y\}), \qquad Y \in \mathcal{S}.$$

The one-to-one correspondence between $\mathbf{N}$ and S puts in the same relation $\tilde{P}$ and P.
It can be shown [28] that the distribution of a random measure ξ is completely determined by its *finite-dimensional distributions*, i.e. by the joint distributions, for all finite families of bounded Borel sets $B_1, \ldots, B_k$, of the random variables

$\xi(B_1), \ldots, \xi(B_k)$.
For a point process Φ they are the probabilities of type

$$\mathbb{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k),$$

where $n_1, \ldots, n_k$ are nonnegative integer.

A particular case of finite-dimensional distributions is given by the socalled *void probabilities*

$$\begin{aligned} v_B &:= P(\{\varphi \in S : card(\varphi \cap B) = 0\}) \\ &= \mathbb{P}(\{\omega \in \Omega : \Phi(\omega) \cap B = \emptyset\}) \\ &= \mathbb{P}(\Phi(B) = 0) \end{aligned}$$

with $B \in \mathcal{B}_{\mathbb{R}^d}$.

Remark 1.45 If a point process is simple, then its distribution is completely specifies by the values v_K, with K running through the class $\mathcal{K}$ of compacts in $\mathbb{R}^d$. Such a characterization may be obtained by regarding Φ as a random closed set in $\mathbb{R}^d$ and then applying the Choquet Theorem.

Since a point process is a particular random closed set, definitions of *stationary* and *isotropic* point processes follow immediately by Definitions 1.39 and 1.40.

1.5.3 Moment measures

While moments of a random variable are real numbers, moments of a point process are measures.

Definition 1.46 *The* n-moment measure *of a point process* Φ *is the measure* $\mu^{(n)}$ *on* $\mathcal{B}_{\mathbb{R}^{nd}}$ *so defined:*

$$\begin{aligned} \int_{\mathbb{R}^{nd}} f(x_1, \ldots, x_n) \mu^{(n)}(\mathrm{d}(x_1, \ldots, x_n)) &= \int_S \sum_{x_1, \ldots, x_n \in \varphi} f(x_1, \ldots, x_n) P(\mathrm{d}\varphi) \\ &= \mathbb{E}\left(\sum_{x_1, \ldots, x_n \in \Phi} f(x_1, \ldots, x_n) \right) \end{aligned}$$

where f *is a nonnegative measurable function on* $\mathbb{R}^{nd}$.

Remark 1.47 If f is the characteristic function of the set $B_1 \times \cdots \times B_n$, where $B_1, \ldots, B_n$ are Borel subsets of $\mathbb{R}^d$, we have that

$$\begin{aligned} \mu^{(n)}(B_1 \times \cdots \times B_n) &= \mathbb{E}\left(\sum_{x_1, \ldots, x_n \in \Phi} \mathbf{1}_{B_1}(x_1) \cdots \mathbf{1}_{B_n}(x_n) \right) \\ &= \mathbb{E}(\Phi(B_1) \cdots \Phi(B_n)); \end{aligned}$$

if in particular $B_1 = \ldots = B_n = B$, then

$$\mu^{(n)}(B^n) = \mathbb{E}(\Phi(B)^n).$$

Therefore $\mu^{(n)}$ allows to obtain the n-th moment of the random variable $\Phi(B)$. Moreover, if Φ is stationary, then the moment measures are translation invariant, i.e.

$$\mu^{(n)}(B_1 \times \cdots \times B_n) = \mu^{(n)}((B_1 + x) \times \cdots \times (B_n + x)) \quad \forall x \in \mathbb{R}^d.$$

An important particular case is given by $n = 1$.

Definition 1.48 *The first moment measure of a point process Φ is called* intensity measure.

Denoted by Λ the intensity measure of Φ, by definition it is the measure on $\mathcal{B}_{\mathbb{R}^d}$ so defined:

$$\Lambda(B) := \mathbb{E}(\Phi(B)) = \int \varphi(B) P(\mathrm{d}\varphi).$$

Thus, $\Lambda(B)$ is the mean number of points of Φ in B.
Further, in accordance with Definition 1.46, for any nonnegative measurable function f on $\mathbb{R}^d$, by Fubini's theorem, it follows that

$$\begin{aligned} \mathbb{E}\left(\int f \mathrm{d}\Phi\right) &= \mathbb{E}\left(\sum_{x\in\Phi} f(x)\right) \\ &= \int \sum_{x\in\varphi} f(x) P(\mathrm{d}\varphi) \\ &= \int\int f(x)\varphi(\mathrm{d}x) P(\mathrm{d}\varphi) = \int f(x)\Lambda(\mathrm{d}x). \end{aligned}$$

It is clear that, if Φ is stationary, then Λ is translation invariant. In such a case we known that there exists a positive constant λ, said *intensity* of the point process Φ, so that $\Lambda = \lambda \nu_d$.

1.5.4 Particular examples of point processes

Point processes on the real line

A point process Φ on $\mathbb{R}$ may be taken to model a process of *arrival times* t_i (times in which certain events occur); so, in general, Φ is supposed to assume only nonnegative values.
Such a process can be described in four equivalent ways. The first one is common to every point process, as we have seen in the previous sections; the others follows by the ordering property of $\mathbb{R}$:

i) As *counting measure.*

$$\Phi(A) = \#\{i : t_i \in A\} \qquad A \in \mathcal{B}_{\mathbb{R}}.$$

ii) As *integer-valued increasing step-function.*

This description follows by regarding Φ as function on $\mathbb{R}_+$, rather than as set-function $\Phi(A)$. So we may define

$$\Phi(t) := \Phi([0, t]) \tag{1.16}$$

with $\Phi(0) := 0$. Note that $\Phi(t)$ is a step-function, with unit jumps, and Φ is also called *counting process*.
If the process assumes only positive values, then it is natural to interpret points t_i as arrival times. Otherwise we may extend (1.16) in the following way:

$$\Phi(t) = \begin{cases} \Phi((0,t]) & (t > 0) \\ 0 & (t = 0) \\ -\Phi((t,0]) & (t < 0) \end{cases}$$

$\Phi(t)$ is again a right-continuous and integer-valued function.
Further, $\Phi(t)$ determines $\Phi(A)$ for any Borel set A, and so it describes the point process by a step function.

iii) As *sequence of points.*
By setting

$$t_i = \inf\{t > 0 : \Phi(t) \geq i\} \qquad (i = 1, 2, \ldots), \tag{1.17}$$

we have the following relation:

$$t_i \leq t \quad \Leftrightarrow \quad \Phi(t) \geq i$$

Thus, a sequence of points $\{t_i\}$ specifies the function $\Phi(t)$, and so the point process Φ. If Φ assumes also negative values, it is possible to extend (1.17) in a similar way as in **ii)**:

$$\begin{aligned} t_i &= \inf\{t : \Phi(t) \geq i\} \\ &= \begin{cases} \inf\{t > 0 : \Phi((0,t]) \geq i\} & (i = 1, 2, \ldots) \\ -\inf\{t > 0 : \Phi((-t,0]) \geq -i+1\} & (i = 0, -1, \ldots) \end{cases} \end{aligned}$$

Note that $t_i \leq t_{i+1} \quad \forall i$.

iv) As *sequence of intervals.*
By setting

$$\tau_i = t_i - t_{i-1},$$

where $\{t_i\}$ is the sequence of points defined in **iii)**, the point process Φ is entirely specified by the sequence of intervals $\{\tau_i\}$.

Remark 1.49 If Φ is stationary with intensity λ, we know that its intensity measure Λ is given by $\lambda\nu^1$. In particular the constant λ can be determined by the following limit

$$\lambda = \lim_{h \downarrow 0} \frac{\mathbb{P}(\Phi(h) > 0)}{h}. \tag{1.18}$$

Note that (1.18) is equivalent to say that

$$\begin{aligned} \mathbb{P}(\Phi(t,t+h] > 0) &= \mathbb{P}(\text{at least an arrival in } (t,t+h]) \\ &= \lambda h + o(h) \quad (h \downarrow 0). \end{aligned}$$

Let $\Phi_t := \Phi([0,t])$ be the counting process associated to a point process Φ on $\mathbb{R}_+$. We said that it denotes the number of events which occur up to time

t. The *available information at time* t are described by a sub-σ-algebra $\mathcal{H}_t$ of $\mathcal{F}$; thus we have an increasing family of σ-algebras $\mathcal{H} = \{\mathcal{H}_t,\, 0 \leq t < \infty\}$, i.e. $\mathcal{H}_s \subseteq \mathcal{H}_t$ for any $s \leq t$. The filtration $\mathcal{H}$ is said *history of the process* if Φ_t is $\mathcal{H}_t$-measurable for any $0 \leq t < \infty$, i.e. $\{\Phi_t\}$ is adapted to $\mathcal{H}$.
The natural filtration $\mathcal{I} = \{\mathcal{I}_t\,,\, 0 \leq t < \infty\}$, where $\mathcal{I}_t$ is the σ-algebra generated by $\{\Phi_s : s \leq t\}$, is said *internal history of the process* and it is the smallest σ-algebra which makes Φ adapted.
We recall that if $\mathcal{H}$ is such that $\mathcal{H}_t = \bigcap_{s>t} \mathcal{H}_s \ \forall s$, then it is said to be *right continuous*.
Note that, since a counting process is right continuous and locally finite, necessarily its internal history is right continuous.

Definition 1.50 *The sub-σ-algebra of $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ generated by the sets*

$$(s,t] \times U \qquad con \quad s < t \quad e \quad U \in \mathcal{H}_s$$

is called predictable σ-algebra *of $\mathcal{H}$.*

Definition 1.51 *A process Ψ is said to be $\mathcal{H}$-*predictable *if it is measurable with respect to the predictable σ-algebra of $\mathcal{H}$, as function on $\mathbb{R}^+ \times \Omega$.*

Notation: $\mathcal{H}_{(-)} = \{\mathcal{H}_{t-}\}$,
where $\mathcal{H}_{0-} = \mathcal{H}_0$ and $\mathcal{H}_{t-} = \limsup_{s<t} \mathcal{H}_s = \bigvee_{s<t} \mathcal{H}_s$.

Theorem 1.52 *[28] A $\mathcal{H}$-predictable process is $\mathcal{H}_{(-)}$-adapted.*

We recall that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathcal{H}$ is a filtration on $(\Omega, \mathcal{F})$ and $\{\Psi_t\}_{0 \leq t < \infty}$ is a process $\mathcal{H}$-adapted, such that $\mathbb{E}(|\Psi_t|) < \infty \quad \forall t$, then Ψ is said *$\mathcal{H}$-martingale* if

$$\mathbb{E}(\Psi_t \,|\, \mathcal{H}_s) = \Psi_s \quad \text{a.s.} \qquad for \quad 0 \leq s < t < \infty$$

(*sub-, super- martingale* if $\geq, \leq$ respectively).

Definition 1.53 *Let Φ be a point process on $\mathbb{R}^+$ and $\mathcal{H} = \{\mathcal{H}_t\}$ be a filtration such that:*

i) $\mathcal{H}$ is complete and right continuous;

ii) $\mathcal{I}_t \subset \mathcal{H}_t \quad \forall t \quad$ (i.e. Φ $\mathcal{H}$-adapted);

iii) $\mathbb{E}(\Phi_t) < \infty \quad \forall t$.

The compensator *of Φ with respect to $\mathcal{H}$is a random measure A on $\mathbb{R}^+$ such that:*

1. *$\{A_t\}$ is $\mathcal{H}$-predictable;*

2. *for any $\mathcal{H}$-predictable process $C \geq 0$*

$$\mathbb{E}\left(\int_0^\infty C \mathrm{d}\Phi\right) = \mathbb{E}\left(\int_0^\infty C \mathrm{d}A\right)$$

(For a proof of existence and uniqueness of A, see [46]).

By hypotheses ii) and iii) of Definition 1.53, and since the process Φ_t is not decreasing, it follows that it is a submartingale with respect to $\mathcal{H}$. By Doob-Meyer decomposition theorem, the compensator A is given by a predictable process such that the process $M_t := \Phi_t - A_t$ is a zero-mean martingale.
As a consequence, we have that, for any $U \in \mathcal{H}_s$,

$$\mathbb{E}(\Phi_t - \Phi_s\,;\, U) = \mathbb{E}(A_t - A_s\,;\, U) \tag{1.19}$$

Since A_t is $\mathcal{H}_{t-}$-measurable, then (1.19) may be so interpreted in infinitesimal terms:

$$\mathrm{d}A_t = \mathbb{E}(\mathrm{d}\Phi_t \,|\, \mathcal{H}_{t-}) \tag{1.20}$$

(where $\mathrm{d}A_t = A_t - A_{t-\mathrm{d}t} = A_t - A_{t-}$, and the infinitesimal increment $\mathrm{d}t$ is positive).

As random measure on $\mathbb{R}^+$, the compensator may be absolutely continuous with respect to the Lebesgue measure. In such a case its density is said *stochastic intensity*:

Definition 1.54 *A positive predictable process $\lambda = \{\lambda_t\}$ such that*

$$A_t = \int_0^t \lambda_s \mathrm{d}s$$

is the compensator of Φ is said stochastic intensity of Φ.

By $\mathrm{d}A_t = \lambda_t \mathrm{d}t$ and (1.20), and remembering that Φ is simple, we have the following interpretation of λ_t as the rate of occurring of a new event:

$$\lambda_t \mathrm{d}t = \mathbb{E}(\mathrm{d}\Phi_t \,|\, \mathcal{H}_{t-}) = \mathbb{P}(\mathrm{d}\Phi_t = 1 \,|\, \mathcal{H}_{t-}) = \mathbb{P}(\mathrm{d}\Phi_t > 0 \,|\, \mathcal{H}_{t-}).$$

Poisson point process

The Poisson point process is the most fundamental example of a point process. It is characterized by a deterministic continuous compensator.

Definition 1.55 (Poisson process) *Let Λ be a Radon measure on $\mathbb{R}^d$. A point process Φ on $\mathbb{R}^d$ is called a* Poisson point process *with intensity measure Λ if*

i) whenever $A_1, \ldots, A_k \in \mathcal{B}_{\mathbb{R}^d}$ are disjoint, the random variables $\Phi(A_1), \ldots, \Phi(A_k)$ are independent;

ii) for each $A \in \mathcal{B}_{\mathbb{R}^d}$ and $k \geq 0$,

$$\mathbb{P}(\Phi(A) = k) = e^{-\Lambda(A)} \frac{\Lambda(A)^k}{k!}. \tag{1.21}$$

(By convention, $\Phi(A) = \infty$ a.s. if $\Lambda(A) = \infty$.)

In particular it follows that

$$\begin{aligned}
\mathbb{P}(\Phi(\mathrm{d}x) = 0) &= 1 - \Lambda(\mathrm{d}x) + o(\Lambda(\mathrm{d}x)) \\
\mathbb{P}(\Phi(\mathrm{d}x) = 1) &= \Lambda(\mathrm{d}x) + o(\Lambda(\mathrm{d}x)) \\
\mathbb{P}(\Phi(\mathrm{d}x) > 1) &= o(\Lambda(\mathrm{d}x)).
\end{aligned}$$

We may notice that, by (1.21), the process Φ is not stationary in general.
If Λ is absolutely continuous with respect to ν^d with density $\lambda \in \mathbb{R}_+$, then Φ is said to be a *homogeneous Poisson point process with intensity* λ; in this case Φ is stationary and isotropic, and λ represents the mean number of points per unit volume.

Definition 1.56 *A random point X is said to be* uniformly distributed *in a compact set $W \subset \mathbb{R}^d$ if, for any Borel set $B \subset W$,*

$$\mathbb{P}(x \in B) = \frac{\nu_d(B)}{\nu_d(W)}.$$

A point process $\Phi_W^{(n)}$ given by n independent points $x_1, \ldots, x_n$ uniformly distributed in W is called binomial point process *with n points.*

It follows that, if $B \subset W$, then $\Phi_W^{(n)}(B)$ is a random variable with Binomial distribution $\mathrm{B}(np(B))$, with $p(B) = \frac{\nu_d(B)}{\nu_d(W)}$.
It is easy to prove that if Φ is a stationary Poisson point process, then its restriction to a compact set W, with the condition that $\Phi(W) = n$, is a binomial point process with n points.

Now let us consider the particular case of a Poisson point process on $\mathbb{R}_+$. We recall that a point process Φ on $\mathbb{R}_+$ is said to have *independent increments* if $\Phi([0,s])$ and $\Phi((s,t])$ are independent for all $s, t \in \mathbb{R}_+$ with $s \leq t$. In particular the following assertions are equivalent:

(i) the point process Φ as independent increments;

(ii) for all $k \in \mathbb{N}$ and mutually disjoint sets $B_1, \ldots, B_k \in \mathcal{B}_{\mathbb{R}_+}$, the random variables $\Phi(B_1), \ldots, \Phi(B_k)$ are independent;

(iii) for all $k \in \mathbb{N}$ and mutually disjoint sets $B_1, \ldots, B_k \in \mathcal{B}_{\mathbb{R}_+}$, the point processes $\Phi_{B_1}, \ldots, \Phi_{B_k}$ are independent.

Thus, by Definition 1.55, it follows that a Poisson point process Φ on $\mathbb{R}_+$ has independent increment.
A continuous intensity measure and the independence of increment characterize a Poisson point process:

Proposition 1.57 *Φ is a Poisson point process with intensity measure Λ if and only if Φ has independent increments and $\Lambda(\{t\}) = 0$ $\forall t \in \mathbb{R}^+$.*

By (1.20) it follows that the compensator of a point process with independent increments is deterministic. In particular the following theorem holds.

Theorem 1.58 *[46] A point process Φ has independent increments if and only if it has a deterministic compensator given by the intensity measure Λ.*

As a corollary of the previous theorem, we have that any stationary point process with independent increment is a Poisson point process, since the intensity measure of a stationary point process is necessarily continuous.
Summarizing, Φ is a Poisson point process if one of the following condition is satisfied:

- Φ has a continuous and deterministic compensator;
- Φ has independent increment and continuous intensity measure ;
- Φ has continuous intensity measure Λ and for any bounded Borel set A the void probabilities of the process are given by

$$\mathbb{P}(\Phi(A) = 0) = e^{-\Lambda(A)}.$$

1.5.5 Marked point processes

Definition 1.59 *A* marked point process (MPP) Φ *on* $\mathbb{R}^d$ *with marks in a complete separable metric space* $\mathbf{K}$ *is a point process on* $\mathbb{R}^d \times \mathbf{K}$ *with the property that the* marginal process *(or* nonmarked process*, or* underlying process*)* $\{\Phi(B \times \mathbf{K}) : B \in \mathcal{B}_{\mathbb{R}^d}\}$ *is a point process.*

Note that, by definition, it follows that a point process on a product space is not a marked point process, in general.

Thus, let $\tilde{\Phi} = \sum_i \varepsilon_{X_i}$ be a point process on $\mathbb{R}^d$; a marked point process with underlying process $\tilde{\Phi}$ is any point process on $\mathbb{R}^d \times \mathbf{K}$

$$\Phi = \sum_i \varepsilon_{(X_i, Z_i)}. \tag{1.22}$$

$\mathbf{K}$ is said *mark space*, while the random element Z_i of $\mathbf{K}$ is the *mark associated to* X_i.

Definition 1.60 *A marked point process $\Phi = \{(X_n; Z_n)\}$ is* stationary *if* $\forall y \in \mathbb{R}^d$ *the translated process* $\Phi_y = \{(X_n + y; Z_n)\}$ *has the same distribution of* Φ.

As a consequence, if a marked point process is stationary, than it is so also the marginal process.

Definition 1.61 (Independent marking) *Let $\tilde{\Phi}$ be a point process on $\mathbb{R}^d$ and let Z_i be independent and identically distributed (IID) random elements of* $\mathbf{K}$*, such that Φ and $\{Z_i\}$ are independent. Then the marked point process Φ as in* (1.22) *is said to be obtained from $\tilde{\Phi}$ by* independent marking.

Note that, in general, a mark Z_i does not only depend on X_i, but also on the history of the process or on other factors.

Definition 1.62 (Position-dependent marking) *A marked point process* $\Phi = \{(X_i, Z_i)\}$ *such that the distribution of* Z_i *depends only on* X_i *is said to be obtained by* independent marking.

It is clear that $\Phi(B \times L)$ represents the number of points which are in $B \in \mathcal{B}_{\mathbb{R}^d}$ with marks in $L \in \mathbf{K}$.
Since a marked point process is a particular process on a product space, Definition 1.48 may be extended in a natural way as follows:

Definition 1.63 *The* intensity measure Λ *of a marked point process* Φ *on* $\mathbb{R}^d$ *with marks in* $\mathbf{K}$ *is the* σ*-finite measure on* $\mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbf{K}}$ *so defined:*

$$\Lambda(B \times L) := \mathbb{E}(\Phi(B \times L))$$

Thus, $\Lambda(B \times L)$ represents the mean number of points of Φ in B with marks in L.

Definition 1.64 (Kernel) *[46] Let* $(\mathbf{X}, \mathcal{X})$ *and* $(\mathbf{Y}, \mathcal{Y})$ *be measurable spaces. A* kernel *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a mapping from* $\mathbf{X} \times \mathcal{Y}$ *into* $[0, \infty]$ *such that*
$K(\,\cdot, A)$ *is measurable for all* $A \in \mathcal{Y}$*,*
$K(x, \cdot\,)$ *is a measure on* $\mathbf{Y}$ *for all* $x \in \mathbf{X}$*.*
It is called stochastic *(*substochastic*) if* $K(x, \mathbf{Y}) = 1$ *(*≤ 1*) for all* $x \in \mathbf{X}$*.*

We remind that two measures μ_1 and μ_2 are *equivalent* if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$.
The following theorem is a generalization of the Radon-Nikodym theorem.

Theorem 1.65 (Disintegration theorem) *[46] Let* $(\mathbf{Y}, \mathcal{B}_{\mathbf{Y}})$ *be a complete separable metric space and let* m *be a* σ*-finite measure on* $\mathbf{X} \times \mathbf{Y}$*. Then there exist a* σ*-finite measure* μ *on* $\mathbf{X}$ *equivalent to* $m(\,\cdot \times \mathbf{Y})$*, and a* σ*-finite kernel* K *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *such that*

$$m(B) = \int\!\!\int \mathbf{1}_B(x, y) K(x, \mathrm{d}y) \mu(\mathrm{d}x) \qquad \forall B \in \mathcal{B}_{\mathbf{X}} \otimes \mathcal{B}_{\mathbf{Y}} \tag{1.23}$$

and

$$\int f(x, y) m(\mathrm{d}(x \times y)) = \int\!\!\int f(x, y) K(x, \mathrm{d}y) \mu(\mathrm{d}x)$$

for any integrable function f*.*
If in particular $m(\,\cdot \times \mathbf{Y})$ *is* σ*-finite, then one can choose* $\mu = m(\,\cdot \times \mathbf{Y})$*. In that case* $K(x, \mathbf{Y}) = 1$ *for* μ*-a.e.* x *and the kernel* K *is uniquely determined by* (1.23) *up to a set of* m*-measure zero.*

Since the intensity measure Λ and the mark space $\mathbf{K}$ satisfy the above theorem, we have that

$$\Lambda(\mathrm{d}(x \times y)) = \tilde{\Lambda}(\mathrm{d}x) Q(x, \mathrm{d}y),$$

where $Q(x, \cdot\,)$ turns to be a probability measure on the mark space, said *mark distribution at a point* x.

Remark 1.66 1. If Q is independent of x, then Λ is a product measure on $\mathbb{R}^d \times \mathbf{K}$; so

$$\Lambda = \tilde{\Lambda} \otimes Q,$$

and Q is called *mark distribution*. (For example when Φ is independent marking, or when it is stationary.)

2. If Φ is stationary, then the underlying process $\tilde{\Phi}$ is stationary as well, and

$$\Lambda = \tilde{\lambda}\nu^d \otimes Q,$$

where $\tilde{\lambda}$ is the intensity of $\tilde{\Phi}$.
Moreover, by stationarity it follows that $\Lambda(\,\cdot \times L)$, $L \in \mathcal{B}_{\mathbf{K}}$ is translational invariant; thus there exists a positive constant λ_L, called *intensity of* Φ *with respect to* L such that $\Lambda(\,\cdot \times L) = \lambda_L \nu^d(\cdot)$.

Let $\Phi = \{(T_n, Z_n\})$ be a marked point process on $\mathbb{R}_+$ with mark space $(\mathbf{K}, \mathcal{B}_{\mathbf{K}})$. Since T_n may represent the occurring time of a certain event, while the mark Z_n the associated information, then Φ may be taken as model for a stochastic system which evolves in time. Thus, for any $t \in \mathbb{R}_+$ and $L \in \mathcal{B}_{\mathbf{K}}$,

$$\Phi_t(L) = \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(T_n)\mathbf{1}_L(Z_n)$$

is the number of events during $[0, t]$, with marks in L.
If we set

$$\mathcal{I}_t := \sigma(\Phi_s(L) \, : \, 0 \leq s \leq t \, , \, L \in \mathcal{B}(\mathbf{K})),$$

then $\mathcal{I} = \{\mathcal{I}_t \, : \, t \geq 0\}$ is a filtration and it represents the *internal history of the process.*

Theorem 1.67 *[41] Let Φ be a marked point process such that the underlying process $\tilde{\Phi}$ satisfies the hypotheses of Definition 1.53.*
Then there exists a random measure A on $\mathbb{R}^+ \times \mathbf{K}$ such that
a) for any $L \in \mathcal{B}_{\mathbf{K}}$, the process $A_t(L) = A([0, t] \times L)$ is predictable;
b) for any nonnegative predictable process C,

$$\mathbb{E}\left(\int_{\mathbb{R}^+ \times \mathbf{K}} C \mathrm{d}\Phi\right) = \mathbb{E}\left(\int_{\mathbb{R}^+ \times \mathbf{K}} C \mathrm{d}A\right)$$

If $\{\mathcal{H}_t\}$ is a filtration representing the history of the process, the $\{\mathcal{H}_t\}$-predictable random measure A is called *compensator of* Φ and, $\forall L \in \mathcal{B}_{\mathbf{K}}$, $M_t(L) := \Phi_t(L) - A_t(L)$ is a zero mean martingale.

We may notice that

$$\mathbb{E}(\Phi(\mathrm{d}t \times \mathrm{d}x)) = \mathbb{E}(A(\mathrm{d}t \times \mathrm{d}x)) = \Lambda(\mathrm{d}t \times \mathrm{d}x).$$

Further, if A and $\tilde{A}$ are the compensator of Φ and $\tilde{\Phi}$, respectively, then $A(\,\cdot \times \mathbf{K}) = \tilde{A}$. It can be shown that A of Φ can be factorized as follows:

$$A(\mathrm{d}t \times \mathrm{d}x) = K(t, \mathrm{d}x)\tilde{A}(\mathrm{d}t), \tag{1.24}$$

where K is a $\{\mathcal{H}_t\}$-predictable stochastic kernel from $\Omega \times \mathbb{R}_+$ to $\mathbf{K}$. In particular we have that

$$K(t, B) = \mathbb{P}(Z_1 \in B | \tilde{\Phi}(\mathrm{d}t) = 1, \mathcal{H}_{t-});$$

i.e. $K(t, \cdot)$ represents the conditional probability distribution of a mark, given the history of the process and given that the associated point is born at time t.

Remark 1.68 If Φ is a position-dependent marking, then K does not depend on $\mathcal{H}_{t-}$, and so K is not *random*. In fact, in this case, it is a kernel from $\mathbb{R}_+$ to $\mathbf{K}$, and

$$\mathbb{P}(Z_n \in \mathrm{d}x | \tilde{\Phi}) = K(T_n, \mathrm{d}x)$$

The marked point process Φ is also called a *position-dependent K-marking* of $\tilde{\Phi}$.

The kernel K is not random also for an independent marking process:

Theorem 1.69 *Let $\Phi = \{(T_n, Z_n)\}$ be an independent marking marked point process on $\mathbb{R}^+$ with internal history $\{\mathcal{I}_t\}$ and let Q the mark distribution Q. If $\tilde{\Phi} = \{T_n\}$ is the underlying process with compensator $\tilde{A}$, then*

$$A(\mathrm{d}t \times \mathrm{d}x) = \tilde{A}(\mathrm{d}t) Q(\mathrm{d}x)$$

is the compensator of Φ with respect to $\mathcal{I}$.

Note that if Φ is stationary, then the marks are independent of $\tilde{\Phi}$, and it easily follows that $Q(\cdot) = \mathbb{E}[K(\cdot)]$.

Poisson marked point process on $\mathbb{R}_+$

Definition 1.70 (Marked Poisson process) *([46], p. 18) Let K be a stochastic kernel form $\mathbb{R}_+$ to $\mathbf{K}$ and Φ a position-dependent K-marking of a Poisson process $\tilde{\Phi} := \Phi(\,\cdot \times \mathbf{K})$ with continuous and locally bounded intensity measure $\tilde{\Lambda}$. Then Φ is called a* marked Poisson process.

The name *marked Poisson process* is due to the fact that

$$\mathbb{P}(\Phi(A) = n) = \frac{\Lambda(A)^n}{n!} e^{\Lambda(A)}, \quad n \in \mathbb{N},\ A \in \mathbb{R}_+ \times \mathbf{K},$$

where

$$\Lambda(\mathrm{d}(t, x)) := K(t, \mathrm{d}x) \tilde{\Lambda}(\mathrm{d}t).$$

Since the kernel K in (1.24) associated to a position-dependent marked point process Φ is deterministic (see Remark 1.68), then it follows that the compensator of Φ is deterministic if and only if the compensator of the underlying process is deterministic. As a consequence, we have that a Poisson marked point process has a deterministic compensator (which coincides with the intensity measure Λ of the process).

Finally, if a Poisson marked point process Φ is stationary with mark distribution Q, then we know that its intensity measure Λ is of the type $\lambda dt Q(dx)$, with $\lambda \in \mathbb{R}_+$. In particular, the following theorem holds:

Theorem 1.71 *[38] Let Φ be a marked point process on $\mathbb{R}_+$ with deterministic compensator $A(\mathrm{d}t \times \mathrm{d}x)$. Then Φ is a stationary marked Poisson process if and only if there exists a σ-finite measure $\mu(dx)$ on the mark space such that $A(\mathrm{d}t \times \mathrm{d}x) = \mathrm{d}t\mu(\mathrm{d}x)$.*

Chapter 2

Densities as linear functionals

In the previous chapter we have introduced the concept of density of a subset of $\mathbb{R}^d$ and of density of a measure, together to the notion of rectifiability of a set, which gives information about the "regularity" of the set. Moreover we have observed that the usual Dirac delta function δ_{X_0} associated to a point $X_0 \in \mathbb{R}^d$ may be seen as a singular generalized function, which, in a certain sense, play the role of the density of the counting measure $\mathcal{H}^0(X_0 \cap \cdot)$, associated to X_0. It is clear that any sufficiently regular subset S of $\mathbb{R}^d$ may induce a Radon measure on $\mathbb{R}^d$, and so it can be defined a linear functional associated to S, acting in a similar way to the Dirac delta of a point X_0.

In this chapter we wish to introduce a *Delta formalism*, á la Dirac, for the description of random measures associated with random closed sets of lower dimensions with respect to the environment space $\mathbb{R}^d$, which will provide a natural framework in several applications, as we will see in the sequel (see [26, 24]).
If Θ_n is almost surely a set of locally finite n-dimensional Hausdorff measure, then it induces a random measure μ_{Θ_n} defined by

$$\mu_{\Theta_n}(A) := \mathcal{H}^n(\Theta_n \cap A), \qquad A \in \mathcal{B}_{\mathbb{R}^d}.$$

It is clear that, if $n < d$ and $\mu_{\Theta_n(\omega)}$ is a Radon measure for almost every $\omega \in \Omega$, then it is singular with respect to the d-dimensional Lebesgue measure ν^d, and so its usual Radon-Nikodym derivative is zero almost everywhere. On the other hand, in dependence of the probability law of Θ_n, the expected measure

$$\mathbb{E}[\mu_{\Theta_n}](A) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \qquad A \in \mathcal{B}_{\mathbb{R}^d}$$

may be either singular or absolutely continuous with respect to ν^d.
As a consequence, it seems to arise the problem of introducing a notion of *generalized Radon-Nikodym derivative* of the measure $\mathbb{E}[\mu_{\Theta_n}]$, which we will denote by $\mathbb{E}[\delta_{\Theta_n}]$, i.e. a generalized function (a continuous linear functional on a suitable test space; see Section 1.3), in a similar way as the usual Dirac

delta δ_{X_0} of a point X_0. That is, formally, we might write $\int_A \mathbb{E}[\delta_{\Theta_n}](x)\mathrm{d}x := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]$. In the case $\mathbb{E}[\mu_{\Theta_n}]$ turns to be absolutely continuous with respect to ν^d, $\mathbb{E}[\delta_{\Theta_n}]$ is the usual Radon-Nikodym derivative.
Further, as we may express the usual Dirac delta function in terms of a suitable approximating sequence of classical functions, we will show that it is possible to approximate $\mathbb{E}[\delta_{\Theta_n}]$ by sequences of classical functions.

2.1 The deterministic case

An important link between Radon measures and linear functionals is given by the Riesz Theorem [29], p. 49:

Theorem 2.1 (Riesz Representation Theorem) *Let $L : C_c(\mathbb{R}^d, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(f) \mid f \in C_c(\mathbb{R}^d, \mathbb{R}^m),\ |f| \leq 1,\ spt(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^d$ ($spt(f)$ denotes the support of f).
Then there exists a Radon measure μ on $\mathbb{R}^d$ and a μ-measurable function $\sigma : \mathbb{R}^d \to \mathbb{R}^m$ such that

(i) $|\sigma(x)| = 1$ *for μ-a.e. x,*

(ii) $L(f) = \int_{\mathbb{R}^d} f \cdot \sigma \, \mathrm{d}\mu$

for all $f \in C_c(\mathbb{R}^d, \mathbb{R}^m)$.

As a consequence, we have that if $L : C_c(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R}$ is a nonnegative linear and continuous functional, then there exists a unique positive Radon measure μ on $\mathbb{R}^d$ such that

$$L(f) = \int_{\mathbb{R}^d} f \, \mathrm{d}\mu \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

Thus, by Riesz Theorem, we may say that the space of the Radon measures on $\mathbb{R}^d$ is the dual of the Banach space $C_c(\mathbb{R}^d, \mathbb{R})$.
Accordingly, we have the following notion of weak convergence of measures:

Definition 2.2 *Let μ, μ_k ($k = 1, 2, \ldots$) be Radon measures on $\mathbb{R}^d$. We say that the measures μ_k* weakly* converge *to the measure μ if*

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu_k = \int_{\mathbb{R}^d} f \, \mathrm{d}\mu \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

We remind now the well known criterion on weak* convergence of measures, which will be useful in the following.

Theorem 2.3 *Let μ, μ_k ($k = 1, 2, \ldots$) be Radon measures on $\mathbb{R}^d$. The following statements are equivalent:*

(i) $\displaystyle\lim_{k\to\infty} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu_k = \int_{\mathbb{R}^d} f \, \mathrm{d}\mu \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$

(ii) $\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$ *for each compact set* $K \subset \mathbb{R}^d$.

(iii) $\mu(U) \le \liminf_{k\to\infty} \mu_k(U)$ *for each open set* $U \subset \mathbb{R}^d$.

(iv) $\lim_{k\to\infty} \mu_k(B) = \mu(B)$ *for each bounded Borel set* $B \subset \mathbb{R}^d$ *with* $\mu(\partial B) = 0$.

We know (Section 1.1) that, given a positive Radon measure μ on $\mathbb{R}^d$, by the Besicovitch derivation theorem we have that the limit

$$\delta_\mu(x) := \lim_{r\to 0} \frac{\mu(B_r(x))}{b_d r^d}$$

exists in $\mathbb{R}$ for ν^d-a.e. $x \in \mathbb{R}^d$, and it is a version of the Radon-Nikodym derivative of $\mu_{\ll}$, while $\mu_\perp$ is the restriction of μ to the ν^d-negligible set $\{x \in \mathbb{R}^d : \delta_\mu(x) = \infty\}$.
According to Riesz Theorem, Radon measures in $\mathbb{R}^d$ can be canonically identified with linear and order preserving functionals on $C_c(\mathbb{R}^d, \mathbb{R})$. The identification is provided by the integral operator, i.e.

$$(\mu, f) = \int_{\mathbb{R}^d} f\, d\mu \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

If $\mu \ll \nu^d$, it admits, as Radon-Nikodym density, a classical function δ_μ defined almost everywhere in $\mathbb{R}^d$, so that

$$(\mu, f) = \int_{\mathbb{R}^d} f(x)\delta_\mu(x)\mathrm{d}x \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R})$$

in the usual sense of Lebesgue integral.
If $\mu \perp \nu^d$, we may speak of a density δ_μ only in the sense of distributions (it is almost everywhere trivial, but it is ∞ on a set of ν^d-measure zero). In this case the symbol

$$\int_{\mathbb{R}^d} f(x)\delta_\mu(x)\mathrm{d}x := (\mu, f)$$

can still be adopted, provided the integral on the left hand side is understood in a generalized sense, and not as a Lebesgue integral.
In either cases, from now on, we will denote by (δ_μ, f) the quantity (μ, f).
Accordingly, a sequence of measures μ_n *weakly* converges* to a Radon measure μ if (δ_{μ_n}, f) converges to (δ_μ, f) for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

In the previous chapter we saw the Dirac delta δ_{X_0} at a point X_0 both as a generalized function (Section 1.3), and as the generalized density of the Dirac measure (Remark 1.5). More precisely, it is the continuous linear functional associated to the Dirac measure ε_{X_0}.
Consider the simple case $X_0 \in \mathbb{R}$.
As generalized function on the usual test space $C_c(\mathbb{R}, \mathbb{R})$, we have

$$(\delta_{X_0}, f) := f(X_0), \qquad f \in C_c(\mathbb{R}, \mathbb{R}); \tag{2.1}$$

as "density" of the measure ε_{X_0}, we have

$$\int_{\mathbb{R}} f(x)\delta_{X_0}(x)\mathrm{d}x := \int_{\mathbb{R}} f(x)\varepsilon_{X_0}(\mathrm{d}x) = f(X_0), \qquad f \in C_c(\mathbb{R}, \mathbb{R}).$$

It is clear that we may define the action of δ_{X_0} for any function $f : \mathbb{R} \to \mathbb{R}$, as in (2.1), but the assumption $f \in C_c(\mathbb{R}, \mathbb{R})$ is necessary in order to guarantee the continuity of δ_{X_0} as functional, and to have δ_{X_0} as limit of a sequence of linear functionals.
For example, let $X_0 = 0$ and consider the sequence of linear functionals $\delta_0^{(m)}$ defined by

$$(\delta_0^{(m)}, f) := \int_{\mathbb{R}} f(x)\varphi_m(x)\mathrm{d}x, \tag{2.2}$$

where

$$\varphi_m(x) := \frac{m}{2}\mathbf{1}_{(-\frac{1}{m},\frac{1}{m})}(x).$$

Note that $\varphi_m(x) \to \delta_0(x)$, and $(\delta_0^{(m)}, f) \to (\delta_0, f)$ for any $f \in C_c(\mathbb{R}, \mathbb{R})$, while, if we choose $g(x) = \mathbf{1}_{[0,b]}(x)$, $b > 0$, then $g \notin C_c(\mathbb{R}, \mathbb{R})$ and we have

$$(\delta_0, g) = 1 \qquad \text{but} \qquad (\delta_0^{(m)}, g) \to \frac{1}{2}.$$

Thus, for this particular function g, we should choose another suitable sequence $\{\varphi_m\}$ (e.g. $\varphi_m(x) := m\mathbf{1}_{(-\frac{1}{m},\frac{1}{m})}(x)$).
Let us notice also that $(\delta_0, g) = \varepsilon_0(A)$, where $A = [0, b] \subset \mathbb{R}$, and $0 \in \partial A$, so that $\varepsilon_0(\partial A) \neq 0$. It easily follows that the sequence of measures $\varepsilon_0^{(m)} := \varphi_m \nu^d$

$$\varepsilon_0^{(m)}(A) := \int_A \varphi_m(x)\mathrm{d}x, \qquad A \in \mathcal{B}_{\mathbb{R}},$$

weakly* converge to the measure ε_0, and so it is clear the correspondence between the weak* convergence of functionals $\delta_0^{(m)}$ to δ_0, and the weak* convergence of measures $\varepsilon_0^{(m)}$ to ε_0.

In order to extend such á la Dirac approach to any other dimension $0 \leq n \leq d$, we are going to consider a class of sufficiently regular random closed sets in $\mathbb{R}^d$ of integer dimension $n \leq d$.

Definition 2.4 (n-regular sets) *Given an integer $n \in [0, d]$, we say that a closed subset S of $\mathbb{R}^d$ is* n-regular*, if it satisfies the following conditions:*

(i) $\mathcal{H}^n(S \cap B_R(0)) < \infty$ *for any* $R > 0$;

(ii) $\lim_{r\to 0} \dfrac{\mathcal{H}^n(S \cap B_r(x))}{b_n r^n} = 1$ *for* $\mathcal{H}^n$*-a.e.* $x \in S$.

Remark 2.5 Note that condition (ii) is related to a characterization of the countable $\mathcal{H}^n$-rectifiability of the set S (see Section 1.2).

Let Θ_n be a n-regular closed subset of $\mathbb{R}^d$. Then it follows that

$$\lim_{r\to 0} \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_n r^n} = \begin{cases} 1 & \mathcal{H}^n\text{-a.e. } x \in \Theta_n, \\ 0 & \forall x \notin \Theta_n. \end{cases} \tag{2.3}$$

In fact, since Θ_n^C is open, $\forall x \notin \Theta_n$ $\exists r_0 > 0$ such that $\forall r \leq r_0$ $B_r(x) \subset \Theta_n^C$, that is $\mathcal{H}^n(\Theta_n \cap B_r(x)) = 0$ for all $r \leq r_0$; thus the limit equals 0, $\forall x \in \Theta_n^C$.

Observe that, for a general set A, problems about "$\mathcal{H}^n$-a.e." and "$\forall$" arise when we consider a point $x \in \partial A$ or singular. For example, if A is a closed square in $\mathbb{R}^2$, for all point x on the edges

$$\lim_{r\to 0} \frac{\mathcal{H}^2(A\cap B_r(x))}{b_2 r^2} = \frac{1}{2},$$

while for each of the four vertices the limit equals $1/4$; notice that in both of cases the set of such points has $\mathcal{H}^2$-measure 0.

As a consequence, for $n < d$, (by assuming $0\cdot\infty = 0$), by (2.3) we also have:

$$\lim_{r\to 0} \frac{\mathcal{H}^n(\Theta_n\cap B_r(x))}{b_d r^d} = \lim_{r\to 0} \frac{\mathcal{H}^n(\Theta_n\cap B_r(x))}{b_n r^n}\frac{b_n r^n}{b_d r^d} = \begin{cases} \infty & \mathcal{H}^n\text{-a.e. } x\in\Theta_n, \\ 0 & \forall x \notin \Theta_n. \end{cases}$$

Note that in the particular case $n = 0$, with $\Theta_0 = X_0$ point in $\mathbb{R}^d$ (X_0 is indeed a 0-regular closed set),

$$\lim_{r\to 0} \frac{\mathcal{H}^0(X_0\cap B_r(x))}{b_d r^d} = \begin{cases} \infty & \text{if } x = X_0, \\ 0 & \text{if } x \neq X_0; \end{cases}$$

Note that, if Θ_n is an n-regular closed set in $\mathbb{R}^d$ with $n < d$, then the Radon measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot\,)$$

is a singular measure with respect to ν^d, and so $(D\mu_{\Theta_n})(x) = 0$ ν^d-a.e. $x\in\mathbb{R}^d$. But, in analogy with the Dirac delta function $\delta_{X_0}(x)$ associated with a point $X_0 \in \mathbb{R}^d$, we may introduce the following definition

Definition 2.6 *We call δ_{Θ_n}, the* generalized density *(or, briefly, the* density*) associated with Θ_n, the quantity*

$$\delta_{\Theta_n}(x) := \lim_{r\to 0} \frac{\mathcal{H}^n(\Theta_n\cap B_r(x))}{b_d r^d}, \tag{2.4}$$

finite or not.

In this way $\delta_{\Theta_n}(x)$ can be considered as the *generalized density* (or the *generalized Radon-Nikodym derivative*) of the measure μ_{Θ_n} with respect to ν_d. Note that, by definition, $\delta_{\Theta_n}(x) = \lim_{r\to 0}\frac{\mu_{\Theta_n}(B_r(x))}{b_d r^d}$.

Remark 2.7 In the case $\Theta_0 = X_0$, $\delta_{X_0}(x)$ coincides with the well known delta function at a point X_0, the (generalized) density of the singular Dirac measure ε_{X_0}. As the well known Dirac delta $\delta_{X_0}(x)$ allows the localization of a mass at point X_0, the *delta function* $\delta_{\Theta_n}(x)$ allows the "localization" of the mass associated to the n-regular closed set Θ_n.

Now we are ready to show that for a n-regular set Θ_n, the density $\delta_{\Theta_n}(x)$ can be seen as a linear functional (or as a generalized function) defined by the measure μ_{Θ_n}, in a similar way as for the classical delta function $\delta_{X_0}(x)$ of a point X_0 described above.

Since, for $n < d$, δ_{Θ_n} takes only the values 0 and ∞, it provides almost no information of practical use on Θ_n, or even on μ_{Θ_n}. This is not the case for some natural approximations at the scale r of δ_{Θ_n}, defined below.

Definition 2.8 *Let $r > 0$ and let Θ_n be n-regular. We set*

$$\delta_{\Theta_n}^{(r)}(x) := \frac{\mu_{\Theta_n}(B_r(x))}{b_d r^d} = \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d}$$

and, correspondingly, the associated measures $\mu_{\Theta_n}^{(r)} = \delta_{\Theta_n}^{(r)} \nu^d$:

$$\mu_{\Theta_n}^{(r)}(B) := \int_B \delta_{\Theta_n}^{(r)}(x) \mathrm{d}x, \qquad B \in \mathcal{B}_{\mathbb{R}^d}.$$

Identifying, as usual, measures with linear functionals on $C_c(\mathbb{R}^d, \mathbb{R})$, according to the functional notation introduced before in terms of the respective densities, we may consider the linear functionals associated with the measures $\mu_{\Theta_n}^{(r)}$ and μ_{Θ_n} as follows:

$$(\delta_{\Theta_n}^{(r)}, f) := \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n}^{(r)}(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x) \delta_{\Theta_n}^{(r)}(x) \mathrm{d}x,$$

$$(\delta_{\Theta_n}, f) := \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n}(\mathrm{d}x), \tag{2.5}$$

for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

We may prove the following result.

Proposition 2.9 *For all $f \in C_c(\mathbb{R}^d, \mathbb{R})$,*

$$\lim_{r \to 0} \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n}^{(r)} \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \mu_{\Theta_n} \mathrm{d}x.$$

Proof. Thanks to the quoted criterion on weak* convergence of measures on metric spaces (Theorem 2.3), we may limit ourselves to prove that for any bounded Borel A of $\mathbb{R}^d$ such that $\mu_{\Theta_n}(\partial A) = 0$, the following holds

$$\lim_{r \to 0} \mu_{\Theta_n}^{(r)}(A) = \mu_{\Theta_n}(A).$$

It is clear that, for any fixed $r > 0$ and for any bounded fixed set A, there exists a compact set K containing A such that $\mathcal{H}^n(\Theta_n \cap B_r(x)) = \mathcal{H}^n(\Theta_n \cap K \cap B_r(x))$ for all $x \in A$. Thus, we have

$$\begin{aligned}
\lim_{r \to 0} \mu_{\Theta_n}^{(r)}(A) &= \lim_{r \to 0} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} \mathrm{d}x \\
&= \lim_{r \to 0} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x)}{b_d r^d} \left(\int_{\Theta_n \cap K} \mathbf{1}_{B_r(x)}(y) \mathcal{H}^n(\mathrm{d}y) \right) \mathrm{d}x \\
&= \lim_{r \to 0} \int_{\mathbb{R}^d} \left(\int_{\Theta_n \cap K} \frac{\mathbf{1}_A(x) \mathbf{1}_{B_r(x)}(y)}{b_d r^d} \mathcal{H}^n(\mathrm{d}y) \right) \mathrm{d}x;
\end{aligned}$$

by exchanging the integrals and using the identity $\mathbf{1}_{B_r(x)}(y) = \mathbf{1}_{B_r(y)}(x)$,

$$\begin{aligned}
&= \lim_{r \to 0} \int_{\Theta_n \cap K} \left(\int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x) \mathbf{1}_{B_r(y)}(x)}{b_d r^d} \mathrm{d}x \right) \mathcal{H}^n(\mathrm{d}y) \\
&= \lim_{r \to 0} \int_{\Theta_n \cap K} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y);
\end{aligned}$$

since $\dfrac{\nu^d(A \cap B_r(y))}{b_d r^d} \leq 1$, and by hypothesis we know that $\mathcal{H}^n(\Theta_n \cap K) < \infty$,

$$= \int_{\Theta_n \cap K} \lim_{r \to 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y);$$

by $\mathcal{H}^n(\Theta_n \cap \partial A) = 0$, and $\lim_{r \to 0} \dfrac{\nu^d(A \cap B_r(y))}{b_d r^d} = 0$ for all $y \in (\mathrm{clos} A)^C$,

$$= \int_{\Theta_n \cap \mathrm{int} A} \lim_{r \to 0} \frac{\nu^d(A \cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y) = \mathcal{H}^n(\Theta_n \cap \mathrm{int} A),$$

since, by the Lebesgue density theorem (Theorem 1.6), $\lim_{r \to 0} \dfrac{\nu^d(A \cap B_r(y))}{b_d r^d} = 1$ for every $y \in \mathrm{int} A$.

So, by the condition $\mathcal{H}^n(\Theta_n \cap \partial A) = 0$, we conclude that

$$\lim_{r \to 0} \mu^{(r)}_{\Theta_n}(A) = \mu_{\Theta_n}(A). \tag{2.6}$$

□

Notice that, in the case $n = d$, δ_{S_d} is a classical function, and we will also use the following well known fact:

$$\delta^{(r)}_{S_d}(x) \to \delta_{S_d}(x) \text{ as } r \to 0 \text{ for } \nu^d\text{-a.e. } x \in \mathbb{R}^d.$$

By the proposition above we may claim that the sequence of measures $\mu^{(r)}_{\Theta_n}$ weakly* converges to the measure μ_{Θ_n}; in other words, the sequence of linear functionals $\delta^{(r)}_{\Theta_n}$ weakly* converges to the linear functional δ_{Θ_n}, i.e.

$$(\delta_{\Theta}, f) = \lim_{r \to 0} (\delta^{(r)}_{\Theta_n}, f) \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}). \tag{2.7}$$

We may like to point out that the role of the sequence $\{\varphi_m(x)\}$ for $n = 0$ in (1.9), is played here, for any $n \in \{0, 1, \ldots, d\}$, by $\left\{ \dfrac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d} \right\}$, by taking $r = 1/m$. We notice that if $n = 0$ and $\Theta_0 = X_0$, then

$$\frac{\mathcal{H}^0(X_0 \cap B_r(x))}{b_d r^d} = \frac{\mathbf{1}_{B_r(X_0)}(x)}{b_d r^d} = \frac{\mathbf{1}_{X_{0 \oplus r}}(x)}{b_d r^d},$$

which is the usual "enlargement" of the point X_0 by Minkowski addition with the closed ball $B_r(0)$ (see definition below); in the case $d = 1$ we have in particular that

$$\delta^{(r)}_{X_0}(x) = \frac{1}{2r} \mathbf{1}_{[X_0 - r, X_0 + r]}(x),$$

in accordance with (1.9).

Definition 2.10 (Minkowski addition) *The* Minkowski addition $A \oplus B$ *of two subsets A and B of $\mathbb{R}^d$ is the set so defined:*

$$A \oplus B := \{a + b \,:\, a \in A,\, b \in B\}.$$

We denote by $A_{\oplus r}$ the Minkowski addition of a set A with the closed ball $B_r(0)$, i.e.

$$A_{\oplus r} = \{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, A) \leq r\}.$$

Remark 2.11 The convergence result shown in Proposition 2.9 can also be understood noticing that $\delta^{(r)}_{\Theta_n}(x)$ is the convolution (e.g. [4]) of the measure μ_{Θ_n} with the kernel

$$\rho_r(y) := \frac{1}{b_d r^d}\mathbf{1}_{B_r(0)}(y).$$

In analogy with the classical Dirac delta, we may regard the continuous linear functional δ_{Θ_n} as a generalized function on the usual test space $C_c(\mathbb{R}^d, \mathbb{R})$, and, in accordance with the usual representation of distributions in the theory of generalized functions, we formally write

$$\int_{\mathbb{R}^d} f(x)\delta_{\Theta_n}(x)\mathrm{d}x := (\delta_{\Theta_n}, f). \tag{2.8}$$

By the weak convergence of $\delta^{(r)}_{\Theta_n}$ to δ_{Θ_n}, if we rewrite (2.7) with the notation in (2.8), we have a formal exchange of limit and integral

$$\lim_{r\to 0}\int_{\mathbb{R}^d} f(x)\frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d}\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\lim_{r\to 0}\frac{\mathcal{H}^n(\Theta_n \cap B_r(x))}{b_d r^d}\mathrm{d}x.$$

Further, we notice that the classical Dirac delta $\delta_{X_0}(x)$ associated to a point X_0 follows now as a particular case.

Remark 2.12 If Θ is a piecewise smooth surface S in $\mathbb{R}^n$ (and so n-regular), then, by the definition in (2.5), it follows that, for any test function f,

$$(\delta_S, f) = \int_S f(x)\ \mathrm{d}S,$$

which is the definition of δ_S in [67] on page 33.

In terms of the above arguments, we may state that $\delta_{\Theta_n}(x)$ is the *(generalized) density* of the measure μ_{Θ_n}, with respect to the usual Lebesgue measure ν^d on $\mathbb{R}^d$ and, formally, we may define

$$\frac{\mathrm{d}\mu_{\Theta_n}}{\mathrm{d}\nu^d}(x) := \delta_{\Theta_n}(x). \tag{2.9}$$

Note that if $n = d$, then μ_{Θ_d} is absolutely continuous with respect to ν^d, and $\frac{\mathrm{d}\mu_{\Theta_d}}{\mathrm{d}\nu^d}(x)$ is the classical Radon-Nikodym derivative.

Finally, according to Riesz Theorem, the linear functional δ_{Θ_n} defines a measure on $\mathbb{R}^d$, and this is a way to construct the Lebesgue measure (see discussion in Example 1.56 in [4]).

2.2 The stochastic case

We consider now random closed sets.

Definition 2.13 (Random n-regular sets) *Given an integer n, with $0 \leq n \leq d$, we say that a random closed set Θ_n in $\mathbb{R}^d$ is* n-regular*, if it satisfies the following conditions:*

(i) for almost all $\omega \in \Omega$, $\Theta_n(\omega)$ is an n-regular closed set in $\mathbb{R}^d$;

(ii) $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_R(0))] < \infty$ *for any* $R > 0$.

(For a discussion of the delicate issue of measurability of the random variables $\mathcal{H}^n(\Theta_n \cap A)$, we refer to [11, 48, 68]).

Suppose now that Θ_n is a random n-regular closed set in $\mathbb{R}^d$. By condition (ii) the random measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot\,)$$

is almost surely a Radon measure, and we may consider the corresponding *expected measure*

$$\mathbb{E}[\mu_{\Theta_n}](\,\cdot\,) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot\,)]. \tag{2.10}$$

In this case $\delta_{\Theta_n}(x)$ is a random quantity, and δ_{Θ_n} is a *random linear functional* in the following sense:

Definition 2.14 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $L(\omega)$ be a linear functional on a suitable test space $\mathcal{S}$ for any $\omega \in \Omega$.*
We say that L is a random linear functional *on $\mathcal{S}$ if and only if (L, s) is a real random variable $\forall s \in \mathcal{S}$; i.e.*

$$\forall s \in \mathcal{S}, \ \forall\, V \in \mathcal{B}_{\mathbb{R}} \qquad \{\omega \in \Omega \ : \ (L(\omega), s) \in V\} \subset \mathcal{F}.$$

Remark 2.15 The definition above is the analogous of the well known definition for Banach valued random variable. We recall the following definitions (see, e.g., [7, 15]):

i) Let B be a separable Banach space with norm $\|\cdot\|$ and Borel σ-algebra $\mathcal{B}$. A *B-valued random variable* defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F} - \mathcal{B}$ measurable function.

   It can be proved that $X : (\Omega, \mathcal{F}) \to (B, \mathcal{B})$ is a B-valued random variable if and only if $x^*(X)$ is a real random variable for every $x^* \in B^*$.

ii) A B-valued random variable X is *weakly integrable* if $x^*(X)$ is integrable for all $x^* \in B^*$ and if there exists an element of B, denoted by $\mathbb{E}[X]$, such that

   $$\mathbb{E}[x^*(X)] = x^*(\mathbb{E}[X]) \qquad \forall x^* \in B^*.$$

   $\mathbb{E}[X]$ is called *weak expectation* of X, or *expected value in the sense of Pettis (or Gelfand-Pettis).*

   It can be proved that X is integrable if and only if X has a Bochner expected value. (The interested reader may refer to [7]).

Note that, if we consider the application

$$L : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{S}^*, \mathcal{B}_{\mathcal{S}^*}),$$

where $\mathcal{S}^*$ is the dual of $\mathcal{S}$ (i.e. for any $\omega \in \Omega$, $L(\omega)$ is a continuous linear functional on a space $\mathcal{S}$), in general $\mathcal{S}^*$ is not a separable Banach space and so we are not in the same situation described in Remark 2.15; even if $\mathcal{S}^*$ is a separable Banach space, we should be able to define $s^{**}(L)$ for any $s^{**} \in \mathcal{S}^{**}$. In general $\mathcal{S}$ is not reflexive and we are directly interested in the action of L on the elements of $\mathcal{S}$; so it is clear that it is more natural to define a random linear functional L as in Definition 2.14, and we consider L "as" a B-valued random variable. (Note that in the case $\mathcal{S}^*$ is a separable Banach space and $\mathcal{S}$ is reflexive, the two definitions coincide).

Now, if L is a random linear functional on $\mathcal{S}$, then it makes sense to compute the expected value of the random variable (L, s) for any $s \in \mathcal{S}$:

$$\mathbb{E}[(L, s)] = \int_\Omega (L(\omega), s) \, \mathrm{d}\mathbb{P}(\omega).$$

If for any $s \in \mathcal{S}$ the random variable (L, s) is integrable, then the map

$$s \in \mathcal{S} \longmapsto \mathbb{E}[(L, s)] \in \mathbb{R}$$

is well defined.
Hence, by extending the definition of expected value of a random operator à la Pettis (or Gelfand-Pettis, [7, 14, 15]), we may define the *expected linear functional* associated with L as follows (see [48]).

Definition 2.16 *Let L be a random linear functional L on $\mathcal{S}$.*
If for any $s \in \mathcal{S}$ the random variable (L, s) is integrable, then we define the expected linear functional *of L as the linear functional $\mathbb{E}[L]$ such that*

$$(\mathbb{E}[L], s) = \mathbb{E}[(L, s)] \qquad \forall s \in \mathcal{S};$$

i.e.

$$\begin{array}{rcl} \mathbb{E}[L] \; : \; \mathcal{S} & \longrightarrow & \mathbb{R} \\ s & \longmapsto & (\mathbb{E}[L], s) := \displaystyle\int_\Omega (L(\omega), s) \, \mathrm{d}\mathbb{P}(\omega). \end{array}$$

Note that $\mathbb{E}[L]$ is well defined since $(\mathbb{E}[L], s) < \infty$ for all $s \in \mathcal{S}$, and if $s = r$, then $(\mathbb{E}[L], s) = (\mathbb{E}[L], r)$. Besides, it easy to check the linearity of $\mathbb{E}[L]$:

$$(\mathbb{E}[L], \alpha s + \beta r) = \alpha(\mathbb{E}[L], s) + \beta(\mathbb{E}[L], r).$$

for any $\alpha, \beta \in \mathbb{R}$ and $s, r \in \mathcal{S}$.

Remark 2.17 A well known example of linear functional on $\mathbb{R}^n$ is the scalar product by a fixed vector.
So, let us consider a random vector $L = (L_1, \ldots, L_n) \in \mathbb{R}^n$; then L defines a random linear functional by:

$$(L, v) := \sum_{i=1}^n L_i v_i, \qquad v \in \mathbb{R}^n.$$

It is well known that $\mathbb{E}[L]$ is defined as the vector with components the expected values $\mathbb{E}[L_i]$ of the components of L. This agrees with our definition of $\mathbb{E}[L]$ as functional; in fact we have that $\mathbb{E}[L]$ is again an element of $\mathbb{R}^n$ and in accordance with Definition 2.16,

$$(\mathbb{E}[L], v) = \sum_{i=1}^{n} \mathbb{E}[L_i]v_i = \mathbb{E}\left[\sum_{i=1}^{n} L_i v_i\right] = \mathbb{E}[(L, v)].$$

Let us now come back to consider the random linear functional δ_{Θ_n}, associated with an n-regular random closed set Θ_n. We have

$$\omega \in (\Omega, \mathcal{F}, \mathbb{P}) \ \longmapsto \ \delta_{\Theta_n}(\omega) \equiv \delta_{\Theta_n(\omega)},$$

and, for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$, (δ_{Θ_n}, f) is an integrable random variable, since certainly an $M \in \mathbb{R}$ exists such that $|f(x)| \leq M$ for any x in the support E of f, and by hypothesis we know that $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap E)] < \infty$.
As before, for the measurability of (δ_{Θ_n}, f) we refer to [11, 68].
Thus, by Definition 2.16, we may define the expected linear functional $\mathbb{E}[\delta_{\Theta_n}]$ on $C_c(\mathbb{R}^d, \mathbb{R})$ by

$$(\mathbb{E}[\delta_{\Theta_n}], f) := \mathbb{E}[(\delta_{\Theta_n}, f)]. \tag{2.11}$$

Remark 2.18 By condition (ii) in Definition 2.13, the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is a Radon measure in $\mathbb{R}^d$; as usual, we may consider the associated linear functional as follows:

$$(\tilde{\delta}_{\Theta_n}, f) := \int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu_{\Theta_n}](\mathrm{d}x), \qquad f \in C_c(\mathbb{R}^d, \mathbb{R}). \tag{2.12}$$

We show that $\mathbb{E}[\delta_{\Theta_n}] = \tilde{\delta}_{\Theta_n}$.

Proposition 2.19 *The linear functionals $\mathbb{E}[\delta_{\Theta_n}]$ and $\tilde{\delta}_{\Theta_n}$ defined in (2.11) and (2.12), respectively, are equivalent.*

Proof. Let us consider a function $f \in C_c(\mathbb{R}^d, \mathbb{R})$. By definition (2.12) we have

$$(\tilde{\delta}_{\Theta_n}, f) := \lim_{k \to \infty} \sum_{j=1}^{k} a_j \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A_j)],$$

where $f_k = \sum_{j=1}^{k} a_j \mathbf{1}_{A_j}$, k=1,2,..., is, as usual, a sequence of simple functions converging to f. (Note that the limit does not depend on the chosen approximating sequence of simple function, and the convergence is uniform.)
For any k,

$$F_k := \sum_{j=1}^{k} a_j \mathcal{H}^n(\Theta_n \cap A_j)$$

is a random variable, and $\lim_{k\to\infty} F_k = (\delta_{\Theta_n}, f)$.
Consider the sequence $\{F_k\}$. We know that an $M \in \mathbb{R}$ exists such that $|f| \leq M$, and so $|a_j| \leq M$ $\forall j$; besides, since A_j is a partition of the support E of f, it follows that $\forall k$

$$F_k \leq M \sum_{j=1}^{k} \mathcal{H}^n(\Theta_n \cap A_j) = M\mathcal{H}^n(\Theta_n \cap E).$$

By hypothesis $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap E)] < \infty$, so that the Dominated Convergence Theorem implies the following chain of equalities:

$$\begin{aligned}
(\tilde{\delta}_{\Theta_n}, f) &= \int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu_{\Theta_n}](\mathrm{d}x) \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} a_j \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A_j)] \\
&= \lim_{k\to\infty} \mathbb{E}\left[\sum_{j=1}^{k} a_j \mathcal{H}^n(\Theta_n \cap A_j)\right] \\
&= \mathbb{E}\left[\lim_{k\to\infty} \sum_{j=1}^{k} a_j \mathcal{H}^n(\Theta_n \cap A_j)\right] \\
&= \mathbb{E}[(\delta_{\Theta_n}, f)] = (\mathbb{E}[\delta_{\Theta_n}], f).
\end{aligned}$$

□

Remark 2.20 Equivalently, by the Riesz duality between continuous functions and Radon measures, the linear functional $\mathbb{E}[\delta_{\Theta_n}]$ corresponds to a Radon measure, say $\tilde{\mu}_n$, which satisfies by (2.11)

$$\mathbb{E}[(\delta_{\Theta_n}, f)] = \int_{\mathbb{R}^d} f(x)\tilde{\mu}_n(\mathrm{d}x) \qquad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}). \tag{2.13}$$

By approximating characteristic functions of bounded open sets by C_c functions, from (2.13), we get

$$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = \mathbb{E}[\mu_{\Theta_n}](A)$$

for any bounded open set A. A simple application of Dynkin's lemma then gives that the identity above holds for all bounded Borel sets A, and provides an alternative possible definition of the expected measure.

As in the deterministic case, we may define the *mean generalized density* $\mathbb{E}[\delta_{\Theta_n}](x)$ of $\mathbb{E}[\mu_{\Theta_n}]$ by the following formal integral representation:

$$\int_A \mathbb{E}[\delta_{\Theta_n}](x)\mathrm{d}x := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)],$$

with

$$\mathbb{E}[\delta_{\Theta_n}](x) := \lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d}.$$

Further, we can provide approximations of mean densities at the scale r, as in the deterministic case.
Let us define

$$\mathbb{E}[\delta^{(r)}_{\Theta_n}](x) := \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d}, \tag{2.14}$$

and denote by $\mathbb{E}[\mu^{(r)}_{\Theta_n}]$ the measure with density the function $\mathbb{E}[\delta^{(r)}_{\Theta_n}](x)$, with respect to the Lebesgue measure ν^d.

Let us introduce the linear functional $\mathbb{E}[\delta^{(r)}_{\Theta_n}]$ associated with the measure $\mathbb{E}[\mu^{(r)}_{\Theta_n}]$, as follows:

$$(\mathbb{E}[\delta^{(r)}_{\Theta_n}], f) := \int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu^{(r)}_{\Theta_n}](\mathrm{d}x), \qquad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

By the same arguments as in the deterministic case, we now show that the measures $\mathbb{E}[\mu^{(r)}_{\Theta_n}]$ weakly* converge to the measure $\mathbb{E}[\mu_{\Theta_n}]$. In fact, the following result, which may be regarded as the stochastic analogue of Proposition 2.9, holds.

Proposition 2.21 *For any bounded Borel set A of $\mathbb{R}^d$ such that $\mathbb{E}[\mu_{\Theta_n}(\partial A)] = 0$ we have*

$$\lim_{r\to 0} \mathbb{E}[\mu^{(r)}_{\Theta_n}(A)] = \mathbb{E}[\mu_{\Theta_n}(A)].$$

Proof. It is clear that, for any fixed $r > 0$ and for any bounded fixed set A, there exists a compact set K containing A such that $\mathcal{H}^n(\Theta_n(\omega)\cap B_r(x)) = \mathcal{H}^n(\Theta_n(\omega)\cap K \cap B_r(x))$ for all $x \in A,\ \omega \in \Omega$; further, the condition $\mathbb{E}[\mu_{\Theta_n}(\partial A)] = 0$ implies

$$\mathbb{P}(\mathcal{H}^n(\Theta_n \cap \partial A) > 0) = 0; \tag{2.15}$$

Thus we have that

$$\begin{aligned}
\lim_{r\to 0} \mathbb{E}[\mu^{(r)}_{\Theta_n}(A)] &= \lim_{r\to 0} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d} \mathrm{d}x \\
&= \lim_{r\to 0} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x)}{b_d r^d} \int_\Omega \int_{\Theta_n(\omega)\cap K} \mathbf{1}_{B_r(x)}(y)\mathcal{H}^n(\mathrm{d}y)\ \mathrm{d}\mathbb{P}(\omega)\ \mathrm{d}x;
\end{aligned}$$

by exchanging the integrals and using the identity $\mathbf{1}_{B_r(x)}(y) = \mathbf{1}_{B_r(y)}(x)$,

$$\begin{aligned}
&= \lim_{r\to 0} \int_\Omega \int_{\Theta_n(\omega)\cap K} \int_{\mathbb{R}^d} \frac{\mathbf{1}_A(x)\mathbf{1}_{B_r(y)}(x)}{b_d r^d} \mathrm{d}x\ \mathcal{H}^n(\mathrm{d}y)\ \mathrm{d}\mathbb{P}(\omega) \\
&= \lim_{r\to 0} \mathbb{E}\left[\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y)\right].
\end{aligned}$$

Note that, by (2.15),

$$\mathbb{E}\left[\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y)\right] = \mathbb{E}\left[\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y) \,|\, \mathcal{H}^n(\Theta_n \cap \partial A) = 0\right],$$

and that

(i) $$\int_{\Theta_n(\omega)\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y) \leq \int_{\Theta_n(\omega)\cap K} \mathcal{H}^n(\mathrm{d}y) = \mathcal{H}^n(\Theta_n(\omega)\cap K);$$

(ii) $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap K)] < \infty$ by hypothesis;

(iii) by (2.6), for any A as in (2.15),

$$\lim_{r\to 0} \int_{\Theta_n(\omega)\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d} \mathcal{H}^n(\mathrm{d}y) = \mathcal{H}^n(\Theta_n(\omega)\cap K\cap A) = \mathcal{H}^n(\Theta_n(\omega)\cap A);$$

thus, by the Dominated Convergence Theorem, we have

$$\begin{aligned}
&\lim_{r\to 0} \mathbb{E}\left[\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d}\mathcal{H}^n(\mathrm{d}y)\right] \\
&= \lim_{r\to 0} \mathbb{E}\left[\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d}\mathcal{H}^n(\mathrm{d}y)\,|\,\mathcal{H}^n(\Theta_n\cap\partial A)=0\right] \\
&= \mathbb{E}\left[\lim_{r\to 0}\int_{\Theta_n\cap K} \frac{\nu^d(A\cap B_r(y))}{b_d r^d}\mathcal{H}^n(\mathrm{d}y)\,|\,\mathcal{H}^n(\Theta_n\cap\partial A)=0\right] \\
&= \mathbb{E}[\mathcal{H}^n(\Theta_n\cap A)].
\end{aligned}$$

□

The quoted criterion on the characterization of weak convergence of sequences of measures (Theorem 2.3, (iv)) implies that the sequence of measures $\mathbb{E}[\mu^{(r)}_{\Theta_n}]$ weakly* converges to the measure $\mathbb{E}[\mu_{\Theta_n}]$, i.e.

$$\lim_{r\to 0}\int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu^{(r)}_{\Theta_n}](\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu_{\Theta_n}](\mathrm{d}x) \qquad \forall f\in C_c(\mathbb{R}^d,\mathbb{R});$$

or, in other words, the sequence of linear functionals $\mathbb{E}[\delta^{(r)}_{\Theta_n}]$ converges weakly* to the linear functional $\mathbb{E}[\delta_{\Theta_n}]$, i.e.

$$(\mathbb{E}[\delta_\Theta], f) = \lim_{r\to 0}(\mathbb{E}[\delta^{(r)}_{\Theta_n}], f) \qquad \forall f\in C_c(\mathbb{R}^d,\mathbb{R}). \tag{2.16}$$

In accordance with the usual representation of distributions in the theory of generalized functions, we formally write

$$\int_{\mathbb{R}^d} f(x)\mathbb{E}[\delta_{\Theta_n}(x)]\mathrm{d}x := (\mathbb{E}[\delta_{\Theta_n}], f).$$

Remark 2.22 In this context, our definition of $\mathbb{E}[\delta_{\Theta_n}]$ given by (2.11) is coherent with the integration of a generalized function with respect to a parameter, known in literature ([40], p. 263):
Let $\varphi(\tau,x)$ be a generalized function of x for all τ in some domain D. If

$$\int_D\left(\int_{\mathbb{R}^d}\varphi(\tau,x)f(x)\mathrm{d}x\right)\mathrm{d}\tau$$

exists for every test function f, then $\int_D\varphi(\tau,x)\mathrm{d}\tau$ is said to exist as a generalized function and is defined by

$$\int_{\mathbb{R}^d} f(x)\left(\int_D\varphi(\tau,x)\mathrm{d}\tau\right)\mathrm{d}x := \int_D\left(\int_{\mathbb{R}^d}\varphi(\tau,x)f(x)\mathrm{d}x\right)\mathrm{d}\tau. \tag{2.17}$$

In our case, $D=\Omega$ and $\varphi(\tau,x)=\delta_{\Theta_n(\omega)}(x)$; thus, formally, we have $\int_D\varphi(\tau,x)\mathrm{d}\tau$ $=\int_\Omega\delta_{\Theta_n(\omega)}(x)\mathrm{d}\mathbb{P}(\omega)=\mathbb{E}[\delta_{\Theta_n}(x)]$, and the definition given by (2.17) is exactly the (2.11).

By using the integral representation of (δ_{Θ_n}, f) and $(\mathbb{E}[\delta_{\Theta_n}], f)$, Eq. (2.11) becomes

$$\int_{\mathbb{R}^d} f(x)\mathbb{E}[\delta_{\Theta_n}](x)\mathrm{d}x = \mathbb{E}\left[\int_{\mathbb{R}^d} f(x)\delta_{\Theta_n}(x)\mathrm{d}x\right]; \tag{2.18}$$

so that, formally, we may exchange integral and expectation.
Further, by (2.16), as for the deterministic case, we have the formal exchange of limit and integral

$$\lim_{r\to 0}\int_{\mathbb{R}^d} f(x)\frac{\mathbb{E}[\mathcal{H}^n(\Theta_n\cap B_r(x))]}{b_d r^d}\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\lim_{r\to 0}\frac{\mathbb{E}[\mathcal{H}^n(\Theta_n\cap B_r(x))]}{b_d r^d}\mathrm{d}x.$$

Remark 2.23 When $n = d$, integral and expectation in (2.18) can be really exchanged by Fubini's theorem, since in this case both μ_{Θ_d} and $\mathbb{E}[\mu_{\Theta_d}]$ are absolutely continuous with respect to ν^d and $\delta_{\Theta_d}(x) = \mathbf{1}_{\Theta_d}(x)$, ν^d-a.s.
In particular $\delta_{\Theta_d}(x) = \mathbf{1}_{\Theta_d}(x)$, ν^d-a.s. implies that

$$\mathbb{E}[\delta_{\Theta_d}](x) = \mathbb{P}(x \in \Theta_d), \quad \nu^d\text{-a.s.},$$

and it is well known the following chain of equalities according with our definition of $\mathbb{E}[\delta_{\Theta_n}]$ ([44], p.46):

$$\begin{aligned}
\mathbb{E}[\nu^d(\Theta_d\cap A)] &= \mathbb{E}\left(\int_{\mathbb{R}^d}\mathbf{1}_{\Theta_d\cap A}(x)\mathrm{d}x\right)\\
&= \mathbb{E}\left(\int_A \mathbf{1}_{\Theta_d}(x)\mathrm{d}x\right)\\
&= \int_A \mathbb{E}(\mathbf{1}_{\Theta_d}(x))\mathrm{d}x\\
&= \int_A \mathbb{P}(x\in\Theta_d)\mathrm{d}x.
\end{aligned}$$

Again, we may formally state that (see (2.9))

$$\frac{\mathrm{d}\mathbb{E}[\mu_{\Theta_n}]}{\mathrm{d}\nu^d}(x) := \mathbb{E}[\delta_{\Theta_n}](x). \tag{2.19}$$

We know that $\mathbb{E}[\mathcal{H}^n(\Theta_n\cap\cdot)]$ is singular with respect to ν^d if and only if its density equals zero almost everywhere, i.e., by our notations, if and only if $\frac{\mathrm{d}\mathbb{E}[\mu_{\Theta_n}]}{\mathrm{d}\nu^d}(x) = 0$ ν^d-a.e. In this case $\mathbb{E}[\delta_{\Theta_n}](x)$ has the same role of a Dirac delta, so, as in the deterministic case, we may interpret $\mathbb{E}[\delta_{\Theta_n}]$ as a generalized function on the usual test space $C_c(\mathbb{R}^d,\mathbb{R})$, the mean Delta function of the random closed set Θ_n, or, in term of the measure $\mathbb{E}[\mu_{\Theta_n}]$, as its generalized density.
On the other hand, if Θ_n is not a pathological set, i.e. $\mathcal{H}^n(\Theta_n)(\omega) > 0$ for $\mathbb{P}$-a.e. $\omega\in\Omega$ ($n < d$), we may notice that, even though for a.e. realization θ_n of Θ_n the measure μ_{θ_n} is positive and singular (and so it is not absolutely continuous), the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ may be absolutely continuous with respect to ν^d. (See Chapter 3.)

Example: Consider the case $n = 0$. Let $\Theta_0 = X_0$ be a random point in $\mathbb{R}^d$; then, in this case, $\mathcal{H}^0(X_0\cap A) = \mathbf{1}_A(X_0)$, and so

$$\mathbb{E}[\mathcal{H}^0(X_0\cap A)] = \mathbb{P}(X_0\in A).$$

If X_0 is a continuous random point with pdf p_{X_0}, then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot\,)]$ is absolutely continuous and, in this case, $\mathbb{E}[\delta_{X_0}](x)$ is just the probability density function $p_{X_0}(x)$, so $\int_A \mathbb{E}[\delta_{X_0}](x)\nu^d(\mathrm{d}x)$ is the usual Lebesgue integral.
Note that we formally have

$$\begin{aligned}\mathbb{E}[\delta_{X_0}](x) &= \int_{\mathbb{R}^d} \delta_y(x) p_{X_0}(y)\nu^d(\mathrm{d}y)\\ &= \int_{\mathbb{R}^d} \delta_x(y) p_{X_0}(y)\nu^d(\mathrm{d}y)\\ &= p_{X_0}(x);\end{aligned}$$

and, in accordance with (2.18),

$$\begin{aligned}\int_A \mathbb{E}[\delta_{X_0}](x)\nu^d(\mathrm{d}x) &= \int_A p_{X_0}(x)\nu^d(\mathrm{d}x)\\ &= \mathbb{P}(X_0 \in A)\\ &= \mathbb{E}[\mathcal{H}^0(X_0 \cap A)]\\ &= \mathbb{E}\left[\int_A \delta_{X_0}(x)\nu^d(\mathrm{d}x)\right].\end{aligned}$$

If instead X_0 is discrete, i.e. $X_0 = x_i$ with probability p_i only for an at most countable set of points $x_i \in \mathbb{R}$, then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)]$ is singular and, as in the previous case, we have that $\mathbb{E}[\delta_{X_0}](x)$ coincides with the probability distribution p_{X_0} of X_0.
In fact, in this case $p_{X_0}(x) = \sum_i p_i \delta_{x_i}(x)$, and by computing the expectation of δ_{X_0}, we formally obtain

$$\mathbb{E}[\delta_{X_0}](x) = \delta_{x_1}(x)p_1 + \delta_{x_2}(x)p_2 + \cdots = \sum_i p_i \delta_{x_i}(x) = p_{X_0}(x).$$

We show this by a simple example. Let X_0 be a random point in $\mathbb{R}$ such that it equals 1 with probability 1/4 and equals 3 with probability 3/4, then

$$\begin{aligned}\mathbb{E}\left[\int_{[0,2]} \delta_{X_0}(x)\mathrm{d}x\right] &= \mathbb{P}(X_0 \in [0,2]) = \frac{1}{4}\\ &= \int_{[0,2]} \left(\frac{1}{4}\delta_1(x) + \frac{3}{4}\delta_3(x)\right)\mathrm{d}x = \int_{[0,2]} \mathbb{E}[\delta_{X_0}(x)]\mathrm{d}x.\end{aligned}$$

Besides, since X_0 is discrete, $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)] \not\ll \nu^d$; in fact, if $A = \{3\}$, then $\nu^d(\{3\}) = 0$, but $\mathbb{E}[\mathcal{H}^0(X_0 \cap \{3\})] = 3/4$.
As a consequence, it is clear that

$$\mathbb{E}[\mathcal{H}^0(X_0 \cap A)] \neq \mathcal{H}^0(\mathbb{E}[X_0] \cap A),$$

or, in an equivalent form, by using the functional notation defined above,

$$\mathbb{E}[(\delta_{X_0}, \mathbf{1}_A)] \neq (\delta_{\mathbb{E}(X_0)}, \mathbf{1}_A);$$

as we may expect by observing that if X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ is a general function not necessarily linear, then $(\delta_X, f) = f(X)$, and it is well known that $\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$.

Remark 2.24 By Remark 2.23 and the considerations on the above example, we may claim that, in the cases $n = d$ and $n = 0$ with X_0 continuous, the expected linear functionals $\mathbb{E}[\delta_{\Theta_d}]$ and $\mathbb{E}[\delta_{X_0}]$ are defined by the function $\rho(x) := \mathbb{P}(x \in \Theta_d)$ and by the pdf p_{X_0} of X_0, respectively, as follows:

$$(\mathbb{E}[\delta_{\Theta_d}], f) := \int_{\mathbb{R}^d} f(x)\rho(x)\mathrm{d}x$$

and

$$(\mathbb{E}[\delta_{X_0}], f) := \int_{\mathbb{R}^d} f(x)p_{X_0}(x)\mathrm{d}x.$$

In fact, let us consider the random point X_0; in accordance with the definition in (2.11):

$$\begin{aligned}
(\mathbb{E}[\delta_{X_0}], f) &:= \int_{\mathbb{R}^d} f(x)p_{X_0}(x)\mathrm{d}x \\
&= \mathbb{E}[f(X_0)] \\
&= \mathbb{E}[(\delta_{X_0}, f)].
\end{aligned}$$

Chapter 3

Discrete, continuous and absolutely continuous random closed sets

Since for our purposes we have to deal with mean generalized densities of random sets, just introduced, and an interesting property of a random set in $\mathbb{R}^d$ is whether its mean generalized density associated is a classical function, or a generalized function (that is, whether the expected measure induced by the random set is absolutely continuous or not with respect to the d-dimensional Lebesgue measure ν^d), a concept of absolute continuity of random closed set in $\mathbb{R}^d$ arises in a natural way, related to the absolute continuity of the associated expected measure. To this end we want to introduce in this chapter definitions of *discrete*, *continuous* and *absolutely continuous* random closed set, coherently with the classical 0-dimensional case, in order to propose an extension of the standard definition of discrete, continuous, and absolutely continuous random variable, respectively.
(This chapter may be seen as a completion of [26, 25].)

3.1 Discrete and continuous random closed sets

For real-valued random variables it is well known the distinction between *discrete, continuous* and *absolutely continuous* random variable, defined in terms of the associated probability measure, with respect to the usual Lebesgue measure, since both are acting on the same Borel sigma algebra on the real line; so, by the well known decomposition theorem of a measure, it is natural the following definition:

Definition 3.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{B}_{\mathbb{R}}$ the Borel σ-algebra on $\mathbb{R}$. A real-valued random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is said to be*

- discrete*, if its probability law $\mathbb{P}_X$ is concentrated on an at most countable subset D of $\mathbb{R}$; i.e. the set of its realizations is discrete (so that its* *cumulative distribution function $F_X : \mathbb{R} \to \mathbb{R}$ is a càdlàg function);*

$$\mathbb{P}_X(\{x\}) = \mathbb{P}(X = x) > 0, \text{for } x \in D, \text{ and } \mathbb{P}_X(D) = 1;$$

- continuous *if $\mathbb{P}_X(\{x\}) = \mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$ (so that F_X is a continuous function);*
- absolutely continuous *if $\mathbb{P}_X$ is absolutely continuous with respect to the usual Lebesgue measure on $\mathbb{R}$ (so that F_X is an absolutely continuous function with respect to such measure).*

As a consequence, we may observe that, whenever a random variable is absolutely continuous, it admits a probability density, while if it is discrete its probability law is singular, with respect to the usual Lebesgue measure.

Remark 3.2 The most general case (also called mixed case) includes probability laws that are the sum of all the three cases above. We shall ignore, for sake of simplicity, the mixed case.

The definition above may be extended to a random vector X in $\mathbb{R}^d$ (i.e. a random point in $\mathbb{R}^d$).
In the case of $\mathbb{R}^d$-valued random variables, the concept of absolute continuity is expressed with respect to the usual Lebesgue measure on $\mathbb{R}^d$. We might give many simple, and still interesting, examples about the distribution of a random point X in $\mathbb{R}^d$; for example, when X is a random point uniformly distributed on a surface $S \subseteq \mathbb{R}^d$, then X is continuous but singular, since its probability distribution is concentrated on an uncountable subset of $\mathbb{R}^d$ with Lebesgue measure zero.

Hence, while the concepts of discrete and continuous random point in $\mathbb{R}^d$ are intrinsic properties of the associated probability law $\mathbb{P}_X$, the definition of absolute continuity requires a reference measure, which in this case is the usual Lebesgue measure on $\mathbb{R}^d$.
Problems arise when we refer to the probability law $\mathbb{P}_\Theta$ of a random closed set Θ in $\mathbb{R}^d$. We remember that a random closed set in $\mathbb{R}^d$ is a measurable map

$$\Theta : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ is the family of closed sets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the sigma algebra generated by the hit-or-miss topology (see Section 1.4).
On $\sigma_{\mathbb{F}}$ we do not have a typical reference measure; so, we first limit our analysis to the concepts of discrete and continuous random closed sets. Later we shall introduce a concept of absolute continuity in terms of measures on $\mathbb{R}^d$, hence with respect to the usual Lebesgue measure ν^d, which is consistent with relevant interpretations from both theoretical and applied point of view.

Definition 3.3 *Let Θ be a random closed set in $\mathbb{R}^d$. We say that Θ is*

- discrete *if its probability law $\mathbb{P}_\Theta$ is concentrated an at most countable subset of $\mathbb{F}$; i.e there exist a family $\theta_1, \theta_2, \dots$ of closed subsets of $\mathbb{R}^d$, and a family of real numbers $p_1, p_2, \dots \in [0, 1]$ such that $\mathbb{P}(\Theta = \theta_i) = p_i$ and $\sum_i p_i = 1$;*

- continuous *if*

$$\mathbb{P}(\Theta = \theta) = 0, \qquad \forall \theta \in \mathbb{F}. \tag{3.1}$$

Note that the definition given is consistent with the case that Θ is a random variable or a random point in $\mathbb{R}^d$. In this case, since the possible realizations of X are points in $\mathbb{R}^d$, then $\mathbb{P}(X = \theta) = 0$ for every subset θ of $\mathbb{R}^d$ which is not a point, and so we say that X is continuous if and only if $\mathbb{P}(X = x) = 0$ for any $x \in \mathbb{R}^d$ (that is the usual definition).

As a simple example of discrete random closed set, consider a random square Θ in $\mathbb{R}^2$ which may assume only four realizations with probabilities p_1, p_2, p_3 and p_4, respectively, as in the figure below.

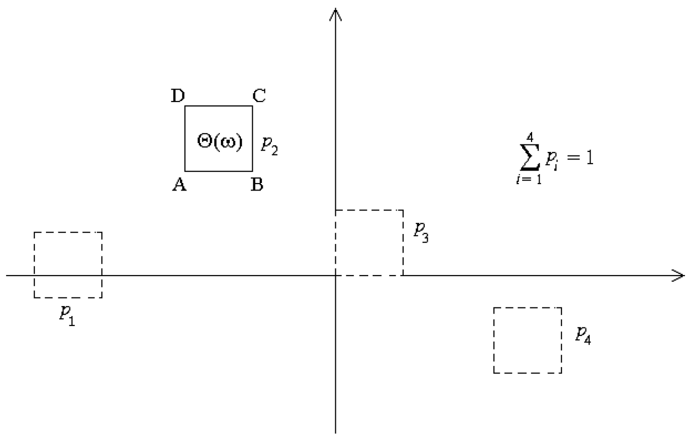

Figure 3.1: An example of discrete random closed set in $\mathbb{R}^2$.

As a simple example of continuous random closed set, consider a random unit square Θ with edges A, B, C and D, as in Fig. 3.2, where $A = (a, 0)$ and a is a real-valued random variable uniformly distributed in $[0, 10]$.

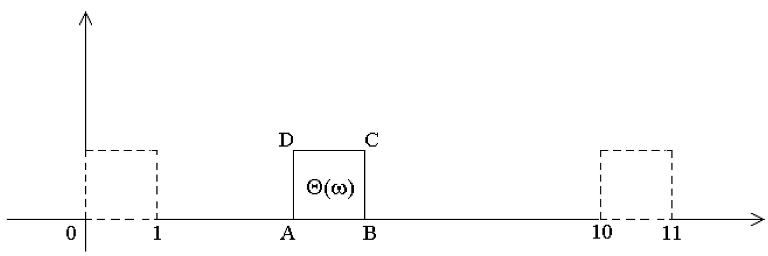

Figure 3.2: An example of continuous random closed set in $\mathbb{R}^2$. Here Θ is a unit square with two edges on the x-axis, and $A = (a, 0)$ with $a \sim U[0, 10]$.

In a large number of cases, an equivalent condition to (3.1) is:

$$\mathbb{P}(\partial\Theta = \partial\theta) = 0, \qquad \forall \theta \in \mathbb{F}. \tag{3.2}$$

(We remind that the map $\Theta \longmapsto \partial\Theta$ is measurable (see [49], p. 46)).
Now we prove that, when Θ is a random closed set, (3.2) is a sufficient condition for the continuity of Θ, and we show under which further condition on Θ, it is equivalent to (3.1).

Lemma 3.4 *Let A and B be two subsets of $\mathbb{R}^d$. Then*

$$A = B \quad \iff \quad \text{int}A = \text{int}B,\ \partial A \cap A = \partial B \cap B. \tag{3.3}$$

Proof. $\Leftarrow$
$A = \text{int}A \cup (\partial A \cap A) = \text{int}B \cup (\partial B \cap B) = B.$

$\Rightarrow$
• By absurd let $\text{int}A \neq \text{int}B$, with $x \in \text{int}A,\ x \notin \text{int}B$. So,

$$x \in \text{int}A,\ x \in (B^C \cup (\partial B \cap B)).$$

We have two possibilities: $x \in B^C$, or $x \in \partial B \cap B$.
If $x \in B^C$, then we have an absurd, since, by hypothesis, $A = B$, and so $x \in A \Rightarrow x \in B$.
If $x \in \partial B \cap B$, then, $\forall r > 0$, $B_r(x) \cap B^C \neq \emptyset$.
Since $x \in \text{int}A$, then $\exists \bar{r} > 0$ such that, $\forall r < \bar{r}$, $B_r(x) \cap A^C = \emptyset$.
It follows that, $\forall r < \bar{r}$, $B_r(x) \cap A^C = \emptyset$ and $B_r(x) \cap B^C \neq \emptyset$; but this is an absurd because, by hypothesis, $A = B$.
• By absurd let $\partial A \cap A \neq \partial B \cap B$, with $x \in \partial A \cap A,\ x \notin \partial B \cap B$. So,

$$x \in \partial A \cap A,\ x \in (B^C \cup \text{int}B).$$

We have two possibilities: $x \in B^C$, or $x \in \text{int}B$.
By proceeding in the same way as above, the thesis follows. □

Proposition 3.5 *Let Θ be a random closed set in $\mathbb{R}^d$. Then*

$$\mathbb{P}(\partial\Theta = \partial A) = 0, \quad \forall A \subset \mathbb{R}^d \implies \mathbb{P}(\Theta = A) = 0, \quad \forall A \subset \mathbb{R}^d.$$

Further, if Θ satisfies one of the following conditions:

1. $\text{int}\Theta = \emptyset$*;*
2. $\text{int}\Theta \neq \emptyset$*, and for a.e. $\omega \in \Omega$, if $\partial A = \partial\Theta(\omega)$, then there exist countably infinitely many closed (distinct) sets $\{A_i\}$ such that $\partial A_i = \partial A (= \partial\Theta(\omega))$;*

then

$$\mathbb{P}(\Theta = A) = 0, \quad \forall A \subset \mathbb{R}^d \implies \mathbb{P}(\partial\Theta = \partial A) = 0, \quad \forall A \subset \mathbb{R}^d.$$

Proof. Let $A \subset \mathbb{R}^d$.
If A is not closed, since by hypothesis $\Theta(\omega)$ is closed for all $\omega \in \Omega$, we have $\mathbb{P}(\Theta = A) = 0$.

Let A be closed. Then $\partial A \cap A = \partial A$, and $\partial\Theta \cap \Theta = \partial\Theta$.
By (3.3) it follows:

$$\begin{aligned}\mathbb{P}(\Theta = A) &= \mathbb{P}(\mathrm{int}\Theta = \mathrm{int}A,\ \partial\Theta \cap \Theta = \partial A \cap A)\\ &= \mathbb{P}(\mathrm{int}\Theta = \mathrm{int}A,\ \partial\Theta = \partial A)\\ &\leq \mathbb{P}(\partial\Theta = \partial A) = 0.\end{aligned}$$

Let $\mathrm{int}\Theta = \emptyset$.
Then, $\Theta = \partial\Theta$. By absurd $\exists A \subset \mathbb{R}^d$ such that $\mathbb{P}(\partial\Theta = \partial A) > 0$. Let $B = \partial A$. Then $\mathbb{P}(\Theta = B) = \mathbb{P}(\partial\Theta = \partial A) > 0$; that is absurd.

Let Θ be a random closed set with $\mathrm{int}\Theta \neq \emptyset$, such that it satisfies the condition 2, and let A be a closed subset of $\mathbb{R}^d$.
Let $A \subset \mathbb{R}^d$ be such that there exists a sequence $A_1, A_2, \ldots, A_n, \ldots$ of closed subsets of $\mathbb{R}^d$ such that $\partial A_i = \partial A\ \forall i$, and $A_i \neq A_j\ \forall i \neq j$. Then, by hypotheses, it follows

$$\mathbb{P}(\partial\Theta = \partial A) = \sum_i \underbrace{\mathbb{P}(\Theta = A_i)}_{=0} = 0.$$

□

In conclusion, by Proposition 3.5, we may observe that:

- If Θ is a random closed set in $\mathbb{R}^d$ with dimension less than d, then (3.1) and (3.2) are equivalent conditions for the continuity of Θ.
- The class of random closed set Θ with $\mathrm{int}\Theta \neq \emptyset$, satisfying condition 2, is very large. Elements of this class are, for example, the Boolean models [64] and all random closed sets Θ such that, for a.e. $\omega \in \Omega$,

  $$\Theta(\omega) = \mathrm{clos}(\mathrm{int}\Theta(\omega)).$$

- In the case that Θ is a random set with $\mathrm{int}\Theta \neq \emptyset$ and such that condition 2 is satisfied, the hypothesis that Θ is closed is necessary. In fact, as a counterexample, let us consider the random set Θ defined by

  $$\Theta(\omega) := \mathrm{int}B_r(0) \cup P(\omega), \quad \omega \in \Omega,\ (r > 0),$$

  where P is a random point uniformly distributed on $\partial B_r(0)$.
  Then it is clear that Θ is not closed, it is continuous, and, for all $\omega \in \Omega$, $\partial\Theta = \partial B_r(0)$. Besides, the only closed sets $A_i \subset \mathbb{R}^d$ such that $\partial A_i = \partial B_r(0)$ are: $B_r(0)$, $\partial B_r(0)$ and $(\mathrm{int}B_r(0))^C$.
  Thus, even if the condition 2 is satisfied, we have $\mathbb{P}(\partial\Theta = \partial B_r(0)) = 1 > 0$.

Finally, we show by a **counterexample** that, in general, if Θ is a random closed set in $\mathbb{R}^d$, then

$$\mathbb{P}(\Theta = A) = 0, \quad \forall A \subset \mathbb{R}^d \ \not\Longrightarrow \mathbb{P}(\partial\Theta = \partial A) = 0, \quad \forall A \subset \mathbb{R}^d.$$

Let us consider a sequence $\{x_i\}_{i\in I}$ of points in $\mathbb{R}$ such that $\bigcup_i x_i = C$, where C is the Cantor set. (Note that I is uncountable).

Let X be a random variable uniformly distributed on C. Then $\mathbb{P}(X = x_i) = 0$ for all $i \in I$.
Let Θ be the random set so defined: if $X(\omega) = x_i$, then

$$\Theta(\omega) := \underbrace{\{x_1\}}_{=\{0\}} \cup \{x_2\} \cup \cdots \cup \{x_{i-1}\} \cup [x_i, x_{i+1}] \cup \{x_{i+2}\} \cup \cdots \cup \{1\}. \tag{3.4}$$

It follows that, for all $\omega \in \Omega$, $\Theta(\omega)$ is closed and $\partial\Theta(\omega) = C$. In particular, since $C = \partial C$, we have that $\mathbb{P}(\partial\Theta = \partial C) = 1 > 0$.
Let A be a subset of $\mathbb{R}$.
If A is not of the type (3.4), then $\mathbb{P}(\Theta = A) = 0$.
If A is of the type (3.4), then $\mathbb{P}(\Theta = A) = \mathbb{P}(X = x_i) = 0$.
Summarizing, for any $A \subset \mathbb{R}$, $\mathbb{P}(\Theta = A) = 0$, but there exists a subset C of $\mathbb{R}$ such that $\mathbb{P}(\partial\Theta = \partial C) > 0$.

3.1.1 A comparison with current literature

In current literature (see [49], p.45 and followings) definitions of continuity are given, but, in many applications, such definitions do not provide a sufficient insight about the structure of the relevant random closed set. Further they do not say whether or not a mean density can be introduced for sets of lower Hausdorff dimensions, with respect to the usual Lebesgue measure on $\mathbb{R}^d$. In particular we remind the following definition:

Definition 3.6 (Matheron)

1. *A random closed set* Θ *in* $\mathbb{R}^d$ *is* $\mathbb{P}$-continuous at a point $x \in \mathbb{R}^d$ *if*

$$\lim_{y\to x} \mathbb{P}(\{x \in \Theta\} \cap \{y \notin \Theta\} = \lim_{y\to x} \mathbb{P}(\{x \notin \Theta\} \cap \{y \in \Theta\} = 0.$$

   Θ *is* $\mathbb{P}$-continuous *if it is* $\mathbb{P}$*-continuous at every* $x \in \mathbb{R}^d$.

2. *A random closed set* Θ *in* $\mathbb{R}^d$ *is* a.s. continuous at $x \in \mathbb{R}^d$ *if*

$$\mathbb{P}(x \in \partial\Theta) = 0, \tag{3.5}$$

   and Θ *is* a.s. continuous *if it is continuous at any* $x \in \mathbb{R}^d$.

Note that "a.s." reflects the fact that the definition is given in terms of the probability law of Θ, and not the fact that the condition (3.5) should be satisfied by a.e. $x \in \mathbb{R}^d$; in fact, in the definition above, Θ is a.s. continuous if and only if (3.5) holds $\forall x \in \mathbb{R}^d$.

Proposition 3.7 *[49]*

i) Let Θ *be a random closed set with associated capacity functional* T_Θ*. Then* Θ *is* $\mathbb{P}$*-continuous at* $x \in \mathbb{R}^d$ *if and only if*

$$x = \lim x_n \text{ in } \mathbb{R}^d \;\Rightarrow\; T_\Theta(\{x\}) = \lim T_\Theta(\{x_n\}) \tag{3.6}$$

ii) a.s. continuity $\Longrightarrow$ $\mathbb{P}$*-continuity.*

Notation: if a random closed set is a.s. continuous in the sense of Matheron, we say that it is *M-continuous*.

The definition of continuity proposed by Matheron concerns the possibility of exchanging the limit in (3.6). This kind of continuity is related to the continuity of the functional T_Θ for any sequence $\{x_n\}$ converging to a point x in $\mathbb{R}^d$.
But we may observe that, if Θ is a M-continuous random closed set, then, in general, its capacity functional T_Θ is not continuous for any sequence $\{K_n\}$ of compact sets converging (in the Hausdorff metric) to a compact set K. For example, consider a random point X uniformly distributed on $K = \partial B_1(0)$. Clearly X is M-continuous, since $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}^d$. Let $K_n = \partial B_{1+\frac{1}{n}}(0)$. Then

$$\lim K_n = K, \quad \text{but} \quad \lim T_X(K_n) \neq T_X(K),$$

since $T_X(K_n) = 0 \ \forall n$, and $T_X(K) = 1$.

With the aim of giving suitable definitions of discrete, continuous and absolutely continuous random closed set, in order to obtain analogous relations, well known for measures (see Remark 1.2), this suggests our definition of continuity of a random closed set. In particular, the main difference between our definition of continuity and the one proposed by Matheron is:
to know that the random set Θ is not continuous by our definition implies that, as in the case of a random variable, it may assume some configuration with probability bigger than 0; to know that the random set Θ is not M-continuous does not give this kind of information.
We show this with an **example**:
Let us consider in $\mathbb{R}^2$ a line through the origin 0 which turns around 0 uniformly, and let Θ be a fixed segment on it with extremes A and B as in Fig. 3.3; i.e. imagine a segment $[A, B]$, with $A, B > 0$ on the x-axis, which turns around the origin uniformly. Note that here $\Theta = \partial\Theta$.

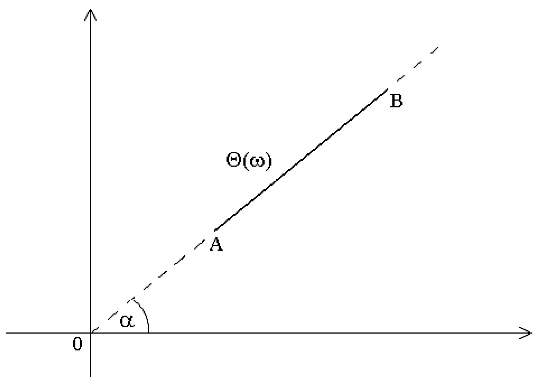

Figure 3.3: A continuous random segment in $\mathbb{R}^2$. Here α is a r.v. with distribution $U[0, 2\pi]$.

Clearly, $\mathbb{P}(\Theta = \theta) = 0$ for any subset θ of $\mathbb{R}^d$, and so we say that it is continuous.

The same holds also in the case that $A = 0$, and so even if $0 \in \Theta(\omega)$ $\forall \omega \in \Omega$ (see Fig. 3.4).

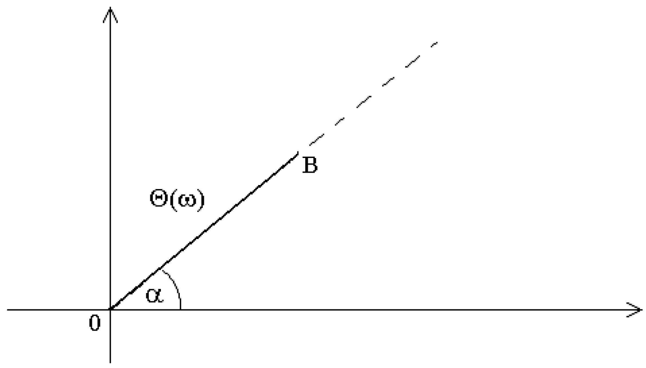

Figure 3.4: A continuous but not M-continuous random segment in $\mathbb{R}^2$. Here α is a r.v. with distribution $U[0, 2\pi]$.

Now, if we want to apply the Definition 3.6, when $A \neq 0$, we have that, $\forall x \in \mathbb{R}^2$, $\mathbb{P}(x \in \partial\Theta) = 0$, and so Θ is considered M-continuous; while if $A = 0$, then $\mathbb{P}(0 \in \partial\Theta) = 1$, and so in this case it is not M-continuous.
In this example the *not continuity* is due to the fixed point $0 \in \Theta$; but we may exhibit a similar example in which the random set Θ has no fixed points, and obtain the same conclusions:
Let $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$ and $D = (0, 1)$ be the edges of a square and let us consider the two random sets Θ_1 and Θ_2 moving the square in a prefixed direction (i.e. for any ω, $\Theta_1(\omega)$ and $\Theta_2(\omega)$ are both a square). Let Θ_1 be the random square with edge A uniformly distributed on a subset of the x-axis as in Fig. 3.2 so that, if $A = (a, 0)$, then a is a random variable with uniform distribution in $[0, 10]$;

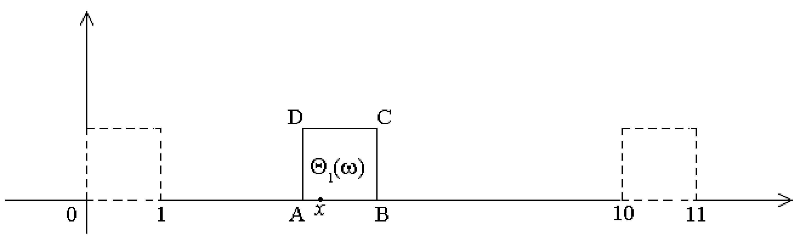

Figure 3.5: An example of continuous but not M-continuous random closed set in $\mathbb{R}^2$. Here Θ_1 is a unit square with two edges on the x-axis, and $A = (a, 0)$ with $a \sim U[0, 10]$.

and let Θ_2 be the same square, but such that A is uniformly distributed on a segment on the line $y = x$, i.e. $A = (a, a)$ (see Fig. 3.6).

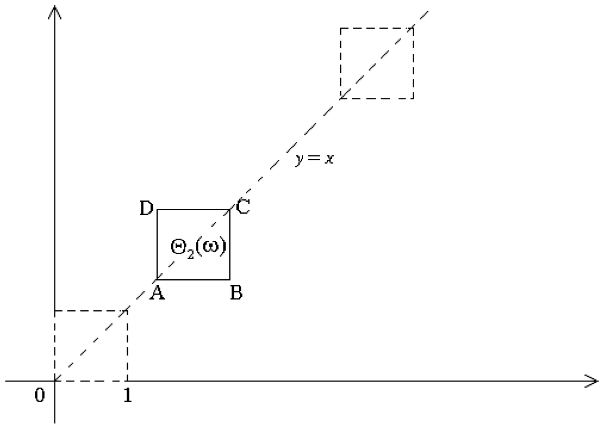

Figure 3.6: An example of random closed set in $\mathbb{R}^2$ which is both continuous and M-continuous. Here Θ_2 is a unit square with two edges on the line $y = x$, and $A = (a, a)$ with $a \sim U[0, 10]$.

By Definition 3.8, we say that Θ_1 and Θ_2 are continuous. Instead, by Matheron's definition, we have that $\mathbb{P}(x \in \partial\Theta_2) = 0$ for all $x \in \mathbb{R}^2$, and so Θ_2 is M-continuous; while for any $x = (s, 0)$, or $x = (s, 1)$, with $s \in [1, 10]$, we have that $\mathbb{P}(x \in \partial\Theta_1) = 1/10$, and so Θ_1 cannot be considered M-continuous.
Thus, although both Θ_1 and Θ_2 move in only one direction, it seems that the reason why Θ_2 is continuous, while Θ_1 is not M-continuous, depends on the specific direction of the movement of the square.

We may notice that, if Θ is M-continuous, since any point $x \in \mathbb{R}^d$ is a compact set with empty interior, it follows that

$$\mathbb{P}(\partial\Theta \cap x \neq \emptyset,\, \partial\Theta \cap \mathrm{int}x = \emptyset) = \mathbb{P}(x \in \partial\Theta) = 0,$$

then any $x \in \mathbb{R}^d$ belongs to the class $\mathcal{S}_T$ of $\partial\Theta$ (see Section 1.4.3). Thus, Matheron's definition of continuity seems to be related to the definition of the class $\mathcal{S}_T$, and so to the problem of the weak convergence of random sets. But note that, even if for all $x \in \mathbb{R}^d$ we have $\mathbb{P}(x \in \partial\Theta) = 0$, it does not imply that $\mathcal{K}^d$ coincides with $\mathcal{S}_T$. Thus, the knowledge that a random set Θ is M-continuous does not determine the class $\mathcal{S}_T$.

3.2 Absolutely continuous random closed sets

As we have previously observed, the concept of absolute continuity requires a reference measure. Because in general on $\sigma_{\mathbb{F}}$ we do not have one, we are considering random sets in $\mathbb{R}^d$ and in a lot of real applications it is of interest to study their expected measures, it is natural to consider as reference measure the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

For any lower dimensional random n-regular closed set Θ_n in $\mathbb{R}^d$, while it is clear that $\mu_{\Theta_n(\omega)}$ is a singular measure, it may well happen that the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is absolutely continuous with respect to ν^d, and so the Radon-Nikodym theorem ensures the existence of a density of this measure with respect to ν^d. So it is natural the following definition.

Definition 3.8 (Absolute continuity in mean) *Let Θ be a random closed set in $\mathbb{R}^d$ such that its associated expected measure $\mathbb{E}[\mu_\Theta]$ is a Radon measure. We say that Θ is* absolutely continuous in mean *if the expected measure $\mathbb{E}[\mu_\Theta]$ is absolutely continuous with respect to ν^d.*

Note that such a definition provides only information on the absolute continuity of the expected measure $\mathbb{E}[\mu_\Theta]$ associated to the random set, but it may not give information on the geometric stochastic properties (e.g. arrangement) of Θ in $\mathbb{R}^d$.
In fact, while it is easy to check that the above definition is consistent with the case in which Θ is a real random variable or a random point in $\mathbb{R}^d$, corresponding to $n = 0$, if Θ has Hausdorff dimension d, then the expected measure $\mathbb{E}[\mathcal{H}^d(\Theta \cap \cdot)]$ is, obviously, always absolutely continuous with respect to the d-dimensional Lebesgue measure ν^d, even if Θ is a deterministic set. So, the absolute continuity of the expected measure $\mathbb{E}[\mathcal{H}^d(\Theta \cap \cdot)]$ with respect to ν^d does not provide information about the continuity of Θ.
However, not only the d-dimensional case give no any kind of information about the continuity of the random set; there exist also some *pathological* sets of lower dimension which may be deterministic or discrete and, in the same time, their expected measure is absolutely continuous. For example, consider a random closed set Θ_n such that, for almost every $\omega \in \Omega$, $\mathcal{H}^n(\Theta_n(\omega)) = 0$; than it follows that $\mathbb{E}[\mathcal{H}^n(\Theta_n)] = 0$, and so it is clear that $\mathbb{E}[\mu_{\Theta_n}] \ll \nu^d$. Therefore, any random set with this property is absolutely continuous in mean, independently of its probability law. For example, consider the case of a 2-dimensional Brownian path in a fixed plane in $\mathbb{R}^3$ (i.e. consider a realization θ_2 of a planar Brownian motion, as deterministic closed set in a plane in $\mathbb{R}^3$); it is known that $\dim_{\mathcal{H}}(\theta_2) = 2$, but $\mathcal{H}^2(\theta_2) = 0$. Thus, there exist deterministic random closed sets which are absolutely continuous in mean.

We have observed that the well known definitions for a random variable X in Definition 3.1 are given in terms of its probability measure, and so, by the quoted relations between measures in Remark 1.2, we may claim that

$$\begin{aligned} X \text{ absolutely continuous} \quad &\Rightarrow \quad X \text{ continuous, but not the reverse;} \qquad (3.7)\\ X \text{ discrete} \quad &\Rightarrow \quad X \text{ singular, but not the reverse.} \end{aligned}$$

Since a random variable is a particular random closed set of Hausdorff dimension 0, we wish to introduce now a stronger definition of absolute continuity for random closed sets, in order to obtain analogous relations for random sets with Hausdorff dimension bigger than 0, so that (3.7) follows as particular case.

In Section 3.1 we have proposed a definition of discrete and continuous random set in terms of its probability law, and we have observed, also thanks to examples, how such definitions take into account the "random arrangement" of the set in the space $\mathbb{R}^d$. As it emerges from Proposition 3.5 and from the

examples given, the random variability of a random closed set is related to the random variability of its boundary. This is the reason why we are going to introduce a stronger definition of absolute continuity of a random closed set in terms of the expected measure induced by its boundary.
Further, since we want that this definition is coherent with the relation "absolute continuity implies continuity", and since we wish to propose a concept of absolute continuity such that it may contain some relevant information on the random set, it is clear that we have to exclude "pathological" sets like the 2-dimensional Brownian path discussed above. But this will be not a particular restriction, because these kind of sets (i.e. sets with null Hausdorff measure) are not interesting in many real applications. In fact, which kind of information could we have on a random set Θ_n by the knowledge of its expected measure $\mathbb{E}[\mu_{\Theta_n}]$, if $\mathbb{E}[\mathcal{H}^n(\Theta_n)] = 0$? Thus, we introduce now a class of random sets, which contains all the random sets which appear in most of the real applications we will consider in the following.

Definition 3.9 ($\mathcal{R}$ class) *We say that a random closed set* Θ *in* $\mathbb{R}^d$ belongs to the class $\mathcal{R}$ *if*

$$\dim_{\mathcal{H}}(\partial\Theta) < d \quad \textit{and} \quad \mathbb{P}(\mathcal{H}^{\dim_{\mathcal{H}}(\partial\Theta)}(\partial\Theta) > 0) = 1.$$

Definition 3.10 (Strong absolute continuity) *We say that a random closed set* Θ *is* (strong) absolutely continuous *if* $\Theta \in \mathcal{R}$ *and*

$$\mathbb{E}[\mu_{\partial\Theta}] \ll \nu^d \tag{3.8}$$

on $\mathcal{B}_{\mathbb{R}^d}$.

Notation: Without any other specification, in the following we will write "absolutely continuous random set" to mean a "strongly absolutely continuous random set".

Remark 3.11 If Θ is a random closed set in $\mathcal{R}$ such that $\dim_{\mathcal{H}}(\Theta) = s < d$, then $\partial\Theta = \Theta$; therefore $\mathbb{E}[\mu_\Theta] = \mathbb{E}[\mu_{\partial\Theta}]$ and, by definition, it follows that there is no distinction between absolute continuity *strong* and *in mean*.

On the other hand, any random closed sets in $\mathcal{R}$ with Hausdorff dimension d is absolutely continuous in mean, but in general it is not absolutely continuous in the strong sense (for instance all discrete random closed sets in $\mathcal{R}$ with Hausdorff dimension d).
Note that, if $\Theta \in \mathcal{R}$ with $\dim_{\mathcal{H}}(\Theta) = d$ is sufficiently regular so that $\dim_{\mathcal{H}}(\partial\Theta) = d - 1$, then it is absolutely continuous if $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \cdot)] \ll \nu^d(\cdot)$.

Remark 3.12 In the particular case that $\Theta = X$ is a random variable, the Definition 3.10 coincides with the usual definition of absolute continuity of a random variable. In fact, $\dim_{\mathcal{H}} X = 0$, $\partial X = X$, and $\mathbb{E}[\mathcal{H}^0(X)] = \mathbb{P}(X \in \mathbb{R}^d) = 1$, so $X \in \mathcal{R}$ and then the condition (3.8) is equivalent to

$$\mathbb{E}[\mathcal{H}^0(X \cap \cdot)] = \mathbb{P}(X \in \cdot) \ll \nu^d.$$

Note that, if Θ_1 and Θ_2 are random squares as in the example in Section 3.1, by Definition 3.10, it follows that Θ_2 is absolutely continuous, while Θ_1 is not. In fact, we may notice that the expected measure given by $\mathbb{E}[\mathcal{H}^1(\partial\Theta_1 \cap \cdot)]$ is not absolutely continuous with respect to ν^2: if, for example, A is the segment $[0, 11]$ on the x-axis, then we have that $\nu^2(A) = 0$, while $\mathbb{E}[\mathcal{H}^1(\partial\Theta_1 \cap A)] = 1$. We have just observed that also by Matheron's definition of a.s. continuity of random sets it follows that Θ_2 is a.s. continuous, while Θ_1 is not so. Since that definition gives a condition on the law of the boundary, one might wonder whether Matheron's definition of continuity is equivalent to our notion of absolute continuity. Really, it is not so; in fact, in the particular case when $\Theta = X$ is a M-continuous random variable, then we have that $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$, but this does not imply that X is absolutely continuous in the usual sense. So it does not say whether or not a mean density can be introduced for sets of lower Hausdorff dimension, with respect to the usual Lebesgue measure on $\mathbb{R}^d$.

At this point, since in the previous section we have noticed that the points $x \in \mathbb{R}^d$ which satisfy $\mathbb{P}(x \in \partial\Theta) = 0$ belong to the class $\mathcal{S}_T$, one might wonder if a good definition of absolutely continuity could be the request that the class $\mathcal{S}_T$ coincides with the whole family $\mathcal{K}^d$ of compacts in $\mathbb{R}^d$. But this is not the case; for example, let us consider

$$\Theta(\omega) = B_r(X(\omega)) \subset \mathbb{R}^2, \qquad \omega \in \Omega,\ r \in \mathbb{R}_+,$$

where X is a random point in $\mathbb{R}^2$ with uniform distribution in some compact set W. Surely, there exists a line l such that $\mathbb{P}(\partial\Theta \cap l \neq \emptyset) > 0$. Since l is a compact set in $\mathbb{R}^2$ with empty interior, it follows that $\mathcal{S}_T \neq \mathcal{K}^d$. Note that, even if we consider only the compact sets $K \in \mathcal{K}^d$ with $\mathrm{int}K \neq \emptyset$, again, we have not a good definition of absolute continuity, since we do not reobtain the particular case of a random variable (e.g. consider a random variable X in $\mathbb{R}$ with distribution concentrated on the Cantor set).

There are situations in which the absolute continuity of the random set can be related to some relevant parameters characterizing the set. For example, consider as random closed set Θ the family of balls in $\mathbb{R}^d$ with fixed radius and random centre given by a spatial point process; so, in particular, Θ may be a Boolean model. Note that, if the intensity measure of the point process is discrete, than Θ turns to be discrete, while if the intensity measure is diffuse, then Θ is an absolutely continuous random set.

It is interesting to notice that also the geometry of the random set plays a crucial role in the correspondence between the absolute continuity of the intensity measure and the absolute continuity of the resulting random closed set Θ as in the above example. In other words, we might say that Θ is discrete if and only if the intensity measure of the process of the centers is discrete since each ball has fixed radius. Thus, in this case, the knowledge of the geometry of the process let us study the random measure $\mathcal{H}^{d-1}(\partial\Theta \cap \cdot)$, or related quantities, in terms of the stochastic intensity of the point process.

Note that Θ can be seen as a particular marked point process in $\mathbb{R}^d$, where the underlying point process is given by the Poisson point process, and the marks by the length of the radius (see e.g. [37]). Now we present some other examples in order to better clarify the role of the two main parameters of the marked point process determining the random closed set Θ; the intensity measure of the process giving the random spatial position of the centers, and the distribution law of the marks (the "objects" associated with each centre). Let Θ be the random closed set given by a marked point process in $\mathbb{R}^d$ with independent marks, such that the underlying point process has intensity measure Λ, while the marks are given by cubes in $\mathbb{R}^d$ with edges of random length R, such that $\Theta \in \mathcal{R}$;

- if Λ is a discrete measure, and the distribution law of R is discrete as well, than the resulting random set Θ is discrete;
- if Λ is discrete, while the distribution law of R is absolutely continuous, then it follows that Θ is an absolute continuous random closed set;
- if both Λ and R are absolutely continuous, then Θ is absolute continuous.
- if Λ is absolutely continuous, while R is discrete, than Θ is absolutely continuous;
- if Λ is continuous, but not absolutely continuous, and R is discrete, than Θ is not discrete, but we can say nothing about its continuity in general.

Note that the absolute continuity of the intensity measure does not always imply the absolute continuity of Θ. For example let us consider the case in which the marks depend on the underlying process: the edge length of each cube may depend on the coordinates of its centre, so that an edge of every cube lies on a prefixed plane π. As a consequence we have that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \pi)] \neq 0$ with $\nu^d(\pi) = 0$, and so Θ is not absolutely continuous.

Obviously, the examples we have shown here (some of which are a bit *pathological*) are given in order to clarify the meaning of our definitions of discrete, continuous, and absolutely continuous random closed set.
An example of absolutely continuous time-dependent random closed set is discussed in Section 6.2.7.

3.2.1 A condition for the absolute continuity of a random closed set

In order to verify if a random closed set Θ is absolutely continuous, we give now a sufficient condition for (3.8).

Proposition 3.13 *Let Θ_s be a random closed set such that $\dim_{\mathcal{H}}\Theta_s = s < d$ (and so $\Theta_s = \partial\Theta_s$). Then*

$$\mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0, \quad \forall A \subset \mathbb{R}^d \implies \mathbb{E}[\mathcal{H}^s(\Theta_s \cap \cdot)] \ll \nu^d. \tag{3.9}$$

Proof. By absurd, suppose that $\exists A \subset \mathbb{R}^d$ such that $\nu^d(A) = 0$ and $\mathbb{E}[\mathcal{H}^s(\Theta_s \cap A)] > 0$.
We may notice that

$$\nu^d(A) = 0 \implies \mathrm{int}A = \emptyset. \tag{3.10}$$

In fact, by absurd let $\text{int}A \neq \emptyset$; then $\exists x \in \text{int}A$ and $\exists B_r(x) \subset A$, with $r > 0$. Thus $\nu^d(A) \geq \nu^d(B_r(x)) = b_d r^d > 0$.
It is clear that

$$\mathbb{E}[\mathcal{H}^s(\Theta_s \cap A)] > 0 \implies \mathbb{P}(\mathcal{H}^s(\Theta_s \cap A) > 0) > 0 \tag{3.11}$$

Hence, we have the following chain of implications:

$$A \subseteq \text{clos}A = \partial A \cup \text{int}A \overset{(3.10)}{=} \partial A$$

$$\Downarrow$$

$$\Theta_s \cap A \subseteq \Theta_s \cap \partial A$$

$$\Downarrow$$

$$\mathcal{H}^s(\Theta_s \cap A) \leq \mathcal{H}^s(\Theta_s \cap \partial A)$$

$$\Downarrow$$

$$\mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) \geq \mathbb{P}(\mathcal{H}^s(\Theta_s \cap A) > 0) \overset{(3.11)}{>} 0,$$

but this is an absurd, since $\mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0$. □

Remark 3.14 Note that the condition

$$\mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0, \quad \forall A \subset \mathbb{R}^d$$

is sufficient, but not necessary. The opposite implication holds under the additional condition that $\nu^d(\partial A) = 0$. In fact, $\mathbb{E}[\mathcal{H}^s(\Theta_s \cap \cdot)] \ll \nu^d$ does not imply, in general, that, for any $A \subset \mathbb{R}^d$, $\mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0$ holds.
For example, let X be an absolutely continuous random variable. Then, in this case, $d = 1$, $\dim_{\mathcal{H}}(X) = 0$, and $\mathbb{E}[\mathcal{H}^0(X \cap \cdot)] = \mathbb{P}(X \in \cdot) \ll \nu^d$.
Let $A = \mathbb{Q}$; then $\partial A = \mathbb{R}$. So, even if $\mathbb{E}[\mathcal{H}^0(X \cap \cdot)]$ is absolutely continuous, we have that $\mathbb{P}(\mathcal{H}^0(X \cap \partial A) > 0) = \mathbb{P}(X \in \mathbb{R}) = 1 > 0$.

Proposition 3.15 *Let Θ_s be a random closed set such that $\dim_{\mathcal{H}}\Theta_s = s < d$. Then*

$$\mathbb{E}[\mathcal{H}^s(\Theta_s \cap \cdot)] \ll \nu^d \implies \mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0, \quad \forall A \subset \mathbb{R}^d \text{ s.t. } \mathcal{H}^d(\partial A) = 0.$$

Proof. By absurd, $\exists A \subset \mathbb{R}^d$ with $\nu^d(\partial A) = 0$, such that

$$\mathbb{P}(\underbrace{\mathcal{H}^s(\Theta_s \cap \partial A) > 0}_{H}) > 0. \tag{3.12}$$

Then

$$\begin{aligned}
\mathbb{E}[\mathcal{H}^s(\Theta_s \cap \partial A)] &= \mathbb{E}[\mathcal{H}^s(\Theta_s \cap \partial A); H] + \mathbb{E}[\mathcal{H}^s(\Theta_s \cap \partial A); H^C] \\
&= \mathbb{E}[\mathcal{H}^s(\Theta_s \cap \partial A); H] \\
&\overset{(3.12)}{>} 0.
\end{aligned}$$

Hence, we may claim that there exists $K = \partial A \subset \mathbb{R}^d$ such that $\nu^d(K) = 0$ and $\mathbb{E}[\mathcal{H}^s(\Theta_s \cap K)] > 0$; that is $\mathbb{E}[\mathcal{H}^s(\Theta_s \cap \cdot)]$ is not absolutely continuous with respect to ν^d. □

Corollary 3.16 *Let Θ_s be a random closed set such that* $\dim_{\mathcal{H}}\Theta_s = s < d$. *Then,* $\forall A \subset \mathbb{R}^d$ *such that* $\mathcal{H}^d(\partial A) = 0$,

$$\mathbb{E}[\mathcal{H}^s(\Theta_s \cap \cdot)] \ll \nu^d \iff \mathbb{P}(\mathcal{H}^s(\Theta_s \cap \partial A) > 0) = 0.$$

As we have just observed before, if $\dim_{\mathcal{H}}(\Theta) = d$, then $\mathbb{E}[\mathcal{H}^d(\Theta \cap \cdot)] \ll \nu^d$, in accordance with the proposition above. In fact, if A is a subset of $\mathbb{R}^d$ such that $\nu^d(\partial A) = 0$, then $\mathcal{H}^d(\Theta \cap \partial A) \leq \mathcal{H}^d(\partial A) = 0$, and, as a consequence, $\mathbb{P}(\mathcal{H}^d(\Theta \cap \partial A) > 0) = 0$.

3.2.2 A remark on the boundary of an absolutely continuous random closed set

We remind that, by Matheron's definition of a.s. continuity (see Definition 3.6), if Θ is M-continuous, then

$$\mathbb{P}(x \in \partial\Theta) = 0 \qquad \forall x \in \mathbb{R}^d.$$

As we have just observed, the requirement that such a property holds for all x in $\mathbb{R}^d$ seems to be too strong in our context. Now we show that also by our definition of absolute continuity, *in mean* and *strong*, a similar property on $\partial\Theta$ may be obtained. In particular, the following holds.

Proposition 3.17 *Let Θ_n be a random closed set with Hausdorff dimension $n < d$. Then*

$$\mathbb{P}(x \in \Theta_n) = 0 \qquad \textit{for } \nu^d\textit{-a.e. } x \in \mathbb{R}^d.$$

Proof. Since $n < d$ then $\mathbb{E}[\nu^d(\Theta_n)] = 0$.
By Fubini's theorem we have

$$\mathbb{E}[\nu^d(\Theta_n)] = \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}_{\Theta_n}(x)\mathrm{d}x\ ) = \int_{\mathbb{R}^d} \int_\Omega \mathbf{1}_{\Theta_n}(x)\mathbb{P}(\mathrm{d}\omega)\ \mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{P}(x \in \Theta_n)\mathrm{d}x.$$

Thus, necessarily, we have that $\mathbb{P}(x \in \Theta_n)$ for ν^d-a.e. $x \in \mathbb{R}^d$. □

Note that the above proposition applies to any lower dimensional set, independently of its probability law. If we suppose now that the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ associated with Θ_n ($n < d$) is absolutely continuous with respect to ν^d, then the set of points $x \in \mathbb{R}^d$ such that $\mathbb{P}(x \in \Theta_n) = 0$ becomes "bigger".

Proposition 3.18 *Let Θ_n be a random closed set with Hausdorff dimension $n < d$ and such that $\mathbb{E}[\mu_{\Theta_n}] \ll \nu^d$.*
Then for any $\mathcal{H}^n$-measurable subset A of $\mathbb{R}^d$ with $\mathcal{H}^s(A) > 0$, $\nu^d(A) = 0$, we have that

$$\mathbb{P}(x \in \Theta_n) = 0 \qquad \mathcal{H}^n\textit{-a.e. } x \in A.$$

Proof. By absurd there exists a $\mathcal{H}^n$-measurable subset A of $\mathbb{R}^d$ with $\mathcal{H}^s(A) > 0$, $\nu^d(A) = 0$, such that $\mathbb{P}(x \in \Theta_n) > 0$ $\mathcal{H}^n$-a.e. $x \in A$.

By Corollary 2.10.48 in [32] we have that the measure $\mathcal{H}^n$ is σ-finite on A, so that we may use Fubibi's theorem. Thus it follows that

$$\begin{aligned} 0 < \int_A \mathbb{P}(x \in \Theta_n)\mathcal{H}^n(\mathrm{d}x) &= \int_A \int_\Omega \mathbf{1}_{\Theta_n}(x)\mathbb{P}(\mathrm{d}\omega)\ \mathcal{H}^n(\mathrm{d}x) \\ &= \int_\Omega \int_A \mathbf{1}_{\Theta_n}(x)\mathcal{H}^n(\mathrm{d}x)\ \mathbb{P}(\mathrm{d}\omega) = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] \end{aligned}$$

But this is in contrast with the assumption $\mathbb{E}[\mu_{\Theta_n}] \ll \nu^d$, because $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]$ should be equal to 0. $\square$

Note that, if Θ is a random closed set with Hasudorff dimension less than d, then $\Theta = \partial\Theta$, while if $\Theta \in \mathcal{R}$, than $\dim_{\mathcal{H}}(\partial\Theta) < d$.

Corollary 3.19 *Let Θ be a random closed set in $\mathbb{R}^d$ with $\dim_{\mathcal{H}}(\partial\Theta) = n < d$. If one of the following two conditions is satisfied:*

(i) $\partial\Theta$ is absolutely continuous in mean,

(ii) Θ is absolutely continuous,

then

$$\mathbb{P}(x \in \partial\Theta) = 0 \qquad \mathcal{H}^n\text{-a.e. } x \in A,$$

for any $\mathcal{H}^n$-measurable subset A of $\mathbb{R}^d$ with $\mathcal{H}^s(A) > 0$, $\nu^d(A) = 0$.

As a simple example, consider the random segment with an extreme coinciding with the origin as in Fig. 3.4. Clearly, it is absolutely continuous, in this case $n = 1$, and the only point belonging to the random set with probability bigger than 0 is the origin, which has null $\mathcal{H}^1$ measure, as we expected.

3.2.3 Relations between discrete, continuous, and absolutely continuous random closed sets

Throughout this section Θ is supposed to belong to the class $\mathcal{R}$.

In analogy with measure theory, we give the following definition:

Definition 3.20 *We say that a random closed set Θ in $\mathbb{R}^d$ is* singular *if and only if it is not absolutely continuous (in the sense of Definition 3.10).*

In this way, the relations between the continuous and singular parts of a measure (see Remark 1.2) hold also for a random closed set, although in a different context. So, in terms of our definitions, we may claim the following.

Proposition 3.21

$$\begin{aligned} \Theta \text{ absolutely continuous} &\quad\Rightarrow\quad \Theta \text{ continuous, but not the reverse}; \qquad (3.13) \\ \Theta \text{ discrete} &\quad\Rightarrow\quad \Theta \text{ singular, but not the reverse.} \qquad (3.14) \end{aligned}$$

Proof. A simple counterexample to (3.13) and (3.14) is given by the continuous, but not absolutely continuous, random square Θ_1 introduced in Section 3.1 and mentioned in Section 3.2.
Let us prove the first implication; the second one follows in a similar way. By absurd, let Θ be not continuous; then there exists $\theta \subset \mathbb{R}^d$ such that $\mathbb{P}(\Theta = \theta) > 0$, with $\mathcal{H}^{\dim_{\mathcal{H}}(\partial\theta)}(\partial\theta) > 0$. Thus, we have that $\mathcal{H}^d(\partial\theta) = 0$, but

$$\begin{aligned}
&\mathbb{E}[\mathcal{H}^{\dim_{\mathcal{H}}(\partial\Theta)}(\partial\Theta \cap \partial\theta)] \\
&= \mathbb{E}[\mathcal{H}^{\dim_{\mathcal{H}}(\partial\Theta)}(\partial\Theta \cap \partial\theta); \{\omega : \Theta = \theta\}] + \mathbb{E}[\mathcal{H}^{\dim_{\mathcal{H}}(\partial\Theta)}(\partial\Theta \cap \partial\theta); \{\omega : \Theta \neq \theta\}] \\
&\geq \mathbb{E}[\mathcal{H}^{\dim_{\mathcal{H}}(\partial\Theta)}(\partial\Theta \cap \partial\theta); \{\omega : \Theta = \theta\}] \\
&= \mathcal{H}^{\dim_{\mathcal{H}}(\partial\theta)}(\partial\theta) \\
&> 0.
\end{aligned}$$

Hence, by Definition 3.10, Θ is not absolutely continuous, that is absurd. □

Now we wonder in what relation a random set Θ is with its subsets. In particular, if Θ, for example, is an absolutely continuous random set with Hausdorff dimension d, then may one claim that any of its subsets of lower dimension is absolutely continuous, too? This is not true as we show by the following counterexample.

Definition 3.22 *Let Θ and Q be random closed sets in $\mathbb{R}^d$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $Q \subseteq \Theta$ if and only if*

$$\forall \omega \in \Omega \qquad Q(\omega) \subseteq \Theta(\omega).$$

We may claim that:

- Θ absolutely continuous $\not\Rightarrow$ every $Q \subset \Theta$ is absolutely continuous.

For example, let us consider the random closed set Θ in $\mathbb{R}^2$ given by

$$\Theta(\omega) = B_r(X(\omega)), \qquad \omega \in \Omega,\ r \in \mathbb{R}_+,$$

where X is a random point uniformly distributed in the unit square centered in 0. Then Θ is absolutely continuous.
Denoting by x the x-axis, let $\Omega' := \{\omega \in \Omega \,:\, \Theta(\omega) \cap x \neq \emptyset\}$. Then $\mathbb{P}(\Omega') > 0$.
If we define

$$Q(\omega) := \begin{cases} \Theta(\omega) \cap x & \text{if } \omega \in \Omega' \\ \emptyset & \text{if } \omega \in \Omega \setminus \Omega', \end{cases}$$

it follows that $Q \subset \Theta$, $\dim_{\mathcal{H}} Q = 1$, $\mathbb{E}[\mathcal{H}^1(Q)] > 0$, and $Q(\omega) \subset x$ for any $\omega \in \Omega$. As a consequence, since $\nu^2(x) = 0$ and $\mathbb{E}[\mathcal{H}^1(Q \cap x)] = \mathbb{E}[\mathcal{H}^1(Q)] > 0$, we may claim that Q is not absolutely continuous. On the other hand, it is also true that

- Θ singular $\not\Rightarrow$ every $Q \subset \Theta$ is singular.

For example, let $\Theta = \Theta_1 \cup \Theta_2$, where Θ_1 is singular, and Θ_2 is absolutely continuous, and let $Q = \Theta_2$. Then, obviously, Θ is singular, while $Q \subset \Theta$ is absolutely continuous.

But the following holds.

Proposition 3.23 *If* $\dim_{\mathcal{H}}\Theta < d$ *and* Θ *is discrete, then any subset* $Q \in \mathcal{R}$ *of* Θ *is singular.*

Proof. By definition, there exist $\theta_1, \theta_2, \ldots$ closed subsets of $\mathbb{R}^d$, and $p_1, p_2, \ldots \in [0,1]$, such that $\mathbb{P}(\Theta = \theta_i) = p_i$ and $\sum_i p_i = 1$.
Since $\mathcal{H}^d(\theta_i) = 0$ for all i, it follows that any countable union of them has Lebesgue measure zero.
Thus, let $A = \bigcup_i \theta_i$; then $\nu^d(A) = 0$ and, $\forall \omega \in \Omega$, $\Theta(\omega) \subseteq A$.
As a consequence, $\forall Q \subseteq \Theta$, we have that, $\forall \omega \in \Omega$, $Q(\omega) \subseteq A$; in particular, if $\dim_{\mathcal{H}} Q = s$, $(s \in [0, \dim_{\mathcal{H}}\Theta])$:

$$\mathbb{E}[\mathcal{H}^s(Q \cap A)] = \mathbb{E}[\mathcal{H}^s(Q)] > 0,$$

i.e. Q is not absolutely continuous. □

Note that the hypothesis $\dim_{\mathcal{H}}\Theta < d$ is crucial. As a simple counterexample, consider a unit cube C in $\mathbb{R}^3$; let $\Theta(\omega) = C\ \forall \omega \in \Omega$, and let X be a random point uniformly distributed in C. Then, obviously, $\dim_{\mathcal{H}}\Theta = 3$, Θ is deterministic (and so discrete), and $X \subset \Theta$; but X is, by definition, absolutely continuous.

In conclusion, we may notice that when we pass from the case of a random point X in $\mathbb{R}^d$, to the case of a general random closed set Θ with dimension bigger than zero, the analysis of the random set becomes more complex.
An intuitive reason of this lies in the fact that a point in $\mathbb{R}^d$ has no proper subsets, while any connected subset of $\mathbb{R}^d$ whose dimension is not zero, has uncountably many proper sets, and it can not be written as a countable union of its subsets in a unique way.

We may summarize the relations between continuity, absolute continuity and M-continuity by the following proposition.

Proposition 3.24 *i) M-continuity* $\substack{\Longrightarrow \\ \not\Longleftarrow}$ *continuity;*

ii) absolute continuity $\substack{\Longrightarrow \\ \not\Longleftarrow}$ *continuity;*

iii) M-continuity $\substack{\not\Longrightarrow \\ \not\Longleftarrow}$ *absolute continuity.*

Proof. i) Proof of $\Rightarrow$
By absurd let Θ be M-continuous, but not continuous. Then there exists $\theta \subset \mathbb{R}^d$ such that $\mathbb{P}(\Theta = \theta) > 0$.
If we set

$$\Omega' := \{\omega : \Theta(\omega) = \theta\},$$

then $\mathbb{P}(\Omega') > 0$ and $\partial\Theta(\omega) = \partial\theta$ for all $\omega \in \Omega'$.
Let $x \in \partial\theta$; so we have that, $\forall \omega \in \Omega'$, $x \in \partial\Theta(\omega)$.
In conclusion it follows that

$$\mathbb{P}(x \in \partial\Theta) \geq \mathbb{P}(\Omega') > 0,$$

i.e. Θ is not M-continuous, that is absurd.

Counterexample for $\not\Leftarrow$: the two random squares Θ_1 and Θ_2 described in Section 3.1.1

ii) This point is just the implication (3.13).

iii) Counterexample for $\not\Rightarrow$: let X be a real-valued random variable continuous, but not absolutely continuous. Then the random set $\Theta = X$ is M-continuous, but it is not absolutely continuous.
Counterexample for $\not\Leftarrow$: let Θ the random ball in $\mathbb{R}^3$ given by

$$\Theta(\omega) := B_1(X(\omega)),$$

where X is a random point uniformly distributed on $\partial B_1(0)$. It follows that Θ is absolutely continuous, but it is not M-continuous since $\mathbb{P}(0 \in \partial\Theta) = 1$. $\square$

Remark 3.25 In the particular case in which $\Theta = X$ is a random point, then

$$\text{M-continuity} \quad \Longleftrightarrow \quad \text{continuity}. \tag{3.15}$$

As we have noticed previously, in this case our definition of continuity coincides with the classical one. The same holds for M-continuity; in fact $\partial X = X$, and

$$\mathbb{P}(x \in \partial X) = \mathbb{P}(X = x) = 0 \qquad \forall x \in \mathbb{R}^d.$$

All the examples we have presented suggest that the reason of the equivalence in (3.15) lies in the fact that a random point has no proper subsets.

Really, there may exist other situations in which M-continuity and continuity are equivalent. For example whenever Θ is a stationary random closed set with $\nu^d(\partial\Theta) = 0$.
It is clear that any stationary random closed set is continuous. On the other hand, the following result holds (see [49], p. 48).

Proposition 3.26 *Let Θ be a random closed set in $\mathbb{R}^d$. If the mapping*

$$x \longrightarrow \mathbb{P}(x \in \partial\Theta)$$

is continuous on $\mathbb{R}^d$, then

$$\Theta \text{ is M-continuous } \Longleftrightarrow \nu^d(\partial\Theta) = 0 \ \mathbb{P}\text{-a.s.}$$

Proof. If Θ is M-continuous, then, by definition, $\mathbb{P}(x \in \partial\Theta) = 0$ for all $x \in \mathbb{R}^d$. As a consequence, by Fubini's theorem it follows that

$$\mathbb{E}[\nu^d(\partial\Theta)] = \int_{\mathbb{R}^d} \mathbb{P}(x \in \partial\Theta)\mathrm{d}x = 0,$$

and so $\nu^d(\partial\Theta) = 0$ $\mathbb{P}$-a.s.
If $\nu^d(\partial\Theta) = 0$ $\mathbb{P}$-a.s., then $\mathbb{E}[\nu^d(\partial\Theta)] = 0$. But $\mathbb{E}[\nu^d(\partial\Theta)] = 0$ if and only if $\mathbb{P}(x \in \partial\Theta) = 0$ for ν^d-a.e. $x \in \mathbb{R}^d$. By the continuity of the mapping

$x \to \mathbb{P}(x \in \partial\Theta)$ we have that $\mathbb{P}(x \in \partial\Theta) = 0$ for all $x \in \mathbb{R}^d$, i.e. Θ is M-continuous. $\square$

Note that if Θ is stationary, then $\partial\Theta$ is stationary as well, and so the mapping $x \to \mathbb{P}(x \in \partial\Theta)$ is continuous, since $\mathbb{P}(x \in \partial\Theta)$ is constant on $\mathbb{R}^d$. Thus it immediately follows that

Corollary 3.27 *If Θ is a stationary random closed set with $\nu^d(\partial\Theta) = 0$ $\mathbb{P}$-a.s., then Θ is M-continuous.*

Therefore we may claim that

$$\Theta \text{ stationary with } \nu^d(\partial\Theta) = 0 \ \mathbb{P}\text{-a.s.}$$

$$\Downarrow$$

$$\Theta \text{ is both M-continuous and continuous.}$$

Note that an uncountable union of singular random sets may be, in general, an absolutely continuous random set. For example, let us consider a vertical random bar Θ in $\mathbb{R}^2$, and let A and B be its extremes. So, A and B are two random points. We assume that $A = (X, 0)$, and $B = (X, 1)$, where X is a random variable (on the same probability space) with uniform distribution in $[0, 10]$. It is clear that $\mathbb{E}[\mathcal{H}^1(\Theta \cap \cdot)] \ll \nu^d$, and so Θ is absolutely continuous. A and B, as any other random point $P_c \subset \Theta$ of the type $P = (X, c)$, with $c \in (0, 1)$, are singular. In fact, if l is the line $y = c$,then $\mathbb{P}(\mathcal{H}^0(P_c \cap l) > 0) = 1$, by Proposition 3.15 it follows that P is not absolutely continuous. Note that $\Theta = \bigcup_{c\in[0,1]} P_c$.

Therefore, the subdivision of a random set in singular and absolutely continuous subsets depends on the specific situation, since the knowledge of Θ does not give, in general, sufficient information about its subsets (except particular cases, or when Θ is discrete).

Let us consider Θ_n discrete (i.e. $\mathbb{P}(\Theta_n = \theta_n^i) = p_i$, with $\sum_{i\in I} p_i = 1$) with $0 \le n < d$; then $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot)]$ is singular with respect to ν^d and, formally, in accordance with (2.18), by $\mathbb{E}[\delta_{\Theta_n}(x)] = \sum_i \delta_{\theta_n^i} p_i$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_A \delta_{\Theta_n}(x)\mathrm{d}x\right] &= \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] \\
&= \sum_i \mathcal{H}^n(\theta_n^i \cap A) p_i \\
&= \sum_i p_i \int_A \delta_{\theta_n^i}(x)\mathrm{d}x \\
&= \int_A \sum_i \delta_{\theta_n^i}(x) p_i \mathrm{d}x \qquad (3.16) \\
&= \int_A \mathbb{E}[\delta_{\Theta_n}(x)]\mathrm{d}x.
\end{aligned}$$

Note that the last three integrals have to be regarded in a generalized sense. We give an intuitively explanation in terms of the generalized densities (delta functions) $\delta^i_{\theta_n}$ associated with Θ_n.
Since, for any i, θ^i_n is a lower dimensional subset of $\mathbb{R}^d$, as we have seen in the previous sections, the (deterministic) measure $\mu_{\theta^i_n}$ is singular with respect to ν^d, and so the $\delta_{\theta^i_n}$'s are "usual Dirac delta"'s, that is, they are zero ν^d-a.e. and have to be considered as generalized functions. Now, in the sum

$$\sum_{i\in I}\delta_{\theta^i_n}(x)p_i(x)$$

which appears in (3.16), I is a countable set, and so the resulting delta function $\mathbb{E}[\delta_{\Theta_n}(x)]$ is again a "usual Dirac delta". Instead, in the case that Θ_n is not discrete, I is not countable, so we may not claim a priori that $\mathbb{E}[\delta_{\Theta_n}(x)]$ (uncountable sum of "usual Dirac delta"'s) is a "usual Dirac delta" or a "usual function". In fact, there are situations in which this sum turns to be an usual function (e.g. see Example on p. 52 with X_0 continuous random point); in these cases the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is absolutely continuous and its Radon-Nikodym derivative is just the density $\mathbb{E}[\delta_{\Theta_n}(x)]$.
Please note that, if $n = d$, as we have observed previously, $\mathbb{E}[\mu_{\Theta_d}]$ is absolutely continuous with respect to ν^d even if Θ_d is discrete, because $\mathcal{H}^d(\Theta_d \cap \cdot\,)$ is the restriction of ν^d to the set Θ_d. So, in the discrete case, the $\delta_{\theta^i_d}$'s are not usual Dirac delta, but indicator functions:

$$\delta_{\theta^i_d}(x) = \mathbf{1}_{\theta^i_d}(x) \quad \nu^d\text{-a.e.}$$

As a consequence, the Radon-Nikodym derivative is the step function given by the countable sum in (3.16).

Finally, if Θ_n can be written as disjoint union of Θ^1 and Θ^2, then

$$\delta_{\Theta_n} = \delta_{\Theta^1} + \delta_{\Theta^2}.$$

As a consequence, we may have a decomposition of $\mathbb{E}[\delta_{\Theta_n}]$ in terms of the densities of Θ^1 and Θ^2. In particular, as a simple example, let us consider the case Θ^1 is fixed, while Θ^2 is random, with $\Theta^1 \cap \Theta^2 = \emptyset$. Then we have

$$\begin{aligned}\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] &= \mathbb{E}[\mathcal{H}^n(\Theta^1 \cap A)] + \mathbb{E}[\mathcal{H}^n(\Theta^2 \cap A)]\\ &= \mathcal{H}^n(\Theta^1 \cap A) + \mathbb{E}[\mathcal{H}^n(\Theta^2 \cap A)]\\ &= (\delta_{\Theta^1}, \mathbf{1}_A) + (\mathbb{E}[\delta_{\Theta^2}], \mathbf{1}_A),\end{aligned}$$

that is to say

$$\mathbb{E}[\delta_{\Theta_n}] = \delta_{\Theta^1} + \mathbb{E}[\delta_{\Theta^2}].$$

Chapter 4

Approximation of the mean densities

In many real applications such as fiber processes, n-facets of random tessellations of dimension $n \leq d$, etc., several problems are related to the estimation of the mean density of the expected measure $\mathbb{E}[\mu_\Theta]$ of a random closed set Θ in $\mathbb{R}^d$. In order to face such problems in the general setting of spatially inhomogeneous processes, we suggest and analyze here an approximation of mean densities for sufficiently regular random closed sets. We will show how some known results in literature follow as particular cases.

4.1 The particular cases $n = 0$ and $n = d$

In the previous chapter we have observed that any d-regular random set is absolutely continuous in mean, and it suffices to apply Fubini's theorem (in $\Omega \times \mathbb{R}^d$, with the product measure $\mathbb{P} \times \nu^d$) to obtain that the mean density is given by (see Remark 2.23)

$$\mathbb{E}[\delta_{\Theta_d}](x) = \mathbb{P}(x \in \Theta_d) \qquad \text{for } \nu^d\text{-a.e. } x \in \mathbb{R}^d.$$

Therefore in this case it is easy to estimate the mean density by the estimation of the hitting functional (Section 1.4.1) at a point x:

$$T_{\Theta_d}(x) = \mathbb{P}(\Theta_d \cap x \neq \emptyset) = \mathbb{P}(x \in \Theta_d).$$

It is clear that, given a sample of the random closed set Θ_d, i.e. given a sequence $\Theta_d^1, \Theta_d^2, \ldots$ of random closed sets IID as Θ_d, for any fixed point $x \in \mathbb{R}^d$, an unbiased estimator of $\mathbb{E}[\delta_{\Theta_d}](x)$ is given by

$$\widehat{\lambda}^{(N)}_{\Theta_d}(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\Theta_d^i}(x).$$

If $n = 0$ and $\Theta_0 = X$ is a random point, then $\mathbb{E}[\mathcal{H}^0(\Theta_0 \cap \cdot)] = \mathbb{P}(X \in \cdot)$. Therefore Θ_0 is absolutely continuous in mean if and only if the law of X is absolutely continuous. In this case $\mathbb{E}[\delta_X]$ coincides with the pdf p_X of X and so we may estimate $\mathbb{E}[\delta_X]$ by the well known estimation by means of histograms ([58] § VII.13)

Estimation of densities by means of histograms

We recall the estimation of probability densities of real-valued random variables by means of functions that are called "histograms".

An interval in $\mathbb{R}$ will be denoted by I, and its length by $|I|$.

Definition 4.1 *A sequence of intervals $I_1, I_2, \dots$ is called an* interval partition *of $\mathbb{R}$ if*

i) $I_p \cap I_q = \emptyset$ *if* $p \neq q$,

ii) $\mathbb{R} = \bigcup_{p\in\mathbb{N}} I_p$.

Such a sequence of intervals will be denoted by $\mathcal{I}$ ($\mathcal{I} := \{I_p\}_{p\in\mathbb{N}}$).
In particular, we set

$$|\mathcal{I}| := \sup_p |I| \quad \text{and} \quad \gamma(\mathcal{I}) := \inf_p |I|.$$

It is assumed that $0 < \gamma(\mathcal{I}) \leq |\mathcal{I}| < \infty$.

Definition 4.2 *If a step function*

$$h = \sum_p c_p \mathbf{1}_{I_p}$$

is a probability density (i.e. $\int_{\mathbb{R}} h = 1$), then we say that h is a histogram.

Now, let $X_1, \dots, X_n$ be a sample of a population having an absolutely continuous probability distribution with density f. Furthermore, let $\mathcal{I}$ be a prefixed interval partition of $\mathbb{R}$. Then with every given outcome $x_1, \dots, x_n$ of the sample we associate a histogram given by

$$h = h(\mathcal{I}; x_1, \dots, x_n) := \sum_p c_p \mathbf{1}_{I_p}, \tag{4.1}$$

where

$$c_p := \frac{\#\{i : x_i \in I_p\}}{n|I_p|}.$$

It is easily verified that the step function h defined above is a probability density, hence it is indeed a histogram, and it can be used as an estimation for f. In fact, we may observe that the coefficients c_p depend on the sample outcome $x_1, \dots, x_n$; thus $h(\mathcal{I}; X_1, \dots, X_n)$ as in (4.1), with

$$c_p = c_p(X_1, \dots, X_p) = \frac{\#\{i : x_i \in I_p\}}{n|I_p|},$$

is a statistic for f. Note that

$$c_p(X_1, \ldots, X_p) = \frac{1}{n|I_p|} \sum_{i=1}^{n} \mathbf{1}_{I_p}(X_i),$$

and the variables $\mathbf{1}_{I_p}(X_1), \ldots, \mathbf{1}_{I_p}(X_n)$ are IID with a Bernoulli distribution with parameter given by

$$\mathbb{P}(\mathbf{1}_{I_p}(X_i) = 1) = \mathbb{P}(X_i \in I_p) = \int_{I_p} f(x)\mathrm{d}x.$$

As a consequence, the following holds ([58], p. 367).

Proposition 4.3 *We have*

i) $\mathbb{E}[c_p(X_1, \ldots, X_n)] = \dfrac{1}{|I_p|} \displaystyle\int_{I_p} f(x)\mathrm{d}x,$

ii) $$\mathrm{var}(c_p(X_1, \ldots, X_n)) = \frac{1}{n|I_p|^2} \left(\int_{I_p} f(x)\mathrm{d}x \right) \left(\int_{I_p^c} f(x)\mathrm{d}x \right) \leq \frac{1}{n|I_p|} \sup_{x \in I_p} f(x).$$

For any fixed $x \in \mathbb{R}$, the expression

$$h(\mathcal{I}; X_1, \ldots, X_n)(x)$$

represents a real-valued statistic. The following theorem is proved in [58], p. 367.

Theorem 4.4 *Let $X_1, \ldots, X_n$ be an infinite sample from an absolutely continuous population with probability density f. Suppose that $\mathcal{I}_1, \mathcal{I}_2, \ldots$ is a sequence of partitions of $\mathbb{R}$ such that*

$$|\mathcal{I}_n| \longrightarrow 0 \quad \textit{and} \quad n\gamma(\mathcal{I}_n) \longrightarrow \infty.$$

Then in every point of continuity x of f we have

$$\lim_{n \to \infty} h(\mathcal{I}_n; X_1, \ldots, X_n)(x) = f(x) \quad \textit{in probability.}$$

4.2 An approximation of the mean densities

Given an n-regular random closed set Θ_n, even if a natural sequence of approximating functions of the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is given by $\mathbb{E}[\delta^{(r)}_{\Theta_n}]$ defined by (2.14), problems might arise in the estimation of $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]$, as the computation of the Hausdorff measure is typically non-trivial even in the deterministic case. Since points and lines are ν^2-negligible, it is natural to make use of their 2-D box approximation. As a matter of fact, a computer graphics representation of them is anyway provided in terms of pixels, which can only offer a 2-D box approximation of points in $\mathbb{R}^2$. Therefore we are led to consider a new approximation, based on the Lebesgue measure (much more robust and computable) of the enlargement of the random set, so we suggest unbiased estimators for densities of random sets of lower dimensions in a given d-dimensional space, by means of their approximation in terms of their d-dimensional enlargement by Minkowski addition. This procedure is obviously consistent with the usual histogram estimation of probability densities of a random variable. A crucial result is given in the following proposition.

Proposition 4.5 *[2] Let Θ_n be a random n-regular set, and let $A \in \mathcal{B}_{\mathbb{R}^d}$. If*

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \tag{4.2}$$

then

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]. \tag{4.3}$$

Proof. For a random closed set Ξ in $\mathbb{R}^d$, Fubini's theorem gives

$$\mathbb{E}[\nu^d(\Xi \cap A)] = \int_A \mathbb{P}(x \in \Xi)\mathrm{d}x.$$

Therefore, the following chain of equalities holds:

$$\begin{aligned} \lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x \quad &= \quad \lim_{r\to 0} \frac{1}{b_{d-n} r^{d-n}} \int_A \mathbb{P}(x \in \Theta_{n_{\oplus r}})\mathrm{d}x \\ &= \quad \lim_{r\to 0} \frac{1}{b_{d-n} r^{d-n}} \mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)] \\ &\overset{(4.2)}{=} \quad \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] \end{aligned}$$

□

Motivated by the previous proposition, we define

$$\delta_n^{\oplus r}(x) := \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}}$$

and, accordingly, the absolutely continuous Radon measures $\mu^{\oplus r} = \delta_n^{\oplus r} \nu^d$, i.e.

$$\mu^{\oplus r}(B) := \int_B \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x \qquad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

We may notice that

$$\mathbb{P}(x \in \Theta_{n_{\oplus r}}) = \mathbb{P}(\Theta_n \cap B_r(x) \neq \emptyset) = T_{\Theta_n}(B_r(x)),$$

thus making explicit the reference to T_{Θ_n}, the capacity functional characterizing the probability law of the random set Θ_n.

Corollary 4.6 *Let Θ_n be a random n-regular set, and assume that* (4.2) *holds for any bounded open set A such that $\mathbb{E}[\mu_{\Theta_n}](\partial A) = 0$. Then the measures $\mu^{\oplus r}$ weakly* converge to the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ as $r \to 0$.*

In other words, we may say that the sequence of linear functionals $\delta_n^{\oplus r}$, associated with the measures $\mu^{\oplus r}$ as follows

$$(\delta_n^{\oplus r}, f) := \int_{\mathbb{R}^d} f(x)\mu^{\oplus r}(\mathrm{d}x),$$

converge weakly* to the linear functional $\mathbb{E}[\delta_{\Theta_n}]$, i.e.

$$\lim_{r\to 0}(\delta_n^{\oplus r}, f) = (\mathbb{E}[\delta_{\Theta_n}], f) \qquad \forall f \in C_b(\mathbb{R}^d, \mathbb{R}).$$

Note that, if Θ_n is absolutely continuous in mean, we have

$$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x,$$

where λ_{Θ_n} is the density of its associated expected measure $\mathbb{E}[\mu_{\Theta_n}]$. So, in this case, we can rephrase (4.3) as

$$\lim_{r\to 0}\int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}}\mathrm{d}x = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x. \tag{4.4}$$

In particular, if Θ_n is a stationary random closed set, then $\delta_n^{\oplus r}(x)$ is independent of x and the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is motion invariant, i.e. $\lambda_{\Theta_n}(x) = L \in \mathbb{R}_+$ for ν^d-a.e. $x \in \mathbb{R}^d$. It follows that

$$\lim_{r\to 0}\int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}}\mathrm{d}x = \lim_{r\to 0}\frac{\mathbb{P}(x_0 \in \Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}}\nu^d(A) \qquad \forall x_0 \in \mathbb{R}^d,$$
$$\int_A \lambda(x)\mathrm{d}x = L\nu^d(A);$$

so by (4.4) we infer

$$\lim_{r\to 0}\frac{\mathbb{P}(x_0 \in \Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}} = L \qquad \forall x_0 \in \mathbb{R}^d.$$

Recall that all these conclusions hold under the assumption, made in Proposition 4.5, that (4.2) holds. So, the main *problem* is to find conditions on Θ_n ensuring that this condition holds. If Θ_n is such that almost every realization $\Theta_n(\omega)$ has Minkowski content equal to the Hausdorff measure, i.e.

$$\lim_{r\to 0}\frac{\nu^d(\Theta_{n_{\oplus r}}(\omega))}{b_{d-n}r^{d-n}} = \mathcal{H}^n(\Theta_n(\omega)), \tag{4.5}$$

then it is clear that, taking the expected values on both sides, (4.2) is strictly related to the possibility of exchanging limit and expectation. So we ask whether (5.11) implies a similar result when we consider the intersection of $\Theta_{n_{\oplus r}}(\omega)$ with an open set A in $\mathbb{R}^d$, and for which kind of random closed sets the convergence above is dominated, so that exchanging limit and expectation is allowed.

We remind that the n-dimensional Minkowski content of a closed set $S \subset \mathbb{R}^d$ is defined by

$$\lim_{r\to 0}\frac{\nu^d(S_{\oplus r})}{b_{d-n}r^{d-n}}$$

whenever the limit exists.
We quote the following result from [4], p.110:

Theorem 4.7 *Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that for all $x \in S$*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall r \in (0,1) \tag{4.6}$$

holds for some $\gamma > 0$ and some Radon measure η in $\mathbb{R}^d$ absolutely continuous with respect to $\mathcal{H}^n$. Then

$$\lim_{r \to 0} \frac{\nu^d(S_{\oplus r})}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S).$$

(Note that it makes no difference to consider the enlargement $S_{\oplus r}$ of S with the open or the closed ball of radius r centered in 0).
The following result is a local version of Theorem 4.7.

Lemma 4.8 *[2] Let S be a compact subset of $\mathbb{R}^d$ satisfying the hypotheses of Theorem 4.7. Then, for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that*

$$\mathcal{H}^n(S \cap \partial A) = 0, \tag{4.7}$$

the following holds

$$\lim_{r \to 0} \frac{\nu^d(S_{\oplus r} \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap A). \tag{4.8}$$

Proof. If $n = d$, then equality (4.8) is easily verified. Thus, let $n < d$.
We may notice that, by the definition of rectifiability, if $C \subset \mathbb{R}^d$ is closed, then the compact set $S \cap C$ is still countably $\mathcal{H}^n$-rectifiable; besides (4.6) holds for all point $x \in S \cap C$ (since it holds for any point $x \in S$). As a consequence, by Theorem 4.7, we may claim that for any closed subset C of $\mathbb{R}^d$, the following holds

$$\lim_{r \to 0} \frac{\nu^d((S \cap C)_{\oplus r})}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap C). \tag{4.9}$$

Let A be as in the assumption.
- Let $\varepsilon > 0$ be fixed. We may observe that the following holds:

$$S_{\oplus r} \cap A \subset (S \cap \mathrm{clos}A)_{\oplus r} \cup (S \cap \mathrm{clos}A_{\oplus \varepsilon} \setminus \mathrm{int}A)_{\oplus r} \qquad \forall r < \varepsilon.$$

Indeed, if $x \in S_{\oplus r} \cap A$ then there exists $y \in S$ with $|x - y| \leq r$, and $y \in \mathrm{clos}A_{\oplus \varepsilon}$. Then, if $x \notin (S \cap \mathrm{clos}A)_{\oplus r}$, we must have $y \in S \setminus \mathrm{clos}A$, hence $y \in S \cap \mathrm{clos}A_{\oplus \varepsilon} \setminus \mathrm{clos A}$.

By (4.9), since $\mathrm{clos}A$ and $\mathrm{clos}A_{\oplus \varepsilon} \setminus \mathrm{int}A$ are closed, we have

$$\lim_{r \to 0} \frac{\nu^d((S \cap \mathrm{clos}A)_{\oplus r})}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap \mathrm{clos}A) \overset{(4.7)}{=} \mathcal{H}^n(S \cap A), \tag{4.10}$$

$$\lim_{r \to 0} \frac{\nu^d(S \cap \mathrm{clos}A_{\oplus \varepsilon} \setminus \mathrm{int}A)_{\oplus r}}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S \cap \mathrm{clos}A_{\oplus \varepsilon} \setminus \mathrm{int}A). \tag{4.11}$$

Thus,

$$\begin{aligned}
&\limsup_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \\
\leq\quad & \limsup_{r\to 0} \frac{\nu^d((S\cap \text{clos}A)_{\oplus r}\cup (S\cap \text{clos}A_{\oplus\varepsilon}\setminus \text{int}A)_{\oplus r})}{b_{d-n}r^{d-n}} \\
\leq\quad & \limsup_{r\to 0} \frac{\nu^d((S\cap \text{clos}A)_{\oplus r})+\nu^d((S\cap \text{clos}A_{\oplus\varepsilon}\setminus \text{int}A)_{\oplus r})}{b_{d-n}r^{d-n}} \\
\stackrel{(4.10),(4.11)}{=}\quad & \mathcal{H}^n(S\cap A)+\mathcal{H}^n(S\cap \text{clos}A_{\oplus\varepsilon}\setminus \text{int}A);
\end{aligned}$$

by taking the limit as ε tends to 0 we obtain

$$\limsup_{r\to 0} \frac{\nu^d(I_r(S)\cap A)}{b_{d-n}r^{d-n}} \leq \mathcal{H}^n(S\cap A)+\underbrace{\mathcal{H}^n(S\cap \partial A)}_{=0} = \mathcal{H}^n(S\cap A).$$

• Now, let B be a closed set well contained in A, i.e. $\text{dist}(A,B)>0$. Then there exists $\tilde{r}>0$ such that $B_{\oplus r}\subset A$, $\forall r<\tilde{r}$. So,

$$\begin{aligned}
\mathcal{H}^n(S\cap B) \quad \stackrel{(4.9)}{=} \quad & \liminf_{r\to 0} \frac{\nu^d((S\cap B)_{\oplus r})}{b_{d-n}r^{d-n}} \\
\leq \quad & \liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap B_{\oplus r})}{b_{d-n}r^{d-n}} \\
\leq \quad & \liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}}.
\end{aligned}$$

Let us consider an increasing sequence of closed sets $\{B_n\}_{n\in\mathbb{N}}$ well contained in A such that $B_n \nearrow \text{int}A$. By taking the limit as n tends to ∞, we obtain that

$$\liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \geq \lim_{n\to\infty} \mathcal{H}^n(S\cap B_n) = \mathcal{H}^n(S\cap \text{int}A) \stackrel{(4.7)}{=} \mathcal{H}^n(S\cap A).$$

We summarize,

$$\mathcal{H}^n(S\cap A) \leq \liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \leq \limsup_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \leq \mathcal{H}^n(S\cap A),$$

and so the thesis follows. □

If we consider the sequence of random variables $\frac{\nu^d(\Theta_{n_{\oplus r}}\cap A)}{b_{d-n}r^{d-n}}$, for r going to 0, we ask which conditions have to be satisfied by a random set Θ_n, so that they are dominated by an integrable random variable. In this way we could apply the Dominated Convergence Theorem in order to exchange limit and expectation in (4.8).

Lemma 4.9 *[2] Let K be a compact subset of $\mathbb{R}^d$ and assume that for all $x\in K$*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall r\in(0,1) \tag{4.12}$$

holds for some $\gamma>0$ and some probability measure η in $\mathbb{R}^d$.
Then, for all $r<2$,

$$\frac{\nu^d(K_{\oplus r})}{b_{d-n}r^{d-n}} \leq \frac{1}{\gamma}2^n 4^d \frac{b_d}{b_{d-n}}.$$

Proof. Since $K_{\oplus r}$ is compact, then it is possible to cover it with a finite number p of closed balls $B_{3r}(x_i)$, with $x_i \in K_{\oplus r}$, such that

$$|x_i - x_j| > 3r \qquad i \neq j.$$

(A first ball is taken off $K_{\oplus r}$, and so on until the empty set is obtained).
As a consequence, there exist $y_1, \dots y_p$ such that

- $y_i \in K$, $i = 1, \dots, p$,
- $|y_i - y_j| > r$, $i \neq j$,
- $K_{\oplus r} \subseteq \bigcup_{i=1}^{p} B_{4r}(y_i)$.

In fact, if $x_i \in K$, then we choose $y_i = x_i$; if $x_i \in K_{\oplus r} \setminus K$, then we choose $y_i \in B_r(x_i) \cap K$. As a consequence, $|y_i - x_i| \leq r$ and $B_{4r}(y_i) \supseteq B_{3r}(x_i)$ for any $i = 1, \dots, p$. So

$$\bigcup_{i=1}^{p} B_{4r}(y_i) \supseteq \bigcup_{i=1}^{p} B_{3r}(x_i) \supseteq K_{\oplus r},$$

and

$$3r \leq |x_i - x_j| \leq |x_i - y_i| + |y_i - y_j| + |y_j - x_j| \leq 2r + |y_i - y_j| \qquad i \neq j.$$

For $r < 2$, $B_{r/2}(y_i) \cap B_{r/2}(y_j) = \emptyset$. Since by hypothesis η is a probability measure satisfying (4.12), we have that

$$1 \geq \eta\left(\bigcup_{i=1}^{p} B_{r/2}(y_i)\right) = \sum_{i=1}^{p} \eta(B_{r/2}(y_i)) \overset{(4.12)}{\geq} p\gamma \left(\frac{r}{2}\right)^n,$$

and so

$$p \leq \frac{1}{\gamma} \frac{2^n}{r^n}.$$

In conclusion,

$$\frac{\nu^d(K_{\oplus r})}{b_{d-n} r^{d-n}} \leq \frac{\nu^d(\bigcup_{i=1}^{p} B_{4r}(y_i))}{b_{d-n} r^{d-n}} \leq \frac{p b_d (4r)^d}{b_{d-n} r^{d-n}} \leq \frac{1}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}}.$$

□

In the following theorem we consider $n \in \{0, 1, \dots, d-1\}$, since the particular case $n = d$ is trivial.

Theorem 4.10 *[2] Let Θ_n be a countably $\mathcal{H}^n$-rectifiable random closed set in $\mathbb{R}^d$ (i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\Theta_n(\omega) \subseteq \mathbb{R}^d$ is a countably $\mathcal{H}^n$-rectifiable closed set), such that $\mathbb{E}[\mu_{\Theta_n}]$ is a Radon measure. Let $W \subset \mathbb{R}^d$ be a compact set and let $\Gamma_W : \Omega \longrightarrow \mathbb{R}$ be the function so defined:*

$$\Gamma_W(\omega) := \max\big\{\gamma \geq 0 \,:\, \exists \text{ a probability measure } \eta \ll \mathcal{H}^n \text{ such that } \eta(B_r(x)) \geq \gamma r^n \quad \forall x \in \Theta_n(\omega) \cap W_{\oplus 1},\ r \in (0,1)\big\}.$$

If there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $1/\Gamma_W(\omega) \le Y(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then, for all $A \in \mathcal{B}_{\mathbb{R}^d}$ such that

$$A \subset \mathrm{int}W \quad \text{and} \quad \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \partial A)] = 0, \tag{4.13}$$

we have

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]. \tag{4.14}$$

Proof. Since $\mathbb{E}[Y] < \infty$, then $Y(\omega) < \infty$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.
Let $A \in \mathcal{B}_{\mathbb{R}^d}$ be satisfying (4.13). Let us define
$\Omega_A := \{\omega \in \Omega : \mathcal{H}^n(\Theta_n(\omega) \cap \partial A) = 0\}$,
$\Omega_T := \{\omega \in \Omega : \Theta_n(\omega)$ is countably $\mathcal{H}^n$-rectifiable and closed$\}$,
$\Omega_Y := \{\omega \in \Omega : Y(\omega) < \infty\}$,
$\Omega_\Gamma := \{\omega \in \Omega : \frac{1}{\Gamma_W(\omega)} \le Y(\omega)\}$;
by hypothesis $\mathbb{P}(\Omega_A) = \mathbb{P}(\Omega_T) = \mathbb{P}(\Omega_Y) = \mathbb{P}(\Omega_\Gamma) = 1$.
Thus, if $\Omega' := \Omega_A \cap \Omega_T \cap \Omega_Y \cap \Omega_\Gamma$, it follows that $\mathbb{P}(\Omega') = 1$.

Let $\omega \in \Omega'$ be fixed. Then

- $\Gamma_W(\omega) > 0$, i.e. a probability measure $\eta \ll \mathcal{H}^k$ exists such that

$$\eta(B_r(x)) \ge \Gamma_W(\omega) r^n \quad \forall x \in \Theta_n(\omega) \cap W_{\oplus 1},\ r \in (0,1),$$

- $\mathcal{H}^n(\Theta_n(\omega) \cap \partial A) = 0$,

so, by applying Lemma 4.8 to $\Theta_n \cap W_{\oplus 1}$, we get

$$\lim_{r\to 0} \frac{\nu^d(\Theta_{n_{\oplus r}}(\omega) \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(\Theta_n(\omega) \cap A);$$

i.e. we may claim that

$$\lim_{r\to 0} \frac{\nu^d(\Theta_{n_{\oplus r}} \cap A)}{b_{d-n} r^{d-n}} = \mathcal{H}^n(\Theta_n \cap A) \quad \text{almost surely.}$$

Further, for all $\omega \in \Omega'$, $\Theta_n(\omega) \cap W_{\oplus 1}$ satisfies the hypotheses of Lemma 4.9, and so

$$\begin{aligned} \frac{\nu^d(\Theta_{n_{\oplus r}}(\omega) \cap A)}{b_{d-n} r^{d-n}} &= \frac{\nu^d((\Theta_n(\omega) \cap W_{\oplus 1})_{\oplus r} \cap A)}{b_{d-n} r^{d-n}} \le \frac{\nu^d((\Theta_n(\omega) \cap W_{\oplus 1})_{\oplus r})}{b_{d-n} r^{d-n}} \\ &\le \frac{1}{\Gamma_W(\omega)} 2^n 4^d \frac{b_d}{b_{d-n}} \le Y(\omega) 2^n 4^d \frac{b_d}{b_{d-n}} \in \mathbb{R}. \end{aligned}$$

Let Z be the random variable so defined:

$$Z(\omega) := Y(\omega) 2^n 4^d \frac{b_d}{b_{d-n}}, \qquad \omega \in \Omega'.$$

By assumption $\mathbb{E}[Z] < \infty$, so that the Dominated Convergence Theorem gives

$$\lim_{r\to 0} \mathbb{E}\left[\frac{\nu^d(\Theta_{n_{\oplus r}} \cap A)}{b_{d-n} r^{d-n}}\right] = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)].$$

□

Notice that in the statement of Theorem 4.10 we introduced the auxiliary function $Y(\omega)$ in order to avoid the non-trivial issue of the measurability of $\Gamma_W(\omega)$; as a matter of fact, in all examples, one can estimate $1/\Gamma_W(\omega)$ from above in a measurable way.

Notice also that if Θ_n satisfies the assumption of the theorem for some closed W, then it satisfies the assumption for all closed $W' \subset W$; analogously, any n-regular random closed set Θ_n' contained almost surely in Θ_n still satisfies the assumption of the theorem.

We summarize as follows.

Theorem 4.11 (Main result) *[2] Let Θ_n be a random n-regular closed set in $\mathbb{R}^d$ and let $\mathbb{E}[\mu_{\Theta_n}]$ be its expected measure. Assume that Θ_n satisfies the density lower bound assumption of Theorem 4.10 for any compact set $W \subset \mathbb{R}^d$. Then*

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}} \mathrm{d}x = \mathbb{E}[\mu_{\Theta_n}](A)$$

for any bounded Borel set $A \subset \mathbb{R}^d$ such that

$$\mathbb{E}[\mu_{\Theta_n}](\partial A) = 0. \tag{4.15}$$

In particular, if Θ_n is absolutely continuous in mean, we have

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}} \mathrm{d}x = \int_A \lambda_{\Theta_n}(x) \mathrm{d}x \tag{4.16}$$

for any bounded Borel set $A \subset \mathbb{R}^d$ with $\nu^d(\partial A) = 0$, where λ_{Θ_n} is the mean density of Θ_n. Finally, if Θ_n is stationary we have

$$\lim_{r\to 0} \frac{\mathbb{P}(x_0 \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}} = \lambda_{\Theta_n} \qquad \forall x_0 \in \mathbb{R}^d. \tag{4.17}$$

Proof. The first statement follows by (4.3) in Proposition 4.5: indeed, the assumption (4.2) of that proposition is fulfilled, thanks to Theorem 4.10.
The second statement is a direct consequence of the first one.
Finally, in the stationary case (4.17) follows directly by (4.16), as explained after Corollary 4.6. □

Note that condition (4.15), when restricted to bounded open sets A, is "generically satisfied" in the following sense: given any family of bounded open sets $\{A_t\}_{t\in\mathbb{R}}$ with $\mathrm{clos} A_s \subseteq A_t$ for $s < t$, the set

$$T := \{t \in \mathbb{R} : \ \mathbb{E}[\mu_{\Theta_n}](\partial A_t) > 0\}$$

is at most countable. This is due to the fact that the sets $\{\partial A_t\}_{t\in T}$ are pairwise disjoint, and all with strictly positive $\mathbb{E}[\mu_{\Theta_n}]$-measure.

Remark 4.12 (Mean density as a pointwise limit) It is tempting to try to exchange limit and integral in (4.16), to obtain

$$\lim_{r\to 0} \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}} = \lambda_{\Theta_n}(x), \tag{4.18}$$

at least for ν^d-a.e. $x \in \mathbb{R}^d$. The proof of the validity of this formula for absolutely continuous (in mean) processes seems to be a quite delicate problem, with the only exception of stationary processes. However, in the extreme cases $n = d$ and $n = 0$ it is not hard to prove it.

A consistent estimator of the mean density

Let us observe that $\mathbb{P}(x \in \Theta_{n_{\oplus r}}) = T_{\Theta_n}(B_r(x))$. The estimation of the hitting functional has been faced in classical literature. Thus, the approximation of the mean densities we have proposed here may be the initial point for the estimation of mean densities of n-regular random sets, at any lower Hasudorff dimension n.

Let us assume that (4.18) holds, and let $\Theta_n^1, \Theta_n^2, \ldots$ be a sequence of random closed sets IID as Θ_n. For any fixed $r > 0$, an unbiased estimator for $\mathbb{P}(x \in \Theta_{n_{\oplus r}})$ is given by

$$\widehat{T}_{\Theta_n}(N, r) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\Theta_n^i \cap B_r(x) \neq \emptyset}.$$

Note that the variables $\mathbf{1}_{\Theta_n^1 \cap B_r(x) \neq \emptyset}, \ldots, \mathbf{1}_{\Theta_n^N \cap B_r(x) \neq \emptyset}$ are IID with a Bernoulli distribution with parameter given by $\mathbb{P}(x \in \Theta_{n_{\oplus r}})$.

As a consequence we have that

$$\mathbb{E}[\frac{\widehat{T}_{\Theta_n}(N, r)}{b_{d-n} r^{d-n}}] = \frac{\mathbb{E}[\mathbf{1}_{\Theta_n^i \cap B_r(x) \neq \emptyset}]}{b_{d-n} r^{d-n}} = \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}}, \tag{4.19}$$

and

$$\mathrm{var}(\frac{\widehat{T}_{\Theta_n}(N, r)}{b_{d-n} r^{d-n}}) = \frac{1}{N (b_{d-n} r^{d-n})^2} \mathbb{P}(x \in \Theta_{n_{\oplus r}})(1 - \mathbb{P}(x \in \Theta_{n_{\oplus r}})). \tag{4.20}$$

It is well known the following result.

Lemma 4.13 *If $Y_1, Y_2, \ldots$ is a sequence of random variables such that*

- $\lim_{N \to \infty} \mathbb{E}[Y_N] = c$,
- $\lim_{N \to \infty} \mathrm{var}(Y_N) = 0$,

then

$$\lim_{N \to \infty} Y_n = c \quad \textit{in probability.}$$

Thus, (4.19) and (4.20) suggest to take as consistent estimator of the mean density $\lambda_{\Theta_n}(x)$ the following

$$\widehat{\lambda}^{(N)}_{\Theta_n}(x) := \frac{\sum_{i=1}^{N} \mathbf{1}_{\Theta_n^i \cap B_{N^{-k}}(x) \neq \emptyset}}{b_{d-n} N^{-k(d-n)+1}}, \tag{4.21}$$

with $0 < k < \frac{1}{d-n}$.

More precisely, we may claim that

Proposition 4.14 *Let $\Theta_n^1, \Theta_n^2, \dots$ be a sequence of random closed sets IID as Θ_n, with $n < d$, such that* (4.18) *holds. Then we have*

$$\lim_{N\to\infty} \widehat{\lambda}^{(N)}_{\Theta_n}(x) = \lambda_{\Theta_n}(x) \qquad \text{in probability.}$$

Proof.

$$\lim_{N\to\infty} \mathbb{E}[\widehat{\lambda}^{(N)}_{\Theta_n}(x)] = \lim_{N\to\infty} \frac{N\mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})}{b_{d-n}N^{-k(d-n)+1}} \overset{(4.18)}{=} \lambda_{\Theta_n}(x).$$

We check now that the variance of $\widehat{\lambda}^{(N)}_{\Theta_n}(x)$ tends to 0.

$$\begin{aligned}
&\lim_{N\to\infty} \operatorname{var}(\widehat{\lambda}^{(N)}_{\Theta_n}(x)) \\
&= \lim_{N\to\infty} \frac{1}{(b_{d-n}N^{-k(d-n)+1})^2} N\mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})(1 - \mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})) \\
&= \frac{1}{b_{d-n}} \lim_{N\to\infty} \frac{1}{N^{-k(d-n)+1}} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})}{b_{d-n}N^{-k(d-n)}}(1 - \mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})).
\end{aligned}$$

Since

$$\lim_{N\to\infty} \mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}}) = 0, \text{ because } n < d,$$

$$\lim_{N\to\infty} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus N^{-k}}})}{b_{d-n}N^{-k(d-n)}} = \lambda_{\Theta_n}(x) \in \mathbb{R}, \text{ by (4.18)},$$

and

$$\lim_{N\to\infty} \frac{1}{N^{-k(d-n)+1}} = 0, \text{ because } 0 < k < \tfrac{1}{d-n},$$

it follows that

$$\lim_{N\to\infty} \operatorname{var}(\widehat{\lambda}^{(N)}_{\Theta_n}(x)) = 0.$$

By Lemma 4.13 the thesis is proved. □

4.2.1 The particular cases $n = 0$ and $n = d$

Even if, of course, the special cases $n = 0$ and $n = d$ can be handled with much more elementary tools, we show now how they are consistent with our framework.

Let us consider the well known case of a random point X in $\mathbb{R}^d$. Thus, we are in the particular case in which $n = 0$ and $\Theta_0 = X$.
First of all we observe that X is a compact 0-regular random set and satisfies the hypotheses of Theorem 4.10 with $\eta := \mathcal{H}^0(X(\omega) \cap \cdot\,)$ for all $\omega \in \Omega$. In fact, for any fixed $\omega \in \Omega$,

- η is absolutely continuous with respect to $\mathcal{H}^0$ with $\eta(\mathbb{R}^d) = 1$;
- $\forall x \in X(\omega)$ (i.e. if $X(\omega) = x$), $\forall r > 0$,

  $$\eta(B_r(x)) = \mathcal{H}^0(X(\omega) \cap B_r(x)) = \mathbf{1}_{B_r(x)}(x) = 1,$$

  and so $\Gamma(\omega) = 1$.

Since $\mathbb{P}(x \in X_{\oplus r}) = \mathbb{P}(X \in B_r(x)) = \mathbb{E}[\mathcal{H}^0(X \cap B_r(x))]$, in this case the linear functional $\delta_0^{\oplus r}$ is just the linear functional $\mathbb{E}[\delta_{\Theta_0}^{(r)}]$. We know by previous arguments (see Section 2.2) that the sequence of linear functionals $\mathbb{E}[\delta_{\Theta_0}^{(r)}]$ converges weakly* to the linear functional $\mathbb{E}[\delta_{\Theta_0}]$, i.e., for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathbb{E}[\mu_{\Theta_0}](\partial A) = 0$,

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(X \in B_r(x))}{b_d r^d} \mathrm{d}x = \mathbb{E}[\mathcal{H}^0(X \cap A)],$$

in accordance with Proposition 4.5.

Now, let us assume that $\mathbb{E}[\mathcal{H}^0(X \cap \cdot\,)] = \mathbb{P}(X \in \cdot\,)$ is absolutely continuous with density $f(x)$ given by the usual Radon-Nikodym derivative, which, in this case, coincides with the probability density distribution of the random point X. By remembering that a version of the Radon-Nikodym derivative of $\mathbb{E}[\mathcal{H}^0(X\cap \cdot\,)]$ is given by the limit $\lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^0(X\cap B_r(x))]}{b_d r^d}$, note that we may exchange limit and integral:

$$\begin{aligned}
\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in B_r(X))}{b_d r^d} \mathrm{d}x &= \mathbb{E}[\mathcal{H}^0(X \cap A)] \\
&= \int_A \lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^0(X \cap B_r(x))]}{b_d r^d} \mathrm{d}x \\
&= \int_A \lim_{r\to 0} \frac{\mathbb{P}(X \in B_r(x))}{b_d r^d} \mathrm{d}x \\
&= \int_A \lim_{r\to 0} \frac{\mathbb{P}(x \in B_r(X))}{b_d r^d} \mathrm{d}x,
\end{aligned}$$

so it follows that

$$\lim_{r\to 0} \delta_0^{\oplus r}(x) = \lim_{r\to 0} \frac{\mathbb{P}(x \in X_{\oplus r})}{b_d r^d} = f(x) \qquad \nu^d\text{-a.e. } x \in \mathbb{R}^d,$$

as expected.

Note that in the case $d = 1$ we have in particular that

$$\mathbb{E}[\delta_0^{\oplus r}](x) = \frac{\mathbb{P}(X \in [x-r, x+r])}{2r}; \tag{4.22}$$

if X has pdf f, we know that $\mathbb{E}[\delta_X](x) = f(x)$, so that (4.22), with Proposition 4.14, lead to the usual estimation of density by means of histograms (see Section 4.1). In fact, in this case, if $X_1, X_2, \dots$ is a sequence of random variables IID as X, then, by Proposition 4.14,

$$\widehat{f}(x) := \widehat{\lambda}_X^{(N)}(x) \overset{(4.21)}{=} \frac{\sum_{i=1}^N \mathbf{1}_{B_{N^{-k}}(x)}(X_i)}{b_1 N^{-k+1}}, \qquad 0 < k < 1$$

converges in probability to $f(x)$ for $N \to \infty$.
Note that

$$\sum_{i=1}^N \mathbf{1}_{B_{N^{-k}}(x)}(X_i) = \#\{i : X_i \in B_{N^{-k}}(x)\}$$

where $B_{N^{-k}}(x)$ is the interval I_p on $\mathbb{R}$ centered in x with length $2N^{-k}$, so that

$$\widehat{f}(x) = \frac{\#\{i : X_i \in I_p\}}{N|I_p|},$$

Thus, if $\mathcal{I}_N$ is a partition of $\mathbb{R}$, where the length of each interval is equal to $2N^{-k}$, with $0 < k < 1$, then, for $N \to \infty$,

$$|\mathcal{I}_N| = 2N^{-k} \longrightarrow 0 \quad \text{and} \quad N\gamma(\mathcal{I}_N) = 2N^{-k+1} \longrightarrow \infty,$$

so Theorem 4.4 may be seen as a particular case of Proposition 4.14.

In the particular case $n = d$, we know that the measure $\mathbb{E}[\mu_{\Theta_d}]$ is always absolutely continuous with density $\lambda_{\Theta_d}(x) = \mathbb{P}(x \in \Theta_d)$. We may like to notice that $\delta_d^{\oplus r} = \mathbb{P}(x \in \Theta_{d_{\oplus r}})$ and by Monotone Convergence Theorem we can exchange limit and integral, and so we reobtain

$$\lim_{r \to 0} \mathbb{P}(x \in \Theta_{d_{\oplus r}}) = \mathbb{P}(x \in \Theta_d) = \lambda_{\Theta_d}(x).$$

4.3 Applications

In many real applications Θ_n is given by a random collection of geometrical objects, so that it may be described as the union of a family of n-regular random closed sets E_i in $\mathbb{R}^d$:

$$\Theta_n = \bigcup_i E_i. \tag{4.23}$$

Here we do not make any specific assumption regarding the stochastic dependence among the E_i's. In fact, if Θ_n is known to be absolutely continuous in mean, a problem of interest is to determine its mean density λ_{Θ_n}, and Theorem 4.10 seems to require sufficient regularity of the E_i's, rather than stringent assumptions about their probability law. As a simple example, consider the case in which, for any i, E_i is a random segment in $\mathbb{R}^d$ such that the mean number of segments which hit a bounded region is finite; then we will show in the sequel that Θ_1 satisfies Proposition 4.5, without any other assumption on the probability law of the E_i's (e.g. the law of the point process associated with the centers of the segments).

Note also that geometric processes like segment-, line-, or surface- processes may be described by the so called *union set of a particle process* (see, for example, [13, 64]):

$$\Theta_n = \bigcup_{K \in \Psi} K,$$

where Ψ is a point process on the state space of $\mathcal{H}^n$-rectifiable closed sets. It is known that the stationarity of Θ_n depends on the stationarity of the point process Ψ.

Other random closed sets represented by unions, as in (4.23), are given by

$$\Theta_n = \bigcup_{i=1}^{\Phi} E_i, \tag{4.24}$$

where Φ is a positive integer valued random variable, representing the random number of geometrical objects E_i.
This kind of representation may be used to model a class of time dependent geometric processes, too. For example, at any fixed time $t \in \mathbb{R}_+$, let Θ^t be given by

$$\Theta^t = \bigcup_{i=1}^{\Phi_t} E_i,$$

where Φ_t is a counting process in $\mathbb{R}_+$; e.g., if Φ_t is a Poisson counting process with intensity λ (see Section 1.5.4), then at any time t the number of random objects E_i is given by a random variable distributed as $\mathrm{Po}(\lambda t)$; in this case the process $\{\Theta^t\}$ is additionally determined by a marked point process $\tilde{\Phi}$ in $\mathbb{R}_+$, with marks in a suitable space: the marginal process is given by the counting process Φ_t, while the marks are given by a family of random closed sets E_i.

In literature, many geometric processes like this are investigated, as point-, line-, segment-, or plane processes, random mosaics, grain processes,...
Note that Θ_n may be unbounded, given by an infinite union of random sets E_i (e.g. as in (4.23) with E_i random line). In such a case, when we consider the restriction of Θ_n to a bounded window $W \subset \mathbb{R}^d$, by the usual assumption that the mean number of E_i's hitting a bounded region is finite, we may represent

$$\Theta_n \cap W = \bigcup_{i=1}^{\Phi} E_i^W,$$

with $E_i^W = E_i \cap W$, the union above being finite almost surely.

We give now some significant simple examples of random sets of this kind, to which the results of the previous sections apply.

Example 1. A class of random sets satisfying hypotheses of Theorem 4.10 is given by all sets Θ_n which are random union of random closed sets of dimension $n < d$ in $\mathbb{R}^d$ as in (4.24), such that

(i) $\mathbb{E}[\Phi] < \infty$,

(ii) $E_1, E_2, \dots$ are IID as E and independent of Φ,

(iii) $\mathbb{E}[\mathcal{H}^n(E)] = C < \infty$ and $\exists \gamma > 0$ such that for any $\omega \in \Omega$,

$$\mathcal{H}^n(E(\omega) \cap B_r(x)) \geq \gamma r^n \qquad \forall x \in E(\omega),\ r \in (0,1). \tag{4.25}$$

We can choose $\eta(\cdot) := \dfrac{\mathcal{H}^n(\Theta_n(\omega) \cap \cdot)}{\mathcal{H}^n(\Theta_n(\omega))}$ for any fixed $\omega \in \Omega$. As a consequence, η is a probability measure absolutely continuous with respect to $\mathcal{H}^n$, and such that

$$\eta(B_r(x)) \geq \frac{\gamma}{\mathcal{H}^n(\Theta(\omega))} r^n \qquad \forall x \in \Theta_n(\omega),\ r \in (0,1).$$

In fact, if $x \in \Theta_n(\omega)$, then there exists an $\bar{\imath}$ such that $x \in E_{\bar{\imath}}(\omega)$; since $\Theta_n(\omega) = \bigcup_{i=1}^{\Phi(\omega)} E_i(\omega)$, we have

$$\eta(B_r(x)) = \frac{\mathcal{H}^n(\Theta_n(\omega) \cap B_r(x))}{\mathcal{H}^n(\Theta_n(\omega))} \geq \frac{\mathcal{H}^n(E_{\bar{\imath}}(\omega) \cap B_r(x))}{\mathcal{H}^n(\Theta_n(\omega))} \geq \frac{\gamma}{\mathcal{H}^n(\Theta(\omega))} r^n.$$

As a result, the function Γ defined as in Theorem 4.10 is such that

$$\frac{1}{\Gamma(\omega)} \leq \frac{\mathcal{H}^n(\Theta_n(\omega))}{\gamma} =: Y(\omega),$$

and so it remains to verify only that $\mathbb{E}[\mathcal{H}^n(\Theta_n)] < \infty$:

$$\begin{aligned}
\mathbb{E}[\mathcal{H}^n(\Theta_n)] &= \mathbb{E}[\mathbb{E}[\mathcal{H}^n(\Theta_n) \mid \Phi]] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\mathcal{H}^n(\bigcup_{i=1}^{k} E_i) \mid \Phi = k]\mathbb{P}(\Phi = k) \\
&\overset{\text{(ii)}}{\leq} \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}[\mathcal{H}^n(E_i)]\mathbb{P}(\Phi = k) \\
&\overset{\text{(iii)}}{=} \sum_{k=1}^{\infty} Ck\mathbb{P}(\Phi = k) \\
&= C\mathbb{E}[\Phi] \overset{\text{(i)}}{<} \infty.
\end{aligned}$$

Note that we have not made any particular assumption on the probability laws of Φ and E. Further, it is clear that the same proof holds even in the case in which the E_i's are not IID, provided that $\mathbb{E}[\mathcal{H}^n(E_i)] \leq C$, $\forall i$, and (4.25) is true for any E_i (with γ independent of ω and i).

By keeping the general assumption that Φ is an integrable positive, integer valued random variable, we may write the probability that a point x belongs to the set $\Theta_{n_{\oplus r}}$ in terms of the mean number of E_i which intersect the ball $B_r(x)$. We prove the following proposition.

Proposition 4.15 *[2] Let $n < d$, let Φ be a positive integer valued random variable with $\mathbb{E}[\Phi] < \infty$, and let $\{E_i\}$ be a collection of random closed sets with dimension n. Let Θ_n be the random closed set so defined:*

$$\Theta_n = \bigcup_{i=1}^{\Phi} E_i.$$

If $E_1, E_2, \dots$ are IID as E and independent of Φ, then, for any $x \in \mathbb{R}^d$ such that $\mathbb{P}(x \in E) = 0$,

$$\lim_{r\to 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} = \lim_{r\to 0} \frac{\mathbb{E}[\#\{E_i \,:\, x \in \mathbb{E}_{i_{\oplus r}}\}]}{b_{d-n} r^{d-n}},$$

provided that at least one of the two limits exists.

Proof. The following chain of equalities holds:

$$\begin{aligned}
\mathbb{P}(x \in \Theta_{n_{\oplus r}}) &= \mathbb{P}(x \in \bigcup_{i=1}^{\Phi} E_{i_{\oplus r}}) \\
&= 1 - \mathbb{P}(x \notin \bigcup_{i=1}^{\Phi} E_{i_{\oplus r}}) \\
&= 1 - \mathbb{P}(\bigcap_{i=1}^{\Phi} \{x \notin E_{i_{\oplus r}}\}) \\
&= 1 - \sum_{k=1}^{\infty} \mathbb{P}(\bigcap_{i=1}^{k} \{x \notin E_{i_{\oplus r}}\}) \mid \Phi = k)\mathbb{P}(\Phi = k);
\end{aligned}$$

since the E_i's are IID and independent of Φ,

$$\begin{aligned}
&= 1 - \sum_{k=1}^{\infty} [\mathbb{P}(x \notin E_{\oplus r})]^k \mathbb{P}(\Phi = k) \\
&= 1 - \mathbb{E}[(\mathbb{P}(x \notin E_{\oplus r}))^{\Phi}] \\
&= 1 - \mathrm{G}(\mathbb{P}(x \notin E_{\oplus r})), \qquad (4.26)
\end{aligned}$$

where G is the probability generating function of the random variable Φ.
Now, let us observe that

$$\begin{aligned}
\mathbb{E}[\#\{E_i \, : \, x \in \mathbb{E}_{i_{\oplus r}}\}] &= \sum_{k=1}^{\infty} \mathbb{E}[\sum_{i=0}^{k} \mathbf{1}_{E_{i_{\oplus r}}}(x) \mid \Phi = k]\mathbb{P}(\Phi = k) \\
&= \sum_{k=1}^{\infty} k\mathbb{P}(x \in E_{\oplus r})\mathbb{P}(\Phi = k) \\
&= \mathbb{P}(x \in E_{\oplus r}) \sum_{k=1}^{\infty} k\mathbb{P}(\Phi = k) \\
&= \mathbb{E}[\Phi]\mathbb{P}(x \in E_{\oplus r}). \qquad (4.27)
\end{aligned}$$

We remind that $\mathbb{E}[\Phi] = \mathrm{G}'(1)$ and $1 = \mathrm{G}(1)$.
In order to simplify the notation, let $s(r) := \mathbb{P}(x \in E_{\oplus r})$. By hypothesis we know that $s(r) \to 0$ as $r \to 0$; thus by (4.26) and (4.27) we have

$$\begin{aligned}
\lim_{r \to 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{\mathbb{E}[\#\{E_i \, : \, x \in E_{i_{\oplus r}}\}]} &= \lim_{r \to 0} \frac{\mathrm{G}(1) - \mathrm{G}(1 - s(r))}{\mathrm{G}'(1)s(r)} \\
&= \frac{1}{\mathrm{G}'(1)} \lim_{r \to 0} \frac{\mathrm{G}(1 - s(r)) - \mathrm{G}(1)}{-s(r)} \\
&= \frac{1}{\mathrm{G}'(1)} \mathrm{G}'(1) = 1. \qquad (4.28)
\end{aligned}$$

In conclusion we obtain

$$\begin{aligned}
\lim_{r \to 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} &= \lim_{r \to 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{\mathbb{E}[\#\{E_i \, : \, x \in E_{i_{\oplus r}}\}]} \frac{\mathbb{E}[\#\{E_i \, : \, x \in E_{i_{\oplus r}}\}]}{b_{d-n} r^{d-n}} \\
&\overset{(4.28)}{=} \lim_{r \to 0} \frac{\mathbb{E}[\#\{E_i \, : \, x \in E_{i_{\oplus r}}\}]}{b_{d-n} r^{d-n}}. \qquad (4.29)
\end{aligned}$$

$\square$

Corollary 4.16 *Under the same assumptions of Proposition 4.15,* (4.27) *and* (4.29) *yield*

$$\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x\in E_{\oplus r})}{b_{d-n}r^{d-n}} \tag{4.30}$$

for any $x\in\mathbb{R}^d$ where at least one of the two limits exists.

Remark 4.17 *1.* Whenever it is possible to exchange limit and integral in (4.4), we can use the fact that A is arbitrary to obtain

$$\lambda_{\Theta_n}(x)=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x\in E_{\oplus r})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]\lambda_E(x),$$

for ν^d-a.e. $x\in R^d$, where λ_{Θ_n} and λ_E are the mean densities of μ_{Θ_n} and μ_E, respectively. In particular, when E is stationary (which implies Θ_n stationary as well), $\lambda_{\Theta_n}(x)\equiv L_{\Theta_n}\in\mathbb{R}^+$ and $\lambda_E(x)\equiv L_E\in\mathbb{R}^+$, so that

$$L_{\Theta_n}=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x_0\in E_{\oplus r})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]L_E.$$

2. Let Θ_n be a random closed set as in Proposition 4.15. By (4.30) we infer that the probability that a point x belongs to the intersection of two or more enlarged sets E_i is an infinitesimal faster than r^{d-n}, i.e.

$$\lim_{r\to 0}\frac{\mathbb{P}(\exists i,j,\ i\neq j\text{ such that }x\in E_{i_{\oplus r}}\cap E_{j_{\oplus r}})}{r^{d-n}}=0. \tag{4.31}$$

In fact, denoting by $F_r(x)$ the event $\{\exists i,j,\ i\neq j\text{ such that }x\in E_{i_{\oplus r}}\cap E_{j_{\oplus r}}\}$, we have

$$\mathbf{1}_{F_r(x)}\leq\sum_i\mathbf{1}_{\{x\in E_{i\oplus r}\}}-\mathbf{1}_{\{x\in\Theta_{n\oplus r}\}},$$

so that, as (4.27) gives

$$\mathbb{E}[\sum_i\mathbf{1}_{\{x\in E_{i\oplus r}\}}]=\mathbb{E}[\Phi]\mathbb{P}(x\in E_{\oplus r}),$$

taking expectations in both sides and dividing by r^{d-n} we get that

$$0\leq\lim_{r\to 0}\frac{\mathbb{P}(F_r(x))}{r^{d-n}}\leq\lim_{r\to 0}\frac{\mathbb{E}[\Phi]\mathbb{P}(x\in E_{\oplus r})-\mathbb{E}[\Phi]\mathbb{P}(x\in E_{\oplus r})}{r^{d-n}}=0.$$

Really, another proof of (4.31), which emphasize the role of the grains E_i's, is the following:

$$
\begin{aligned}
&\mathbb{P}(x \in \bigcup_{i=1}^{\Phi}(E_{i_{\oplus r}}) && (4.32)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(x \in \bigcup_{i=1}^{n} E_{i_{\oplus r}} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(\{\omega : b_r(x) \cap \bigcup_{i=1}^{n} E_i(\omega) \neq \emptyset)\} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(\{\omega : \bigcup_{i=1}^{n}(b_r(x) \cap E_i(\omega)) \neq \emptyset)\} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(\{\omega : \exists i \in \{1, \dots, n\} \text{ such that } b_r(x) \cap E_i(\omega) \neq \emptyset)\} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(\bigcup_{i=1}^{n}\{\omega : b_r(x) \cap E_i(\omega) \neq \emptyset)\} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \mathbb{P}(\bigcup_{i=1}^{n}\{\omega : x \in E_{i_{\oplus r}}(\omega) \neq \emptyset)\} \mid \Phi = n)\mathbb{P}(\Phi = n)\\
&\overset{(i)}{=} \sum_{n=1}^{\infty} \Bigg[\sum_{i=1}^{n} \mathbb{P}(x \in E_{i_{\oplus r}} \mid \Phi = n) - \sum_{i<j} \mathbb{P}(x \in E_{i_{\oplus r}} \cap E_{j_{\oplus r}} \mid \Phi = n)\\
&\qquad + \sum_{i<j<k} \mathbb{P}(x \in E_{i_{\oplus r}} \cap E_{j_{\oplus r}} \cap E_{k_{\oplus r}} \mid \Phi = n) \; - \; \cdots \Bigg] \mathbb{P}(\Phi = n)\\
&= \sum_{n=1}^{\infty} \Bigg[n\mathbb{P}(x \in E_{\oplus r}) - \binom{n}{2}\mathbb{P}(x \in E_{i_{\oplus r}} \cap E_{j_{\oplus r}}) && (4.33)\\
&\qquad + \binom{n}{3}\mathbb{P}(x \in E_{i_{\oplus r}} \cap E_{j_{\oplus r}} \cap E_{k_{\oplus r}}) - \cdots \Bigg] \mathbb{P}(\Phi = n),
\end{aligned}
$$

where in (i) we used the inclusion-exclusion theorem.
In order to simplify the notations, let

$$
p_{i_1,\dots,i_s}(x) := \mathbb{P}(x \in E_{{i_1}_{\oplus r}} \cap \cdots \cap E_{{i_s}_{\oplus r}});
$$

thus, by (4.33) we have that

$$
\mathbb{P}(x \in \bigcup_{i=1}^{\Phi}(E_{i_{\oplus r}})) = \mathbb{E}[\Phi]\mathbb{P}(x \in E_{\oplus r}) - \sum_{n=1}^{\infty} \left[\binom{n}{2} p_{i,j} - \binom{n}{3} p_{i,j,k} + \cdots \right] \mathbb{P}(\Phi = n).
$$

By (4.30), and since the equation above holds for any discrete random variable Φ, so in particular for any deterministic $\Phi \equiv n$, we may conclude that, for any integer $s \geq 2$,

$$
\lim_{r \to 0} \frac{\mathbb{P}(x \in E_{{i_1}_{\oplus r}} \cap \cdots \cap E_{{i_s}_{\oplus r}})}{b_{d-n} r^{d-n}} = 0.
$$

Example 2. (Poisson line process.)
Now, we recall the definition of a Poisson line process, given in [64] as a simple example of applicability of the above arguments. We shall obtain the same result for the mean density of the random length measure.
Line processes are the simplest examples of fibre processes. Such random patterns can be treated directly as random sets; however, they can also be considered as point processes with constituent "points" lying not in the Euclidean space, but in the space of lines in the plane, which can be parameterized as a cylinder $\mathbf{C}^*$ in $\mathbb{R}^3$ (see [64], Ch. 8):

$$\mathbf{C}^* = \{(\cos\alpha,\ \sin\alpha,\ p) \,:\, p \in \mathbb{R},\ \alpha \in (0, \pi]\},$$

where p is the signed perpendicular distance of the line l from the origin 0 (the sign is positive if 0 lies to the left of l and negative if it lies to the right), and α is the angle between l and the x-axis, measured in an anti-clockwise direction.
"A *line process* is a random collection of lines in the plane which is locally finite, i.e. only finitely many lines hit each compact planar set. Formally it is defined as a random subset of the representation space $\mathbf{C}^*$. The process is locally finite exactly when the representing random subset is a random locally finite subset, hence a point process, on $\mathbf{C}^*$. Such point processes are particular cases of point processes on $\mathbb{R}^2$, because, as suggested by the parametrization (p, α), the cylinder can be cut and embedded as the subset $\mathbb{R} \times (0, 2\pi]$ of $\mathbb{R}^2$" ([64], p.248).
A line process $\Theta_1 = \{l_1, l_2, \ldots\}$, when regarded as a point process on $\mathbf{C}^*$, yields an *intensity measure* Λ on $\mathbf{C}^*$:

$$\Lambda(A) = \mathbb{E}[\#\{l :\ l \in \Theta_1 \cap A\}]$$

for each Borel subset A of $\mathbf{C}^*$.
A *Poisson line process* Ξ is the line process produced by a Poisson process on $\mathbf{C}^*$. Consequently it is characterized completely by its intensity measure Λ. Under the assumption of stationarity of Ξ, it follows that there exists a constant $L_\Xi > 0$ such that the intensity measure Λ of Ξ is given by

$$\Lambda(\mathrm{d}(p, \alpha)) = L_\Xi \cdot \mathrm{d}p \cdot \frac{\mathrm{d}\alpha}{2\pi};$$

besides, it is clear that the measure $\mathbb{E}[\mathcal{H}^1(\Xi \cap \cdot)]$ is motion invariant on $\mathbb{R}^2$, and so there exists a constant c such that

$$\mathbb{E}[\mathcal{H}^1(\Xi \cap A)] = c\nu^2(A)$$

for any $A \in \mathcal{B}_{\mathbb{R}^2}$.
Such a constant c can be calculated using the cylinder representation of Ξ; it is shown that $c = L_\Xi$ (see [64], p.249).

We show now that the same statement can be obtained as a consequence of (4.3).
As a matter of fact, by stationarity we know that

$$c\nu^2(A) = \lim_{r \to 0} \frac{\mathbb{P}(0 \in \Xi_{\oplus r})}{2r} \nu^2(A) \tag{4.34}$$

holds for any Borel set A which satisfies condition (4.2), so that it is sufficient to prove that

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^2(\Xi_{\oplus r} \cap A)]}{2r} = \mathbb{E}[\mathcal{H}^1(\Xi \cap A)]$$

holds for a particular fixed A. Let us choose a closed square W in $\mathbb{R}^2$ with edges P_1, P_2, P_3, P_4, and side length h.
Note that $\Theta_1 := \Xi \cap W_{\oplus 1}$ is a countably $\mathcal{H}^1$-rectifiable and compact random set and, by the absolute continuity of the expected measure, $\mathbb{E}[\mathcal{H}^1(\Xi \cap \partial W] = 0$ (so that $\mathbb{P}(\mathcal{H}^1(\Xi \cap \partial W) > 0) = 0$).
For any $\omega \in \Omega$ let us define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\Theta_1(\omega) \cap \cdot\,)}{\mathcal{H}^1(\Theta_1(\omega))}.$$

Then η is a probability measure absolutely continuous with respect to $\mathcal{H}^1$ such that

$$\eta(B_r(x)) \geq \frac{1}{\mathcal{H}^1(\Theta_1)} r \qquad \forall x \in \Theta_1(\omega),\ r \in (0,1),$$

and we may notice that $\mathcal{H}^1(l_i(\omega) \cap W_{\oplus 1}) \leq (h+2)\sqrt{2}$ for any $\omega \in \Omega$, for any i.
Let $I := \{i : l_i \cap W_{\oplus 1} \neq \emptyset\}$, and $\Phi^W := \mathrm{card}(I)$; we know that $\mathbb{E}[\Phi^W] < \infty$, so

$$\mathbb{E}[\mathcal{H}^1(\Theta_1)] = \mathbb{E}[\sum_{i\in I} \mathcal{H}^1(l_i \cap W_{\oplus 1})] \leq (h+2)\sqrt{2}\mathbb{E}[\Phi_W] < \infty.$$

The hypotheses of Theorem 4.10 are satisfied with $A = W$ and $Y = \mathcal{H}^1(\Theta_1)$, thus we obtain

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^2(\Xi_{\oplus r} \cap W)]}{2r} = \mathbb{E}[\mathcal{H}^1(\Xi \cap W)].$$

In conclusion, remembering that the number N_r of lines of Ξ hitting the ball $B_r(0)$ is a Poisson random variable with mean $2rL_\Xi$ ([64], p.250), by (4.34) we obtain

$$\begin{aligned} c &= \lim_{r\to 0} \frac{\mathbb{P}(0 \in \Xi_{\oplus r})}{2r} \\ &= \lim_{r\to 0} \frac{\mathbb{P}(\Xi \cap B_r(0) \neq \emptyset)}{2r} \\ &= \lim_{r\to 0} \frac{\mathbb{P}(N_r \geq 1)}{2r} \\ &= \lim_{r\to 0} \frac{1 - e^{-2rL_\Xi}}{2r} = L_\Xi. \end{aligned}$$

Remark 4.18 In the above example we have chosen the measure η as the restriction of the Hausdorff measure $\mathcal{H}^1$ to a suitable set containing $\Xi \cap W$. As a matter of fact, due to the stochasticity of the relevant random closed set Ξ, problems may arise in identifying a measure η needed for the application of Theorem 4.10. A proper choice of η can be made by referring to another suitable random set containing Ξ. We further clarify such procedure by the following example.

Example 3. (Segment processes.)
Let Θ_1 be a random closed set in $\mathbb{R}^2$ such that

$$\Theta_1 := \bigcup_{i=1}^{\Phi} S_i,$$

where Φ is a counting process (i.e. a positive integer valued random variable) with $\mathbb{E}[\Phi] < \infty$, and $S_1, S_2, \ldots$, are random segments independent of Φ, randomly distributed in the plane with random lengths $\mathcal{H}^1(S_i)$ in $[0, M]$.
Let us consider a realization $\Theta_1(\omega)$ and define $\eta(\cdot) := \dfrac{\mathcal{H}^1(\Theta_1(\omega) \cap \cdot)}{\mathcal{H}^1(\Theta_1(\omega))}$.
Let $x \in \Theta_1(\omega)$; then an $\bar{\imath}$ exists such that $x \in S_{\bar{\imath}}(\omega)$, and so

$$\eta(B_r(x)) = \frac{\mathcal{H}^1(\Theta_1(\omega) \cap B_r(x))}{\mathcal{H}^1(\Theta_1(\omega))} \geq \frac{\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x))}{\mathcal{H}^1(\Theta_1(\omega))}.$$

Fixed $r \in (0,1)$, observe that, if $S_{\bar{\imath}}(\omega) \cap \partial B_r(x) \neq \emptyset$, then $\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x)) \geq r$, while if $S_{\bar{\imath}}(\omega) \subseteq B_r(x)$, then $\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x)) = \mathcal{H}^1(S_{\bar{\imath}}(\omega)) \geq \mathcal{H}^1(S_{\bar{\imath}}(\omega))r$.
Suppose that $\Phi(\omega) = n$ and define

$$L(\omega) := \min_{i=1,\ldots,n} \{\mathcal{H}^1(S_i(\omega))\}.$$

We have that

$$\eta(B_r(x)) \geq \frac{\min\{1, L(\omega)\}}{\mathcal{H}^1(\Theta_1(\omega))} r, \qquad \forall x \in \Theta_1(\omega),\ r \in (0,1).$$

Thus, $\Theta_1(\omega)$ satisfies the hypotheses of Theorem 4.7.
If we want to apply Theorem 4.10, the above is not a good choice for η. In fact,

$$\frac{1}{\Gamma(\omega)} \leq \max\{\mathcal{H}^1(\Theta_1(\omega)), \frac{\mathcal{H}^1(\Theta_1(\omega))}{L(\omega)}\} =: Y(\omega),$$

and we may well have $\mathbb{E}[Y] = \infty$. In this case, a possible solution to the problem is to extend all the segments with length less than 2 (the extension can be done omothetically from the center of the segment, so that measurability of the process is preserved). In particular, for any $\omega \in \Omega$, let

$$\tilde{S}_i(\omega) = \begin{cases} S_i(\omega) & \text{if } \mathcal{H}^1(S_i(\omega)) \geq 2, \\ S_i(\omega) \text{ extended to length 2} & \text{if } \mathcal{H}^1(S_i(\omega)) < 2; \end{cases}$$

and

$$\tilde{\Theta}_1(\omega) := \bigcup_{i=1}^{\Phi(\omega)} \tilde{S}_i(\omega).$$

In this way, for every $x \in \Theta_1(\omega)$, there exists an $\bar{\imath}$ such that $x \in \tilde{S}_{\bar{\imath}}(\omega)$ with $\tilde{S}_{\bar{\imath}}(\omega) \cap \partial B_r(x) \neq \emptyset$ for any $r \in (0,1)$. If we define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\tilde{\Theta}_1(\omega) \cap \cdot)}{\mathcal{H}^1(\tilde{\Theta}_1(\omega))},$$

then

$$\eta(B_r(x)) \geq \frac{1}{\mathcal{H}^1(\tilde{\Theta}_1(\omega))} r \qquad \forall x \in \Theta_1(\omega),\ r \in (0,1),$$

and so in this case we have $Y = \mathcal{H}^1(\tilde{\Theta}_1)$, and

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathcal{H}^1(\bigcup_{i=1}^{\Phi} \tilde{S}_i)] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{\Phi} \mathcal{H}^1(\tilde{S}_i) \mid \Phi]] \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathbb{E}[\mathcal{H}^1(\tilde{S}_i) \mid \Phi = n] \mathbb{P}(\Phi = n) \leq \sum_{n=1}^{\infty} n(M+2)\mathbb{P}(\Phi = n) \\
&= (M+2)\mathbb{E}[\Phi] < \infty.
\end{aligned}$$

(The same holds whenever the S_i's are IID as S with $\mathbb{E}[\mathcal{H}^1(S)] < \infty$.)
Note the peculiar role played by the geometrical properties of the random set.

A particular segment process is the well known *stationary Poisson segment process* in $\mathbb{R}^d$ [13, 64]. In this case each segment S_i is determined by its reference point c_i, length and orientation. The c_i's are given by a stationary Poisson point process Ψ with intensity $\alpha > 0$, while length, say R, and orientation are supposed to be random and independent, with $\mathbb{E}[R] < \infty$. Then the measure $\mathbb{E}[\mu_{\Theta_1}]$ induced by the segment process is stationary, and it can be proved (see e.g. [13], p. 42, [64]) that its density is given by $\lambda(x) = \alpha\mathbb{E}[R] =: L$, for any $x \in \mathbb{R}^d$.
Clearly, the resulting random closed set is not compact and the mean number of segments which intersect a fixed bounded region is finite. Using Theorem 4.11, and because of stationarity, we know that

$$L = \lim_{r \to 0} \frac{\mathbb{P}(0 \in \Theta_{1_{\oplus r}})}{b_{d-1} r^{d-1}}.$$

Let us first consider for simplicity the particular case in which orientation and length R are both fixed. Then, denoted by K the subset of $\mathbb{R}^d$ such that a segment with reference point in K hits the ball $B_r(0)$, it is easy to see that $\nu^d(K) = b_{d-1} r^{d-1} R + b_d r^d$, and

$$\mathbb{P}(0 \in \Theta_{1_{\oplus r}}) = \mathbb{P}(\Psi(K) > 0) = 1 - e^{-\alpha \nu^d(K)},$$

so that we obtain

$$\lim_{r \to 0} \frac{1 - e^{-\alpha(b_{d-1} r^{d-1} R + b_d r^d)}}{b_{d-1} r^{d-1}} = \alpha R.$$

Note that this does not depend on orientation, so, if now we return to consider the case in which length and orientation are random, it easily follows that

$$L = \alpha\mathbb{E}[R],$$

as we expected.

Geometric processes of great interest in applications are the so called **fibre processes**. A fibre process Θ_1 is a random collection of rectifiable curves. A relevant real system which can be modelled as a fibre process is the system of vessels in tumor driven angiogenesis. Estimation of the mean length intensity of such system is useful for suggesting important methods of diagnosis and of dose response in clinical treatments.

It is clear that, as in Example 3, Theorem 4.10 can be applied also to this kind of $\mathcal{H}^1$-rectifiable random closed sets, going to consider as $\tilde{\Theta}_1$ the random closed set given by the union of suitably extended fibres. As a consequence, Proposition 4.5 holds for fibre processes, and we may obtain information about the measure $\mathbb{E}[\mu_{\Theta_1}]$, also under hypotheses of inhomogeneity of the process.

Example 4. (Boolean models.)
Another geometric process, well known in literature, is given by a inhomogeneous Boolean model of spheres (see [64]); i.e. Θ_{d-1} turns out to be a random union of spheres in $\mathbb{R}^d$, and so it may be represented as follows

$$\Theta_{d-1}(\omega) := \bigcup_i \partial B_{R_i(\omega)}(Y_i(\omega)),$$

where Y_i is a random point in $\mathbb{R}^d$, given by a Poisson point process in $\mathbb{R}^d$, and R_i is a positive random variable (e.g. $R \sim U[0, M]$). As a consequence, the mean number of balls which intersect any compact set K is finite. In order to claim that (4.2) holds, we proceed in an analogous way as in the previous example for a stationary Poisson segment process; it is clear that if Theorem 4.10 holds for a random closed set

$$\Xi_{d-1} := \bigcup_{i=1}^{\Phi} \partial B_{R_i(\omega)}(Y_i(\omega)), \tag{4.35}$$

where Φ is a positive integer valued random variable with finite expected value, and Y_i is a random point in $\mathbb{R}^d$, then the thesis follows.
Since in general $1/\mathcal{H}^{d-1}(\partial B_{R_i}(Y_i))$ has not a finite expected value, using the same approach as in the previous example, we are going to consider a suitable random set $\tilde{\Xi}_{d-1}$ containing Ξ_{d-1}.
Let $d = 2$; the case $d > 2$ follows similarly. For any $\omega \in \Omega$, let

$$B_i(\omega) = \begin{cases} B_{R_i(\omega)}(Y_i(\omega)) & \text{if } R_i(\omega) \geq \frac{1}{2}, \\ B_{R_i(\omega)}(Y_i(\omega)) \cup l_{Y_i(\omega)} & \text{if } R_i(\omega) < \frac{1}{2}, \end{cases}$$

where $l_{Y_i(\omega)}$ is a segment centered in $Y_i(\omega)$ with length 3, and

$$\tilde{\Xi}_1(\omega) := \bigcup_{i=1}^{\Phi(\omega)} \partial B_i(\omega).$$

In this way, for every $x \in \Xi_1(\omega)$, there exists an $\bar{\imath}$ such that $x \in \partial B_{\bar{\imath}}(\omega)$ with $\partial B_{\bar{\imath}}(\omega) \cap \partial B_r(x) \neq \emptyset$ for any $r \in (0, 1)$. We define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\tilde{\Xi}_1(\omega) \cap \cdot)}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

Let $r \in (0,1)$ be fixed and observe that, if $R_{\bar{\imath}}(\omega) \geq \frac{r}{2}$, then $\partial B_r(x) \cap \partial B_{R_{\bar{\imath}}(\omega)}(Y_{\bar{\imath}}(\omega)) \neq \emptyset$, and so

$$\eta(B_r(x)) \geq \frac{2r}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

On the other hand, $R_{\bar{\imath}}(\omega) < \frac{r}{2} < 1/2$, let $s := \mathrm{dist}(x, l_{Y_{\bar{\imath}}(\omega)})$, and $m := \mathcal{H}^1(l_{Y_{\bar{\imath}}(\omega)} \cap B_r(x))$; then $s^2 \leq R_{\bar{\imath}}(\omega)^2 \leq \frac{r^2}{4}$ and

$$\eta(B_r(x)) \geq \frac{m}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} = \frac{2\sqrt{r^2 - s^2}}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} \geq \frac{\sqrt{3}r}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

Hence, summarizing, we have

$$\eta(B_r(x)) \geq \frac{\sqrt{3}}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} r \qquad \forall x \in \Xi_1(\omega),\ \forall r \in (0,1),$$

and it is clear that $\mathbb{E}[\mathcal{H}^1(\tilde{\Xi}_1)] < \infty$; thus Theorem 4.10 holds for $\tilde{\Xi}_1$, with $W = \mathbb{R}^d$ and $Y := \mathcal{H}^1(\tilde{\Xi}_1)/\sqrt{3}$.

Consider now the random closed set Ξ_1 defined by (4.35), where Φ, R_i and X_i are chosen as before. It is clear that Theorem 4.10 holds for Ξ_1 as well, since $\Xi_1 \subseteq \tilde{\Xi}_1$.

Remark 4.19 If we consider

$$\Xi := \bigcup_{i=1}^{\Phi} B_{R_i}(X_i), \tag{4.36}$$

(i.e. Ξ is a Boolean model of balls in $\mathbb{R}^d$), than $\partial\Xi \subseteq \bigcup_{i=1}^{\Phi} \partial B_{R_i}(X_i)$. It is clear that Theorem 4.11 holds for $\partial\Xi$ as well.

Chapter 5

First order Steiner formulas for random closed sets

If we consider a d-dimensional random closed set Θ with $\dim_{\mathcal{H}}(\partial\Theta) = d - 1$, sufficiently regular such that $\partial\Theta$ satisfies the conditions of Theorem 4.10, then we may rephrase (4.14) as

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\partial\Theta_{\oplus r} \cap A)]}{r} = 2\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap A)]. \tag{5.1}$$

Note that $\mathbb{E}[\nu^d(\partial\Theta)] = 0$, so that, roughly speaking, we may regard (5.1) as the derivative in $r = 0$ of the expected Hausdorff measure of $\partial\Theta$ with respect to the Minkowski enlargement.
If we enlarge $\partial\Theta$ only in the complement of Θ (i.e. we consider $\Theta_{\oplus r} \setminus \Theta$), we ask if Θ satisfies a *local mean first order Steiner formula*, i.e:

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta_{\oplus r} \setminus \Theta \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap A)].$$

In this Chapter we shall give conditions about regularity of a random closed subset of $\mathbb{R}^d$ in order to satisfy a first order Steiner formula, which will be very useful in the sequel. (See also [42] for further applications.) In particular we say that a random closed set Θ in $\mathbb{R}^d$ *satisfies a first order Steiner formula* if and only if for almost every $\omega \in \Omega$, $\mathcal{H}^{d-1}(\partial\Theta(\omega)) < \infty$ and

$$\lim_{r\to 0} \frac{\mathcal{H}^d(\Theta(\omega)_{\oplus r} \setminus \Theta(\omega))}{r} = \mathcal{H}^{d-1}(\partial\Theta(\omega));$$

while we say that Θ *satisfies a mean first order Steiner formula* if and only if $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)] < \infty$ and

$$\lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \setminus \Theta)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)].$$

By a *Steiner formula*, we mean a polynomial expansion of the volume of the parallel set of a given subset of the Euclidean space $\mathbb{R}^d$. A subset A of $\mathbb{R}^d$ is said to satisfy a Steiner formula if there exist numbers $\Phi_m(A)$, $m = 1, \ldots, d$, such that, for every (sufficiently small) $r > 0$

$$\mathcal{H}^d(A_{\oplus r}) = \sum_{m=0}^{d} r^{d-m} b_{d-m} \Phi_m(A). \tag{5.2}$$

In [31] Federer introduced the class of sets with *positive reach* and showed that for these sets the Steiner formula holds.
We start by recalling the definition of set with positive reach, introduced by Federer. Let $A \subset \mathbb{R}^d$; let $\mathrm{Unp}(A)$ be the set of points having a unique projection (or footpoint) on A:

$$\mathrm{Unp}(A) := \{a \in \mathbb{R}^d \,:\, \exists!\, x \in A \text{ such that } \mathrm{dist}(x, A) = \|a - x\|\}.$$

The definition of $\mathrm{Unp}(A)$ implies the existence of a projection mapping ξ_A : $\mathrm{Unp}(A) \to A$ which assigns to $x \in \mathrm{Unp}(A)$ the unique point $\xi_A(x) \in A$ such that $\mathrm{dist}(x, A) = \|x - \xi_A(x)\|$. For a point $a \in A$ we set:

$$\mathrm{reach}(A, a) = \sup\{r > 0 \,:\, B_r(a) \subset \mathrm{Unp}(A)\}.$$

Definition 5.1 *The* reach of A *is defined by*

$$\mathrm{reach}(A) = \inf_{a \in A} \mathrm{reach}(A, a),$$

and A is said to be a set with positive reach *if* $\mathrm{reach}(A) > 0$.

It is known that for a set with positive reach the Steiner formula (5.2) applies. In fact, Theorem 5.6 in [31] shows that a local Steiner formula holds:

Theorem 5.2 *If $A \subset \mathbb{R}^d$ and $\mathrm{reach}(A) > 0$, then there exist unique Radon measures $\psi_0, \psi_1, \ldots, \psi_d$ over $\mathbb{R}^d$ such that, for $0 \le r < \mathrm{reach}(A)$,*

$$\mathcal{H}^d(\{x \,:\, \mathrm{dist}(x, A) \le r \textit{ and } \xi_A(x) \in B\}) = \sum_{m=0}^{d} r^{d-m} b_{d-m} \psi_m(B),$$

whenever B is a Borel set of $\mathbb{R}^d$.

The Radon measures $\psi_0, \psi_1, \ldots, \psi_d$ are said the *curvature measures associated with* A, and their supports are contained in A.
Note that, when $\psi_0(A), \psi_1(A), \ldots, \psi_d(A)$ are meaningful, for instance in the case A is compact, these numbers are the *total curvatures* of A, and are denoted by $\Phi_0(A), \Phi_1(A), \ldots \Phi_d(A)$, respectively. Thus, the Steiner formula (5.2) is obtained by choosing $B = A$, and the first order Steiner formula follows immediately.

Sets with positive reach include the important subclass of *convex bodies*, i.e. compact convex subsets in $\mathbb{R}^d$. Steiner formulas and curvature measures for these sets are studied in [62]. Let us also mention that an extension of Steiner type formulas to closed sets has been achieved by Hug, Last, and Weil in [37], Theorem 2.1, which involves tools from convex geometry. This result has been

used by Rataj in [59], Theorem 3, to prove a first order Steiner formula for compact unions of sets with positive reach.
In the next section, we provide an alternative proof of the first order Steiner formula for unions of sets with positive reach, based on elementary tools of measure theory, in particular on the inclusion-exclusion theorem, which seems to be more tractable when we pass to consider the stochastic case.

5.1 A first order Steiner formula for finite unions of sets with positive reach

Lemma 5.3 *[1] Let μ be a measure on $\mathcal{B}_{\mathbb{R}^d}$. Let $A_1, \dots, A_n$ be closed subsets of $\mathbb{R}^d$ such that $\mu(\partial A_i) < \infty$ $\forall i = 1, \dots, n$.*
If

$$\mu(\partial A_i \cap \partial A_j) = 0 \quad \forall i \neq j, \tag{5.3}$$

then

$$\mu(\partial(\bigcap_{i=1}^{n} A_i)) = \sum_{i=1}^{n} \mu(\partial A_i \cap \mathrm{int}(\bigcap_{j \neq i} A_j)). \tag{5.4}$$

Proof. We proceed by induction.

Step 1: (5.4) is true for $n = 2$.
$A_1 \cap A_2 = \{x : x \in A_1,\ x \in A_2\}$ can be written as disjoint union of the following sets:

- $\mathrm{int}A_1 \cap \mathrm{int}A_2$
- $\mathrm{int}A_1 \cap \partial A_2$
- $\partial A_1 \cap \mathrm{int}A_2$
- $\partial A_1 \cap \partial A_2$.

Notice that

$$\begin{aligned} x \in \mathrm{int}(A_1 \cap A_2) \quad &\Leftrightarrow \quad \exists r \text{ t.c. } B_r(x) \subset A_1 \cap A_2 \\ &\Leftrightarrow \quad \exists r \text{ t.c. } B_r(x) \subset A_1,\ B_r(x) \subset A_2 \\ &\Leftrightarrow \quad x \in \mathrm{int}A_1,\ x \in \mathrm{int}A_2, \end{aligned}$$

and so $\mathrm{int}(A_1 \cap A_2) = \mathrm{int}A_1 \cap \mathrm{int}A_2$.
Therefore we have

$$\begin{aligned} \partial(A_1 \cap A_2) &= (A_1 \cap A_2) \setminus \mathrm{int}(A_1 \cap A_2) \\ &= (\mathrm{int}A_1 \cap \partial A_2) \ \cup \ (\partial A_1 \cap \mathrm{int}A_2) \ \cup (\partial A_1 \cap \partial A_2). \end{aligned}$$

Since this is a disjoint union and by hypothesis $\mu(\partial A_1 \cap \partial A_2) = 0$, it follows that

$$\mu(\partial(A_1 \cap A_2)) = \mu(\partial A_1 \cap \mathrm{int}A_2) + \mu(\mathrm{int}A_1 \cap \partial A_2).$$

Step 2: We show that if (5.4) is true for k, then it is true also for $k+1$. Since $\partial(\bigcap_{i=1}^k A_i) \subseteq \bigcup_{i=1}^k \partial A_i$, by hypothesis (5.3) we have

$$\mu(\partial(\bigcap_{i=1}^{k} A_i) \cap \partial A_{k+1}) \leq \mu(\bigcup_{i=1}^{k} \partial A_i \cap \partial A_{k+1}) \leq \sum_{i=1}^{k} \mu(\partial A_i \cap \partial A_{k+1}) = 0.$$

Therefore

$$\begin{aligned} \mu(\partial(\bigcap_{i=1}^{k+1} A_i)) &= \mu(\partial(\bigcap_{i=1}^{k} A_i \cap A_{k+1})) \\ &= \mu(\partial(\bigcap_{i=1}^{k} A_i) \cap \mathrm{int} A_{k+1}) + \mu(\partial A_{k+1} \cap \mathrm{int} \bigcap_{i=1}^{k} A_i). \end{aligned}$$

Since $\mu(\partial A_{k+1} \cap \mathrm{int} \bigcap_{i=1}^k A_i)$ is the last term of the sum in (5.4) (with $n = k+1$), the thesis follows if and only if

$$\mu(\partial(\bigcap_{i=1}^{k} A_i) \cap \mathrm{int} A_{k+1}) = \sum_{i=1}^{k} \mu(\partial A_i \cap \mathrm{int}(A_1 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_{k+1})). \tag{5.5}$$

Let ν be the restriction of the measure μ to the set $\mathrm{int} A_{k+1}$, i.e.

$$\nu(\cdot) := \mu_{|\mathrm{int} A_{k+1}}(\cdot) = \mu(\cdot \cap \mathrm{int} A_{k+1});$$

ν is also a Borel measure and clearly

$$\nu(\partial A_i \cap \partial A_j) = 0 \qquad \forall i \neq j.$$

Applying the induction assumption to ν and using the equality $\mathrm{int}(\bigcap A_i) = \bigcap \mathrm{int} A_i$, we have that

$$\begin{aligned} \mu(\partial(\bigcap_{i=1}^{k} A_i) \cap \mathrm{int} A_{k+1}) &= \nu(\partial(\bigcap_{i=1}^{k} A_i)) \\ &= \sum_{i=1}^{k} \nu(\partial A_i \cap \mathrm{int}(\bigcap_{j \neq i} A_j)) \\ &= \sum_{i=1}^{k} \mu(\partial A_i \cap \mathrm{int}(\bigcap_{j \neq i} A_j) \cap \mathrm{int} A_{k+1}), \end{aligned}$$

which concludes the proof. □

Lemma 5.4 *[1] Let μ be a measure on $\mathcal{B}_{\mathbb{R}^d}$. Let $A_1, \ldots, A_n$ be closed subsets of $\mathbb{R}^d$ such that $\mu(\partial A_i) < \infty$ $\forall i = 1, \ldots, n$.*
If

$$\mu(\partial A_i \cap \partial A_j) = 0 \quad \forall i \neq j, \tag{5.6}$$

then

$$\mu(\partial(\bigcup_{i=1}^{n} A_i)) = \sum_{i=1}^{n} \mu(\partial A_i \cap (\bigcup_{j \neq i} A_j)^C). \tag{5.7}$$

Proof. We proceed by induction.

Step 1: (5.7) is true for $n = 2$.
$A_1 \cup A_2$ can be written as disjoint union of the following sets:

- $A_1 \cap A_2$
- $A_1 \cap A_2^C$
- $A_1^C \cap A_2$.

Each of them can be written as disjoint union of sets in the following way:

$$A_1 \cap A_2 = (\text{int}A_1 \cap \text{int}A_2) \cup (\text{int}A_1 \cap \partial A_2) \cup (\partial A_1 \cap \text{int}A_2) \cup (\partial A_1 \cap \partial A_2)$$
$$A_1 \cap A_2^C = (\partial A_1 \cap A_2^C) \cup (\text{int}A_1 \cap A_2^C)$$
$$A_1^C \cap A_2 = (\partial A_2 \cap A_1^C) \cup (\text{int}A_2 \cap A_1^C).$$

Let us notice that:

- $x \in \text{int}A_1 \Rightarrow x \in \text{int}(A_1 \cup A_2)$
- $x \in \text{int}A_2 \Rightarrow x \in \text{int}(A_1 \cup A_2)$
- $x \in \partial A_1 \cap A_2^C \Rightarrow x \notin \text{int}(A_1 \cup A_2)$
  (If $x \in \partial A_1 \cap A_2^C$, then $\forall r > 0$, $B_r(x) \cap A_1^C \neq \emptyset$ and, $\forall r < \tilde{r}$ for a certain $\tilde{r} > 0$, $B_r(x) \cap A_2 = \emptyset$ since A_2^C is open; i.e. $\nexists r_0 > 0$ such that $B_r(x) \subset A_1 \cup A_2\ \forall r < r_0$).
- $x \in \partial A_2 \cap A_1^C \Rightarrow x \notin \text{int}(A_1 \cup A_2)$
- in general, if $x \in \partial A_1 \cap \partial A_2$, it is not possible to say whether x belongs to $\text{int}(A_1 \cup A_2)$ or not.

Thus, $\partial(A_1 \cup A_2)$ is given by the following disjoint union of sets:

$$\partial(A_1 \cup A_2) = (\partial A_1 \cap A_2^C) \cup (\partial A_2 \cap A_1^C) \cup E,$$

where $E := \{x \in \partial A_1 \cap \partial A_2 : x \in \partial(A_1 \cup A_2)\}$, and in particular $E \subseteq \partial A_1 \cap \partial A_2$. Since by hypothesis $\mu(\partial A_1 \cap \partial A_2) = 0$, it follows that

$$\mu(\partial(A_1 \cup A_2)) = \mu(\partial A_1 \cap A_2^C) + \mu(A_1^C \cap \partial A_2).$$

Step 2: We show that if (5.7) is true for k, then it is true also for $k+1$. Since $\partial(\bigcup_{i=1}^k A_i) \subseteq \bigcup_{i=1}^k \partial A_i$, by (5.6) we have

$$\mu(\partial(\bigcup_{i=1}^k A_i) \cap \partial A_{k+1}) \leq \mu(\bigcup_{i=1}^k \partial A_i \cap \partial A_{k+1}) \leq \sum_{i=1}^k \mu(\partial A_i \cap \partial A_{k+1}) = 0.$$

Therefore

$$\begin{aligned} \mu(\partial(\bigcup_{i=1}^{k+1} A_i)) &= \mu(\partial(\bigcup_{i=1}^k A_i \cup A_{k+1})) \\ &= \mu(\partial(\bigcup_{i=1}^k A_i) \cap A_{k+1}^C) + \mu(\partial A_{k+1} \cap (\bigcup_{i=1}^k A_i)^C), \end{aligned}$$

Since $\mu(\partial A_{k+1}\cap(\bigcup_{i=1}^k A_i)^C)$ is the last term of the sum in (5.7) (with $n=k+1$), the thesis follows if and only if

$$\mu(\partial(\bigcup_{i=1}^k A_i)\cap A_{k+1}^C)=\sum_{i=1}^k \mu(\partial A_i\cap(\bigcup_{j\neq i} A_j)^c). \tag{5.8}$$

Let ν be the restriction of the measure μ to the set A_{k+1}^C, i.e.

$$\nu(\cdot)=\mu_{|A_{k+1}^C}(\cdot)=\mu(\cdot\cap A_{k+1}^C)\,;$$

ν is a Borel measure and clearly

$$\nu(\partial A_i\cap\partial A_j)=0\quad\forall i\neq j\,.$$

Applying the induction assumption to ν, we get

$$\begin{aligned}\mu(\partial(\bigcup_{i=1}^k A_i)\cap A_{k+1}^C) &= \nu(\partial(\bigcup_{i=1}^k A_i))\\ &= \sum_{i=1}^k \nu(\partial A_i\cap(\bigcup_{j\neq i} A_j)^C)\\ &= \sum_{i=1}^k \mu(\partial A_i\cap(\bigcup_{j\neq i} A_j)^C)\cap A_{k+1}^C),\end{aligned}$$

and so (5.8). □

In the particular case $\mu\equiv\mathcal{H}^{d-1}$, the following corollary follows:

Corollary 5.5 *Let $A_1,\dots,A_n$ be closed subsets of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial A_i)<\infty$ $\forall i=1,\dots,n$.*
If

$$\mathcal{H}^{d-1}(\partial A_i\cap\partial A_j)=0\quad\forall i\neq j,$$

then

$$\mathcal{H}^{d-1}(\partial(\bigcap_{i=1}^n A_i))=\sum_{i=1}^n \mathcal{H}^{d-1}(\partial A_i\cap\operatorname{int}(\bigcap_{j\neq i} A_j)), \tag{5.9}$$

and

$$\mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i))=\sum_{i=1}^n \mathcal{H}^{d-1}(\partial A_i\cap(\bigcup_{j\neq i} A_j)^C). \tag{5.10}$$

Definition 5.6 *We say that a compact subset S of $\mathbb{R}^d$ with $\mathcal{H}^n(S)<\infty$* admits Minkowski content *if*

$$\lim_{r\to 0}\frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n}r^{d-n}}=\mathcal{H}^n(S). \tag{5.11}$$

We remind that (see Definition 1.16) a subset S of $\mathbb{R}^d$ is *n-rectifiable* (n positive integer) if there exists a Lipschitz function mapping a bounded subset of $\mathbb{R}^d$ onto S.
The following theorem is proved in [32], p. 275.

Theorem 5.7 *If S is a closed n-rectifiable subset of $\mathbb{R}^d$, then S admits Minkowski content.*

Sufficient conditions for (5.11) are given also in [4], p. 110, in terms of countable $\mathcal{H}^n$-rectifiability.

Proposition 5.8 *[1] If A is a compact subset of $\mathbb{R}^d$ with $\operatorname{reach}(A) = r > 0$, then ∂A is $d-1$-rectifiable.*

Proof. Let $s \in (0, r)$ be fixed. Consider the sets

$$A_s := \{x \,|\, \operatorname{dist}(x, A) \leq s\},$$

$$A'_s := \{x \,|\, \operatorname{dist}(x, A) \geq s\},$$

$$\Sigma_s := \{x \,|\, \operatorname{dist}(x, A) = s\}.$$

We show that Σ_s is a $d-1$-rectifiable closed set.
It is clear that Σ_s is closed, and it is compact since A is bounded.
By Theorem 4.8 (5) in [31], $\operatorname{dist}(\cdot, A)$ is continuously differentiable on $\operatorname{int}(\operatorname{Unp}(A)\setminus A)$, which is an open set containing Σ_s; by (3) of the same theorem, we have that $\|\nabla \operatorname{dist}(\cdot, A)\| \equiv 1$ on Σ_s.
By Dini's theorem (implicit function theorem), σ_s is locally the graph of a C^1 function in $(d-1)$-variables. Since Σ_s is compact, we may claim that

$$\Sigma_s = \Sigma_s^1 \cup \cdots \cup \Sigma_s^N,$$

where $\Sigma_s^i = f_i(B_i)$ with B_i bounded subset of $\mathbb{R}^{d-1}$, and f_i Lipschitz function. Without loss of generality, we may suppose that the B_i's are disjoint balls. Let $\tilde{B} = \bigcup_{i=1}^N B_i$, and $R > 0$ be such that $B = B_R(0)$ contains $\tilde{B}$. Then, by Whitney's Extension Theorem (see for instance [29], p. 245), there exists a Lipschitz map

$$F : B \longrightarrow \mathbb{R}^d \quad \text{such that} \quad F \equiv f_i \ \text{ on } B_i, \ \forall i = i \ldots, N.$$

As a consequence, $\Sigma_s = F(\tilde{B})$, and so Σ_s is $(d-1)$-rectifiable.
By Theorem 4.8 (8) in [31], it follows that

$$|\xi_A(x) - \xi_A(y)| \leq \frac{r}{r-s}\|x - y\|, \quad \forall x, y \in A_s.$$

As a consequence ξ_A is a Lipschitz function on A_s.
Let us observe that if $x \notin A$, then $\xi_A(x) \in \partial A$. (In fact, $\xi_A(x) \in A$ since A is closed; on the other hand $\xi_A(x) \in \operatorname{int}(A)$ is a contradiction because for any $z \in \operatorname{int}(A)$ we have $\|x - z\| > \operatorname{dist}(x, A)$.) Thus, in particular, it is defined the Lipschitz map

$$\xi_A : \Sigma_s \longrightarrow \partial A.$$

We show that $\xi_A : \Sigma_s \to \partial A$ is surjective.
By Corollary 4.9 in [31], $\operatorname{reach}(A'_s) \geq s$; in particular the projection map $\xi_{A'_s}$: $A_s \to A'_s$ is defined in $A_s \setminus A$. Let $a \in \partial A$, and $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of points

$a_i \notin A$, such that $\lim_{i\to\infty} a_i = a$, with $\|a_i - a\| < s/2$ for all $i \in \mathbb{N}$. Then, by Corollary 4.9 in [31],

$$\xi_A[\xi_{A'_s}(a_i)] = \xi_A(a_i). \tag{5.12}$$

Observe that $\xi_{A'_s}(a_i) \in \partial A'_s \subseteq \Sigma_s$: since $\mathrm{dist}(a_i, A) \le \|a_i - a\| \le s/2$, then $a_i \notin A'_s$, and so $\xi_{A'_s}(a_i) \in \partial A'_s$. If by contradiction $x \in \partial A'_s$ and $\mathrm{dist}(x, A) > s$, then, by the continuity of $\mathrm{dist}(\cdot, A)$, we have $x \in \mathrm{int}(A'_s)$, i.e. $x \notin \partial A'_s$. As a consequence, $x \in \partial A'_s \Rightarrow \mathrm{dist}(x, A) = s$, that is $\partial A'_s \subseteq \Sigma_s$.
Since Σ_s is compact, there exists $y \in \Sigma_s$ such that $\xi_{A'_s}(a_i) \to y$. By the continuity of ξ_A on A_s, we have

$$\xi_A(y) = \xi_A[\lim_{i\to\infty} \xi_{A'_s}(a_i)] = \lim_{i\to\infty} \xi_A[\xi_{A'_s}(a_i)] \overset{(5.12)}{=} \lim_{i\to\infty} \xi_A(a_i) = \xi_A(a) = a,$$

i.e. ξ_A is surjective.

Summarizing, we have proved that

- there exists a bounded set $\tilde{B} \subseteq \mathbb{R}^{d-1}$ and a Lipschitz function

  $$F : \tilde{B} \longrightarrow \mathbb{R}^d$$

  such that $F(\tilde{B}) = \Sigma_s$;

- there exists a function

  $$\xi_A : \Sigma_s \longrightarrow \partial A$$

  which is surjective and Lipschitz.

Thus, the composition

$$\xi_A \circ F : \tilde{B} \longrightarrow \partial A$$

is Lipschitz, defined on a compact set, and surjective, and so ∂A is $(d-1)$-rectifiable. $\square$

A consequence of the above proposition and Theorem 5.7 is the following

Corollary 5.9 *Let A be a compact subset of $\mathbb{R}^d$ with $\mathrm{reach}(A) = r > 0$. Then ∂A admits Minkowski content.*

Lemma 5.10 *[1] Let $A_1, \dots, A_n$ be compact sets in $\mathbb{R}^d$. For every $\varepsilon > 0$ and for every $r \le \varepsilon/2$, the following inclusion holds:*

$$A_{1_{\oplus r}} \cap \ldots \cap A_{n_{\oplus r}} \subseteq (A_1 \cap \ldots \cap A_n)_{\oplus r} \;\cup\; \bigcup_{i \ne j} (\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)_{\oplus r}.$$

Proof. Let $x \in A_{1_{\oplus r}} \cap \ldots \cap A_{n_{\oplus r}}$. There exist n points $x_1, \dots, x_n$ such that

$$x_i \in A_i, \quad \|x - x_i\| = \mathrm{dist}(x, A_i) \le r, \qquad i = 1, \dots, n.$$

We may have two cases: $x \in A_1 \cap \ldots \cap A_n$, and $\exists i \in \{1, \dots, n\}$ such that $x \notin A_i$. It is easy to check that if $x \notin A_i$, then $x_i \in \partial A_i$.

First case: $x \in A_1 \cap \ldots \cap A_n$.
In this case it is clear that $x \in (A_1 \cap \ldots \cap A_n)_{\oplus r}$.

Second case: $x \notin A_i$.
We have $x_i \in \partial A_i$. If $x \notin (A_1 \cap \ldots \cap A_n)_{\oplus r}$, then $\exists j \neq i$ such that $x_i \notin A_j$. (In fact, if $\forall j \neq i$, $x_i \in A_j$, then $x_i \in A_1 \cap \ldots \cap A_n$, and so $x \in (A_1 \cap \ldots \cap A_n)_{\oplus r}$, since $\|x - x_i\| \leq r$.)
Thus, $x_i \in \partial A_i \setminus \mathrm{int} A_j$. Further, $\|x_i - x_j\| \leq \|x_i - x\| + \|x - x_j\| \leq 2r$, and so $x_i \in A_{j_{\oplus 2r}}$. Therefore $x_i \in (\partial A_i \setminus \mathrm{int} A_j \cap A_{j_{\oplus 2r}})$.
As a consequence of the above argument, either $x \in (A_1 \cap \ldots \cap A_n)_{\oplus r}$, or $\exists j \neq i$ such that

$$x \in (\partial A_i \cap A_{j_{\oplus 2r}} \setminus \mathrm{int} A_j)_{\oplus r}.$$

We conclude that

$$A_{1_{\oplus r}} \cap \ldots \cap A_{n_{\oplus r}} \subseteq (A_1 \cap \ldots \cap A_n)_{\oplus r} \;\cup\; \bigcup_{i \neq j} (\partial A_i \cap A_{j_{\oplus 2r}} \setminus \mathrm{int} A_j)_{\oplus r};$$

since $2r \leq \varepsilon$, the thesis follows. □

We are now ready to prove the main result of this section, claimed in Theorem 5.12. In order to make clearer the idea of the proof, we consider first the particular case of the union of two sets with positive reach.

Proposition 5.11 *If A and B are two compact subsets of $\mathbb{R}^d$ with positive reach such that*

(i) $\mathrm{reach}(A \cap B) > 0$,

(ii) $\mathcal{H}^{d-1}(\partial A \cap \partial B) = 0$,

then

$$\lim_{r \to 0} \frac{\mathcal{H}^d((A \cup B)_{\oplus r}) - \mathcal{H}^d(A \cup B)}{r} = \mathcal{H}^{d-1}(\partial(A \cup B)).$$

Proof. Let

$$R := \min\{\mathrm{reach}(A), \mathrm{reach}(B), \mathrm{reach}(A \cap B)\};$$

then $R > 0$ and, $\forall \rho < R$,

$$\mathcal{H}^d(A_{\oplus \rho}) = \mathcal{H}^d(A) + \rho \mathcal{H}^{d-1}(\partial A) + \sum_{i=0}^{d-2} \rho^{d-i} b_{d-i} \Phi_i(A)$$

$$\mathcal{H}^d(B_{\oplus \rho}) = \mathcal{H}^d(B) + \rho \mathcal{H}^{d-1}(\partial B) + \sum_{i=0}^{d-2} \rho^{d-i} b_{d-i} \Phi_i(B)$$

$$\mathcal{H}^d((A \cap B)_{\oplus \rho}) = \mathcal{H}^d(A \cap B) + \rho \mathcal{H}^{d-1}(\partial(A \cap B)) + \sum_{i=0}^{d-2} \rho^{d-i} b_{d-i} \Phi_i(A \cap B).$$

We remind that

- $\mathcal{H}^d(A \cup B) = \mathcal{H}^d(A) + \mathcal{H}^d(B) - \mathcal{H}^d(A \cap B)$,
- $(A \cup B)_{\oplus r} = A_{\oplus r} \cup B_{\oplus r}$,
- $(A \cap B)_{\oplus r} \subseteq A_{\oplus r} \cap B_{\oplus r}$

In particular, by Lemma 5.10, we may claim that, $\forall\varepsilon > 0$ fixed, $\forall r \le \varepsilon/2$,

$$\mathcal{H}^d(A_{\oplus r} \cap B_{\oplus r}) \ge \mathcal{H}^d((A \cap B)_{\oplus r}) \tag{5.13}$$

$$\mathcal{H}^d(A_{\oplus r}\cap B_{\oplus r}) \le \mathcal{H}^d((A\cap B)_{\oplus r})+\mathcal{H}^d((\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r})+\mathcal{H}^d((\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B)_{\oplus r}). \tag{5.14}$$

Thus, it follows that, for any $r < R$, $r \le \varepsilon/2$,

$$\begin{aligned}
\mathcal{H}^d((A\cup B)_{\oplus r}) &= \mathcal{H}^d(A_{\oplus r} \cup B_{\oplus r}) \\
&= \mathcal{H}^d(A_{\oplus r}) + \mathcal{H}^d(B_{\oplus r}) - \mathcal{H}^d(A_{\oplus r} \cap B_{\oplus r}) \\
&\overset{(5.13)}{\le} \mathcal{H}^d(A_{\oplus r}) + \mathcal{H}^d(B_{\oplus r}) - \mathcal{H}^d((A \cap B)_{\oplus r}) \\
&= \mathcal{H}^d(A) + r\mathcal{H}^{d-1}(\partial A) \\
&\quad + \mathcal{H}^d(B) + r\mathcal{H}^{d-1}(\partial B) \\
&\quad - \mathcal{H}^d(A \cap B) - r\mathcal{H}^{d-1}(\partial(A \cap B)) \\
&\quad + \sum_{i=0}^{d-2} r^{d-i} b_{d-i}[\Phi_i(A) + \Phi_i(B) - \Phi_i(A \cap B)].
\end{aligned}$$

$\mathcal{H}^{d-1}(\partial A\cap\partial B) = 0$ implies that $\mathcal{H}^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A\cap B^C)+\mathcal{H}^{d-1}(\partial A\cap \mathrm{int}B)$, (the same for $\mathcal{H}^{d-1}(\partial B)$), and by Corollary 5.5 we have

$$\begin{aligned}
&\mathcal{H}^{d-1}(\partial A) + \mathcal{H}^{d-1}(\partial B) - \mathcal{H}^{d-1}(\partial(A \cap B)) \\
&= \mathcal{H}^{d-1}(\partial A) + \mathcal{H}^{d-1}(\partial B) - \mathcal{H}^{d-1}(\partial A \cap \mathrm{int}B) - \mathcal{H}^{d-1}(\mathrm{int}A \cap \partial B) \\
&= \mathcal{H}^{d-1}(\partial A \cap B^C) + \mathcal{H}^{d-1}(\partial B \cap A^C) \\
&= \mathcal{H}^{d-1}(\partial(A \cup B)).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\frac{\mathcal{H}^d((A \cup B)_{\oplus r}) - \mathcal{H}^d(A \cup B)}{r} \le \\
&\qquad \mathcal{H}^{d-1}(\partial(A \cup B)) + \sum_{i=0}^{d-2} r^{d-i-1} b_{d-i}[\Phi_i(A) + \Phi_i(B) - \Phi_i(A \cap B)].
\end{aligned}$$

By taking the lim sup for r tending to 0, we obtain

$$\limsup_{r\to 0} \frac{\mathcal{H}^d((A \cup B)_{\oplus r}) - \mathcal{H}^d(A \cup B)}{r} \le \mathcal{H}^{d-1}(\partial(A \cup B)). \tag{5.15}$$

By proceeding in a similar way, we have that

$$\begin{aligned}
\mathcal{H}^d((A\cup B)_{\oplus r}) &= \mathcal{H}^d(A_{\oplus r}\cup B_{\oplus r})\\
&= \mathcal{H}^d(A_{\oplus r})+\mathcal{H}^d(B_{\oplus r})-\mathcal{H}^d(A_{\oplus r}\cap B_{\oplus r})\\
&\overset{(5.14)}{\geq} \mathcal{H}^d(A_{\oplus r})+\mathcal{H}^d(B_{\oplus r})-\mathcal{H}^d((A\cap B)_{\oplus r})\\
&\quad -\mathcal{H}^d((\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r})-\mathcal{H}^d((\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B)_{\oplus r})\\
&= \mathcal{H}^d(A)+r\mathcal{H}^{d-1}(\partial A)\\
&\quad +\mathcal{H}^d(B)+r\mathcal{H}^{d-1}(\partial B)\\
&\quad -\mathcal{H}^d(A\cap B)-r\mathcal{H}^{d-1}(\partial(A\cap B))\\
&\quad +\sum_{i=0}^{d-2} r^{d-i}b_{d-i}[\Phi_i(A)+\Phi_i(B)-\Phi_i(A\cap B)]\\
&\quad -\mathcal{H}^d((\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r})-\mathcal{H}^d((\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B)_{\oplus r}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\mathcal{H}^d((A\cup B)_{\oplus r})-\mathcal{H}^d(A\cup B)}{r} &\geq \mathcal{H}^{d-1}(\partial(A\cup B))\\
&\quad +\sum_{i=0}^{d-2} r^{d-i-1}b_{d-i}[\Phi_i(A)+\Phi_i(B)-\Phi_i(A\cap B)]\\
&\quad -\frac{\mathcal{H}^d((\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r})}{r}-\frac{\mathcal{H}^d((\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B)_{\oplus r})}{r}
\end{aligned}$$

We may observe that $\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A$ is closed and $d-1$-rectifiable, since it is a subset of ∂B which is $d-1$-rectifiable by Proposition 5.8. The same holds for $\partial A\cap B_{\oplus\varepsilon}\setminus\mathrm{int}B$, as well. As a consequence, by Theorem 5.7, we may claim that

$$\lim_{r\to 0}\frac{\mathcal{H}^d((\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r})}{r}=2\mathcal{H}^{d-1}(\partial B\cap A_{\oplus\varepsilon}\setminus \mathrm{int}A),$$

$$\lim_{r\to 0}\frac{\mathcal{H}^d((\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B)_{\oplus r})}{r}=2\mathcal{H}^{d-1}(\partial A\cap B_{\oplus\varepsilon}\setminus \mathrm{int}B).$$

By taking the lim inf for r going to 0 we obtain

$$\begin{aligned}
\liminf_{r\to 0}\frac{\mathcal{H}^d((A\cup B)_{\oplus r})-\mathcal{H}^d(A\cup B)}{r} &\geq \mathcal{H}^{d-1}(\partial(A\cup B))\\
&\quad -2[\mathcal{H}^{d-1}(\partial B\cap A_{\oplus\varepsilon}\setminus\mathrm{int}A)+\mathcal{H}^{d-1}(\partial A\cap B_{\oplus\varepsilon}\setminus\mathrm{int}B)]. \qquad (5.16)
\end{aligned}$$

Observe now that $\{A_{\oplus\varepsilon}\setminus\mathrm{int}A\}_\varepsilon\downarrow\partial A$, and so

$$\lim_{\varepsilon\to 0}\mathcal{H}^{d-1}(\partial B\cap A_{\oplus\varepsilon}\setminus\mathrm{int}A)=\mathcal{H}^{d-1}(\partial B\cap\partial A)\overset{(ii)}{=}0.$$

Finally, by taking the limit for ε tending to 0 in (5.16), we have

$$\liminf_{r\to 0}\frac{\mathcal{H}^d((A\cup B)_{\oplus r})-\mathcal{H}^d(A\cup B)}{r}\geq\mathcal{H}^{d-1}(\partial(A\cup B)). \qquad (5.17)$$

By (5.17) and (5.15) the thesis follows. □

Let us consider now the case of a finite union.

Theorem 5.12 *[1] If $A_1, \ldots, A_n$ are compact subsets of $\mathbb{R}^d$ with positive reach and such that*

(i) for every subset of indices $I \subset \{1, 2, \ldots, n\}$ the set $\bigcap_{i \in I} A_i$ has positive reach,

(ii) $\mathcal{H}^{d-1}(\partial A_i \cap \partial A_j) = 0$, $\forall i \neq j$,

then

$$\lim_{r \to 0} \frac{\mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) - \mathcal{H}^d(\bigcup_{i=1}^n A_i)}{r} = \mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)).$$

Proof. Let

$$R = \min\{\mathrm{reach}(\bigcap_{i \in I} A_i) \, : \, I \subset \{1, \ldots, n\}\}.$$

We know that, $\forall r < R$,

$$\mathcal{H}^d((A_i \cap \ldots \cap A_k)_{\oplus r}) = \mathcal{H}^d(A_i \cap \ldots \cap A_k) + r\mathcal{H}^{d-1}(\partial(A_i \cap \ldots \cap A_k)) + \sum_{i=0}^{d-2} r^{d-i} b_{d-i} \Phi_i(A_i \cap \ldots \cap A_k). \quad (5.18)$$

Further, by Lemma 5.10, for any $\varepsilon > 0$ and $\forall r \leq \varepsilon/2$, we have

$$\mathcal{H}^d(A_{1_{\oplus r}} \cap \ldots \cap A_{k_{\oplus r}}) \geq \mathcal{H}^d((A_1 \cap \ldots \cap A_k)_{\oplus r}) \quad (5.19)$$

$$\mathcal{H}^d(A_{1_{\oplus r}} \cap \ldots \cap A_{k_{\oplus r}}) \leq \mathcal{H}^d((A_1 \cap \ldots \cap A_k)_{\oplus r}) + \sum_{i \neq j} \mathcal{H}^d((\partial A_i \cap A_{j_{\oplus \varepsilon}} \setminus \mathrm{int} A_j)_{\oplus r}) \quad (5.20)$$

In order to simplify the notations, let

$$E_{i,j} := (\partial A_i \cap A_{j_{\oplus \varepsilon}} \setminus \mathrm{int} A_j)_{\oplus r}, \qquad i, j \in \{1, \ldots, n\}.$$

It follows that, for any $r < R$, $r \leq \varepsilon/2$,

$$
\begin{aligned}
& \mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) \\
= \quad & \mathcal{H}^d(\bigcup_{i=1}^n A_{i_{\oplus r}}) \\
= \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) - \sum_{i<j} \mathcal{H}^d(A_{i_{\oplus r}} \cap A_{j_{\oplus r}}) + \sum_{i<j<k} \mathcal{H}^d(A_{i_{\oplus r}} \cap A_{j_{\oplus r}} \cap A_{k_{\oplus r}}) \\
& \qquad + \ldots + (-1)^{n+1} \mathcal{H}^d(A_{1_{\oplus r}} \cap \ldots \cap A_{n_{\oplus r}}) \\
\overset{(5.19),(5.20)}{\leq} \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) \\
& \quad - \sum_{i<j} \mathcal{H}^d((A_i \cap A_j)_{\oplus r}) \\
& \quad + \sum_{i<j<k} [\mathcal{H}^d((A_i \cap A_j \cap A_k)_{\oplus r}) + \mathcal{H}^d(E_{i,j}) + \mathcal{H}^d(E_{i,k}) \\
& \qquad\qquad + \mathcal{H}^d(E_{j,i}) + \mathcal{H}^d(E_{j,k}) + \mathcal{H}^d(E_{k,i}) + \mathcal{H}^d(E_{k,j})] \\
& \quad - \sum_{i<j<k<l} \mathcal{H}^d((A_i \cap A_j \cap A_k \cap A_l)_{\oplus r}) \\
& \quad + \cdots \\
= \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) - \sum_{i<j} \mathcal{H}^d((A_i \cap A_j)_{\oplus r}) + \sum_{i<j<k} \mathcal{H}^d((A_i \cap A_j \cap A_k)_{\oplus r}) \\
& \qquad + \ldots + (-1)^{n+1} \mathcal{H}^d((A_1 \cap \ldots \cap A_n)_{\oplus r}) \qquad (5.21) \\
& \quad + \sum_{i<j<k} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{k,j})] \\
& \quad + \sum_{i<j<k<l<m} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{m,l})] \\
& \quad + \cdots
\end{aligned}
$$

We observe that, by (5.18),

$$\sum_{i=1}^{n}\mathcal{H}^d(A_{i_{\oplus r}}) - \sum_{i<j}\mathcal{H}^d((A_i\cap A_j)_{\oplus r}) + \sum_{i<j<k}\mathcal{H}^d((A_i\cap A_j\cap A_k)_{\oplus r})$$
$$+\ldots+(-1)^{n+1}\mathcal{H}^d(A_1\cap\ldots\cap A_n)_{\oplus r})$$
$$= \sum_{i=1}^{n}\left[\mathcal{H}^d(A_i) + r\mathcal{H}^{d-1}(\partial A_i) + \sum_{m=0}^{d-2} r^{d-m} b_{d-m}\Phi_m(A_i)\right]$$
$$-\sum_{i<j}\left[\mathcal{H}^d(A_i\cap A_j) + r\mathcal{H}^{d-1}(\partial(A_i\cap A_j)) + \sum_{m=0}^{d-2} r^{d-m} b_{d-m}\Phi_m(A_i\cap A_j)\right]$$
$$+\sum_{i<j<k}\left[\mathcal{H}^d(A_i\cap A_j\cap A_k) + r\mathcal{H}^{d-1}(\partial(A_i\cap A_j\cap A_k)) + \sum_{m=0}^{d-2} r^{d-m} b_{d-m}\Phi_m(A_i\cap A_j\cap A_k)\right]$$
$$+\ldots+(-1)^{n+1}\left[\mathcal{H}^d(A_1\cap\ldots\cap A_n) + r\mathcal{H}^{d-1}(\partial(A_1\cap\ldots\cap A_n)) + \sum_{m=0}^{d-2} r^{d-m} b_{d-m}\Phi_m(A_1\cap\ldots\cap A_n)\right]$$
$$= \sum_{i=1}^{n}\mathcal{H}^d(A_i) - \sum_{i<j}\mathcal{H}^d(A_i\cap A_j) + \sum_{i<j<k}\mathcal{H}^d(A_i\cap A_j\cap A_k) + \ldots + (-1)^{n+1}\mathcal{H}^d(A_1\cap\ldots\cap A_n) \tag{5.22}$$
$$+ r\left[\sum_{i=1}^{n}\mathcal{H}^{d-1}(\partial A_i) - \sum_{i<j}\mathcal{H}^{d-1}(\partial(A_i\cap A_j)) + \sum_{i<j<k}\mathcal{H}^{d-1}(\partial(A_i\cap A_j\cap A_k)) + \ldots + (-1)^{n+1}\mathcal{H}^{d-1}(\partial(A_1\cap\ldots\cap A_n))\right] \tag{5.23}$$
$$+\sum_{m=0}^{d-2} r^{d-m} b_{d-m}\left[\sum_{i=1}^{n}\Phi_m(A_i) - \sum_{i<j}\Phi_m(A_i\cap A_j) + \sum_{i<j<k}\Phi_m(A_i\cap A_j\cap A_k) + \ldots + (-1)^{n+1}\Phi_m(A_1\cap\ldots\cap A_n)\right] \tag{5.24}$$

Now, we notice that:

- (5.22) $= \mathcal{H}^d(\bigcup_{i=1}^{n} A_i)$.
- By relation (5.9) on the measure of the boundary of an intersection of sets,

we have

$$
\begin{aligned}
\frac{(5.23)}{r} \quad = \quad & \sum_{i=1}^{n} \mathcal{H}^{d-1}(\partial A_i) \\
& \quad - \sum_{i<j} [\mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int} A_j) + \mathcal{H}^{d-1}(\partial A_j \cap \mathrm{int} A_i)] \\
& \quad + \sum_{i<j<k} [\mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int}(A_j \cap A_k)) + \mathcal{H}^{d-1}(\partial A_j \cap \mathrm{int}(A_i \cap A_k)) \\
& \qquad\qquad\qquad\qquad + \mathcal{H}^{d-1}(\partial A_k \cap \mathrm{int}(A_i \cap A_j))] \\
& \quad - \cdots \\
= \quad & \sum_{i=1}^{n} \Bigg[\mathcal{H}^{d-1}(\partial A_i) - \sum_{j \neq i} \mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int} A_j) \qquad (5.25) \\
& \qquad + \sum_{j,k \neq i; j<k} \mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int}(A_j \cap A_k)) - \cdots \Bigg].
\end{aligned}
$$

Let us observe that

$$
\begin{aligned}
& \mathcal{H}^{d-1}(\partial A_i \cap (\bigcup_{j \neq i} \mathrm{int} A_j)) \\
& = \mathcal{H}^{d-1}(\bigcup_{j \neq i} (\partial A_i \cap \mathrm{int} A_j)) \\
& = \sum_{j \neq i} \mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int} A_j) - \sum_{j,k \neq i; j<k} \mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int} A_j \cap \mathrm{int} A_k) \\
& \quad + \sum_{j,k,l \neq i; j<k<l} \mathcal{H}^{d-1}(\partial A_i \cap \mathrm{int} A_j \cap \mathrm{int} A_k \cap \mathrm{int} A_l) \ - \ \ldots
\end{aligned}
$$

Therefore,

$$
(5.25) = \sum_{i=1}^{n} [\mathcal{H}^{d-1}(\partial A_i) - \mathcal{H}^{d-1}(\partial A_i \cap (\bigcup_{j \neq i} \mathrm{int} A_j))].
$$

Since ∂A_i is given by the disjoint union of $\partial A_i \cap (\bigcup_{j \neq i} \mathrm{int} A_j)$, $\partial A_i \cap \partial(\bigcup_{j \neq i} A_j)$, and $\partial A_i \cap (\bigcup_{j \neq i} A_j)^C$, and from assumption (ii) of the theorem it follows that $\mathcal{H}^{d-1}(\partial A_i \cap \partial(\bigcup_{j \neq i} A_j)) = 0$. Summarizing we have that

$$
\frac{(5.23)}{r} = \sum_{i=1}^{n} \mathcal{H}^{d-1}(\partial A_i \cap (\bigcup_{j \neq i} A_j)^C) \stackrel{(5.10)}{=} \mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^{n} A_i)).
$$

- $(5.24) = o(r)$.

According to the above considerations, from (5.21) we obtain

$$
\begin{aligned}
&\mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) \\
&\leq \mathcal{H}^d(\bigcup_{i=1}^n A_i) + r\mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)) + o(r) \\
&\quad + \sum_{i<j<k} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{k,j})] \\
&\quad + \sum_{i<j<k<l<m} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{m,l})] \\
&\quad + \ \ldots
\end{aligned}
$$

As in the previous proposition, we may claim that every $E_{i,j}$ is a $d-1$-rectifiable closed set, and in particular it admits Minkowski content. By taking the $\limsup$ as r tends to zero we have

$$
\begin{aligned}
&\limsup_{r\to 0} \frac{\mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) - \mathcal{H}^d(\bigcup_{i=1}^n A_i)}{r} \\
&\leq \ \mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)) \\
&\quad + 2\sum_{i<j<k} [\mathcal{H}^{d-1}(\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j) + \ldots + \mathcal{H}^{d-1}(\partial A_k \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)] \\
&\quad + 2\sum_{i<j<k<l<m} [\mathcal{H}^{d-1}(\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j) + \ldots + \mathcal{H}^{d-1}(\partial A_m \cap A_{l_{\oplus\varepsilon}} \setminus \mathrm{int} A_l)] \\
&\quad + \ \ldots
\end{aligned}
$$

for every $\varepsilon > 0$. In a similar way to the previous proposition, by taking the limit for ε going to 0, by assumption (ii), we obtain

$$
\limsup_{r\to 0} \frac{\mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) - \mathcal{H}^d(\bigcup_{i=1}^n A_i)}{r} \leq \ \mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)). \tag{5.26}
$$

For an estimation of the $\liminf$ we proceed in analogous way.

$$
\begin{aligned}
& \mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) \\
= \quad & \mathcal{H}^d(\bigcup_{i=1}^n A_{i_{\oplus r}}) \\
= \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) - \sum_{i<j} \mathcal{H}^d(A_{i_{\oplus r}} \cap A_{j_{\oplus r}}) + \sum_{i<j<k} \mathcal{H}^d(A_{i_{\oplus r}} \cap A_{j_{\oplus r}} \cap A_{k_{\oplus r}}) \\
& \qquad + \ldots + (-1)^{n+1} \mathcal{H}^d(A_{1_{\oplus r}} \cap \ldots \cap A_{n_{\oplus r}}) \\
\overset{(5.19),(5.20)}{\geq} \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) \\
& \quad - \sum_{i<j} [\mathcal{H}^d((A_i \cap A_j)_{\oplus r}) + \mathcal{H}^d(E_{i,j}) + \mathcal{H}^d(E_{j,i})] \\
& \quad + \sum_{i<j<k} \mathcal{H}^d((A_i \cap A_j \cap A_k)_{\oplus r}) \\
& \quad - \sum_{i<j<k<l} [\mathcal{H}^d((A_i \cap A_j \cap A_k \cap A_l)_{\oplus r}) + \mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{l,k})] \\
& \quad + \cdots \\
= \quad & \sum_{i=1}^n \mathcal{H}^d(A_{i_{\oplus r}}) - \sum_{i<j} \mathcal{H}^d((A_i \cap A_j)_{\oplus r}) + \sum_{i<j<k} \mathcal{H}^d((A_i \cap A_j \cap A_k)_{\oplus r}) \\
& \qquad + \ldots + (-1)^{n+1} \mathcal{H}^d((A_1 \cap \ldots \cap A_n)_{\oplus r}) \\
& \quad - \sum_{i<j} [\mathcal{H}^d(E_{i,j}) + \mathcal{H}^d(E_{j,i})] \\
& \quad - \sum_{i<j<k<l} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{l,k})] \\
& \quad - \cdots \\
= \quad & \mathcal{H}^d(\bigcup_{i=1}^n A_i) + r\mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)) + o(r) \\
& \quad - \sum_{i<j} [\mathcal{H}^d(E_{i,j}) + \mathcal{H}^d(E_{j,i})] \\
& \quad - \sum_{i<j<k<l} [\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{l,k})] \\
& \quad - \cdots
\end{aligned}
$$

By taking the lim inf as r tends to 0, before, and the limit as ε tends to 0, then, we obtain

$$\liminf_{r\to 0} \frac{\mathcal{H}^d((\bigcup_{i=1}^n A_i)_{\oplus r}) - \mathcal{H}^d(\bigcup_{i=1}^n A_i)}{r} \geq \mathcal{H}^{d-1}(\partial(\bigcup_{i=1}^n A_i)). \tag{5.27}$$

(5.26) and (5.27) imply the thesis. □

5.2 A first order Steiner formula for sets with Lipschitz boundary

In this section we prove that sets with Lipschitz boundary satisfy a first order Steiner formula.

5.2.1 Distance function and sets of finite perimeter

Let us recall some basic definitions and results concerning sets of finite perimeter, which will be an important ingredient here (for a complete treatment we refer to [4] and references therein).

Let E be a subset of $\mathbb{R}^d$; we denote by $d_E : \mathbb{R}^d \to \mathbb{R}$ the *signed distance function*, so defined:

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, E^C).$$

Note that $d_{E^C}(x) = -d_E(x)$, and, in particular,

$$\{x : x \in \partial E_{\oplus r}\} = \{x : \mathrm{dist}(x, \partial E) \le r\} = \{x : |d_E(x)| \le r\}. \tag{5.28}$$

It is well known that d_E is a Lipschitz function, almost everywhere differentiable in $\mathbb{R}^d$, with $|\nabla d_E(x)| = 1$ for any differentiability point (see [3], p. 11).

We remind that, given a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^k$ and an n-dimensional domain $E \subseteq \mathbb{R}^d$ with $n \ge k$, by Coarea formula we may reduce an integral on E to a double integral, where the first integral is computed on the level set $E \cap \{f = t\}$ with respect to $\mathcal{H}^{n-k}$, and the result is integrated in t with respect to ν^k. (For further details see [4], § 2.12.)
In the particular case $k = 1$ and $n = d$, for any Borel function $g : \mathbb{R}^d \to [0, \infty]$, it holds ([4], (2.74)):

$$\int_E g(x)|\nabla f(x)|\mathrm{d}x = \int_{-\infty}^{+\infty} \left(\int_{E \cap \{f=t\}} g(y)\mathrm{d}\mathcal{H}^{d-1}(y) \right) \mathrm{d}t. \tag{5.29}$$

Thus, we may notice that, by choosing $g(x) \equiv 1$ and $f = d_E$ in the above equation, we have

$$\begin{aligned}
\mathcal{H}^d(\partial E_{\oplus r}) &= \int_{\partial E_{\oplus r}} \mathrm{d}x \\
&= \int_{\partial E_{\oplus r}} |\nabla d_E(x)|\mathrm{d}x \\
&\overset{(5.29)}{=} \int_{-\infty}^{+\infty} \mathcal{H}^{d-1}(\{x : x \in \partial E_{\oplus r}\} \cap \{x : d_E(x) = t\})\mathrm{d}t \\
&\overset{(5.28)}{=} \int_{-r}^{r} \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t
\end{aligned} \tag{5.30}$$

Similarly we have

$$\mathcal{H}^d(E_{\oplus r} \setminus E) = \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t. \tag{5.31}$$

Definition 5.13 *Let $(X, \mathcal{E})$ be a measure space. If μ is a measure, we define its* total variation $|\mu|$ *for every $E \in \mathcal{E}$ as follows:*

$$|\mu(E)| := \sup \left\{ \sum_{i=0}^{\infty} |\mu(E_i)| \ : \ E_i \in \mathcal{E} \text{ pairwise disjoint}, \ E = \bigcup_{i=0}^{\infty} E_i \right\}.$$

It follows that $|\mu|$ is a positive finite measure ([4], p. 4), and μ is absolutely continuous with respect to $|\mu|$. As a consequence the following holds ([4], p. 14):

Proposition 5.14 (Polar decomposition) *Let μ be a $\mathbb{R}^d$-valued measure on the measure space $(X, \mathcal{E})$; then there exists a unique S^{d-1}-valued function $f \in [L^1(X, |\mu|)]^d$ (S^{d-1} is the unit sphere in $\mathbb{R}^d$) such that $\mu = f|\mu|$.*

Definition 5.15 (Sets of finite perimeter) *Let E be a measurable set of $\mathbb{R}^d$. For any open set $\mathcal{D} \subseteq \mathbb{R}^d$ the* perimeter of E in $\mathcal{D}$*, denoted by $\mathcal{P}(E, \mathcal{D})$, is the variation of the characteristic function χ_E in $\mathcal{D}$, i.e.*

$$\mathcal{P}(E, \mathcal{D}) := \sup \left\{ \int_E \mathrm{div}\phi \ \mathrm{d}x \ : \ \phi \in [C_c^1(\mathcal{D})]^d, \ \|\phi\|_\infty \leq 1 \right\}.$$

We say that E is a set of finite perimeter in $\mathcal{D}$ *if $\mathcal{P}(E, \mathcal{D}) < \infty$.*

In the sequel we will write $\mathcal{P}(E)$ instead of $\mathcal{P}(E, \mathbb{R}^d)$

The theory of sets of finite perimeter is closely connected to the theory of functions of bounded variation (see [4]). We denote by $BV(\mathcal{D})$ the family of functions of bounded variations in the open set $\mathcal{D}$.
In particular, if $\nu^d(E \cap \mathcal{D})$ is finite, then $\chi_E \in L^1(\mathcal{D})$, and we can say that E has finite perimeter in $\mathcal{D}$ if and only if $\chi_E \in BV(\mathcal{D})$, and that $\mathcal{P}(E, \mathcal{D})$ coincides with $|D\chi_E|$, the total variation in $\mathcal{D}$ of the distributional derivative of χ_E.

Theorem 5.16 *([4], p. 143) For any set E of finite perimeter in $\mathcal{D}$ the distributional derivative $D\chi_E$ is an $\mathbb{R}^d$-valued finite Radon measure in $\mathcal{D}$. Moreover, $\mathcal{P}(E, \mathcal{D}) = |D\chi_E|(\mathcal{D})$ and a generalized Gauss-Green formula holds:*

$$\int_E \mathrm{div}\phi \ \mathrm{d}x = - \int_{\mathcal{D}} \langle \nu_E, \phi \rangle \ \mathrm{d}|D\chi_E| \qquad \forall \phi \in [C_c^1(\mathcal{D})]^d,$$

where ν_E is the inner unit normal to E, and $D\chi_E = \nu_E |D\chi_E|$ is the polar decomposition of $D\chi_E$.

In dealing with sets with finite measure, it is typically used *convergence in measure*; we recall that $\{E_i\}$ converges to E in measure in $\mathcal{D}$ if $\nu^d(E_i \Delta E)$ converges to 0 as $i \to \infty$ (Δ is the symmetric difference of sets). In many applications it will also be useful the so-called *local* convergence in measure, i.e. convergence in measure in any open set $A \subset \mathcal{D}$. These convergences correspond to $L^1(\mathcal{D})$ and $L^1_{\mathrm{loc}}(\mathcal{D})$ convergences of the characteristic functions.

The following **properties of perimeter** hold:

Theorem 5.17 *([4], p. 144)*

(a) The function $\mathcal{D} \mapsto \mathcal{P}(E, \mathcal{D})$ is the restriction to open sets of a Borel measure in $\mathbb{R}^d$.

(b) $E \mapsto \mathcal{P}(E, \mathcal{D})$ is lower semicontinuous with respect to local convergence in measure in $\mathcal{D}$.

(c) $E \mapsto \mathcal{P}(E, \mathcal{D})$ is local, i.e. $\mathcal{P}(E, \mathcal{D}) = \mathcal{P}(F, \mathcal{D})$ whenever $\nu^d(\mathcal{D} \cap (E \Delta F)) = 0$.

(d) $\mathcal{P}(E, \mathcal{D}) = \mathcal{P}(E^C, \mathcal{D})$ and

$$\mathcal{P}(E \cup F, \mathcal{D}) + \mathcal{P}(E \cap F, \mathcal{D}) \leq \mathcal{P}(E, \mathcal{D}) + \mathcal{P}(F, \mathcal{D}).$$

We recall that a set A has *Lipschitz boundary* if locally ∂A can be seen as the graph of a Lipschitz function.
In particular, the following result concerning the topological boundary can be proved.

Proposition 5.18 *([4], p. 159) Any open set $A \subset \mathbb{R}^d$ satisfying $\mathcal{H}^{d-1}(\partial A) < \infty$ has finite perimeter in $\mathbb{R}^d$ and $|D\chi_A| \leq \mathcal{H}^{d-1}_{|\partial A}$.*
Equality holds if A has Lipschitz boundary.

Remark 5.19 By the above proposition, and by property (c) of Theorem 5.17, if E is a closed set with Lipschitz boundary, then $|D\chi_E| = \mathcal{H}^{d-1}(\partial E)$; by Theorem 5.16 we have that $\mathcal{P}(E) = \mathcal{H}^{d-1}(\partial E)$.

5.2.2 A general theorem for the first order Steiner formula

Lemma 5.20 *Let $\{A_n\}$ and $\{B_n\}$ be sequences in $\mathbb{R}$. If*

- $\limsup_{n\to\infty}(A_n + B_n) \leq A + B$,
- $\liminf_{n\to\infty} A_n \geq A$,
- $\liminf_{n\to\infty} B_n \geq B$,

then

$$\lim_{n\to\infty} A_n = A, \quad \lim_{n\to\infty} B_n = B.$$

Proof. The following holds:

$$\limsup_{n\to\infty} A_n + B \leq \limsup_{n\to\infty} A_n + \liminf_{n\to\infty} B_n \leq \limsup_{n\to\infty}(A_n + B_n) \leq A + B,$$

and so

$$\limsup_{n\to\infty} A_n \leq A.$$

Since by hypothesis $\liminf_{n\to\infty} A_n \geq A$, we have $\lim_{n\to\infty} A_n = A$.
The same holds for the sequence $\{B_n\}$. □

Theorem 5.21 *[1] Let E be a closed subset of $\mathbb{R}^d$ with $\mathcal{H}^{d-1}(\partial E) < \infty$, such that*

(i) $\mathcal{P}(E) = \mathcal{H}^{d-1}(\partial E)$,

(ii) ∂E *admits Minkowski content.*

Then

$$\lim_{r\to 0} \frac{\mathcal{H}^d(E_{\oplus r} \setminus E)}{r} = \mathcal{H}^{d-1}(\partial E).$$

Proof. Let

- $A_r := \dfrac{1}{r}\displaystyle\int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t$,
- $B_r := \dfrac{1}{r}\displaystyle\int_{-r}^0 \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t$,
- $A = B := \mathcal{H}^{d-1}(\partial E)$.

Then we have

$$\begin{aligned}
\limsup_{r\to 0}(A_r + B_r) &= \limsup_{r\to 0} \frac{1}{r}\int_{-r}^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t \\
&= \lim_{r\to 0} 2\cdot\frac{1}{2r}\int_{-r}^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t \\
&\overset{(5.30)}{=} 2\lim_{r\to 0}\frac{\mathcal{H}^d(\partial E_{\oplus r})}{2r} \\
&\overset{(ii)}{=} 2\mathcal{H}^{d-1}(\partial E) \\
&= A + B.
\end{aligned}$$

Further,

$$\begin{aligned}
\liminf_{r\to 0} A_r &= \liminf_{r\to 0}\frac{1}{r}\int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\})\mathrm{d}t \\
&= \liminf_{r\to 0}\int_0^1 \mathcal{H}^{d-1}(\{x : d_E(x) = tr\})\mathrm{d}t;
\end{aligned}$$

by Fatou's Lemma,

$$\geq \int_0^1 \liminf_{r\to 0}\mathcal{H}^{d-1}(\{x : d_E(x) = tr\})\mathrm{d}t;$$

since $\mathcal{P}(E) < \infty$ it is clear that $\mathcal{P}(\{x : d_E(x) < tr\}) < \infty$, thus by Theorem 5.16 and Proposition 5.18,

$$\geq \int_0^1 \liminf_{r\to 0}\mathcal{P}(\{x : d_E(x) < tr\})\mathrm{d}t;$$

by property (b) in Theorem 5.17,

$$
\begin{aligned}
&\geq \int_0^1 \mathcal{P}(E)\mathrm{d}t \\
&\stackrel{(i)}{=} \mathcal{H}^{d-1}(E)\cdot 1 \\
&= A.
\end{aligned}
$$

Similarly:

$$\liminf_{r\to 0} B_r \geq B.$$

By Lemma 5.20, we have

$$\lim_{r\to 0}\frac{\mathcal{H}^d(E_{\oplus r}\setminus E)}{r} \stackrel{(5.31)}{=} \lim_{r\to 0} A_r = A = \mathcal{H}^{d-1}(\partial E).$$

□

Corollary 5.22 *[1] Let E be a subset of $\mathbb{R}^d$ with Lipschitz boundary such that $\mathcal{H}^{d-1}(\partial E)<\infty$; then*

$$\lim_{r\to 0}\frac{\mathcal{H}^d(E_{\oplus r}\setminus E)}{r} = \mathcal{H}^{d-1}(\partial E).$$

Proof. It is clear that ∂E is a $(d-1)$-rectifiable set, and so by Theorem 5.7 it admits Minkowski content. By Remark 5.19 we know that $\mathcal{P}(E)=\mathcal{H}^{d-1}(\partial E)$, thus the thesis follows by Theorem 5.21. □

Corollary 5.23 *[1] Let E be a closed subset of $\mathbb{R}^d$ with $\mathcal{H}^{d-1}(\partial E)<\infty$, such that*

(i) $\mathcal{P}(E)=\mathcal{H}^{d-1}(\partial E)$,

(ii) ∂E admits Minkowski content.

Then, for any $A\in\mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^{d-1}(\partial E\cap\partial A)=0$,

$$\lim_{r\to 0}\frac{\mathcal{H}^d(E_{\oplus r}\setminus E\cap A)}{r} = \mathcal{H}^{d-1}(\partial E\cap A).$$

Proof. It is clear that, since ∂E admits Minkowski content, then for any closed set $C\subset\mathbb{R}^d$ it holds

$$\lim_{r\to 0}\frac{\mathcal{H}^d((\partial E\cap C)_{\oplus r})}{2r} = \mathcal{H}^{d-1}(\partial E\cap C).$$

Thus, by Lemma 4.8 it follows that, for any $A\in\mathcal{B}_{\mathbb{R}^d}$ such that $\mathcal{H}^{d-1}(\partial E\cap\partial A)=0$, we have

$$\lim_{r\to 0}\frac{\mathcal{H}^d(\partial E_{\oplus r}\cap A)}{2r} = \mathcal{H}^{d-1}(\partial E\cap A).$$

The proof can be concluded by repeating the same argument as in the proof of Theorem 5.21, where now

- $A_r := \dfrac{1}{r}\displaystyle\int_0^r \mathcal{H}^{d-1}(\{x \in A : d_E(x) = t\})\mathrm{d}t,$
- $B_r := \dfrac{1}{r}\displaystyle\int_{-r}^0 \mathcal{H}^{d-1}(\{x \in A : d_E(x) = t\})\mathrm{d}t,$
- $A = B := \mathcal{H}^{d-1}(\partial E \cap A).$

□

Remark 5.24 There exist closed sets $E \subset \mathbb{R}^d$ whose boundary is not Lipschitz, but such that they admit Minkowski content and $\mathcal{P}(E) = \mathcal{H}^{d-1}(E)$, i.e. satisfying hypotheses of Theorem 5.21.
A simple example is given by the union of two tangent balls (see also Appendix B). Note also that in this case the first order Steiner formula is a consequence of Theorem 5.12 as well.

5.3 A mean first order Steiner formula for random closed sets

Let us consider a random closed set Θ satisfying a first order Steiner formula, in other words it holds

$$\lim_{r\to 0}\frac{\mathcal{H}^d(\Theta(\omega)_{\oplus r} \setminus \Theta(\omega))}{r} = \mathcal{H}^{d-1}(\partial\Theta(\omega)), \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{5.32}$$

It is well known that almost sure convergence does not imply L^1-convergence, so that, in general, (5.32) does not imply

$$\lim_{r\to 0}\frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \setminus \Theta)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)]. \tag{5.33}$$

On the other hand, if the family of random variables $X_r := \frac{\mathcal{H}^d(\Theta_{\oplus r}\setminus\Theta)}{r}$ is uniformly integrable, or we may apply the Dominated Convergence Theorem, we obtain (5.33).
Taking this fact into account, we consider the case in which Θ is given by a finite union of random closed sets A_i with positive reach. Using Theorem 5.12, it is easy to find sufficient conditions for Θ such that (5.33) holds.

Proposition 5.25 *[1] Let $A_1, \dots A_n$ be random closed sets in K, compact subset of $\mathbb{R}^d$, such that*

(i) for $\mathbb{P}$-a.e. $\omega \in \Omega$, each possible intersection of the $A_i(\omega)$'s is a set with reach $\geq R > 0$*;*

(ii) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathcal{H}^{d-1}(\partial A_i \cap \partial A_j) = 0$, $\forall i \neq j$;

(iii) $\forall i = 1, \dots, n$, it holds $\lim_{r\to 0}\dfrac{\mathbb{E}[\mathcal{H}^d((\partial A_i)_{\oplus r})]}{2r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial A_i)].$

Then, the random closed set $\Theta = \bigcup_{i=1}^n A_i$ satisfies the mean first order Steiner formula (5.33).

Proof. By proceeding in a similar way as in proof of Theorem 5.12, for any fixed $\varepsilon > 0$ and $r < R \wedge \varepsilon/2$, we have

$$\begin{aligned}
&\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r})] \\
&\leq \mathbb{E}[\mathcal{H}^d(\Theta)] + r\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)] \\
&\quad + \sum_{m=0}^{d-2} r^{d-m} b_{d-m} \mathbb{E}[\sum_{i=1}^n \Phi_m(A_i) - \sum_{i<j} \Phi_m(A_i \cap A_j) + \sum_{i<j<k} \Phi_m(A_i \cap A_j \cap A_k) \\
&\qquad\qquad + \ldots + (-1)^{n+1} \Phi_m(A_1 \cap \ldots \cap A_n)] \\
&\quad + \sum_{i<j<k} \mathbb{E}[\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{k,j})] \\
&\quad + \sum_{i<j<k<l<m} \mathbb{E}[\mathcal{H}^d(E_{i,j}) + \ldots + \mathcal{H}^d(E_{m,l})] \\
&\quad + \ \ldots
\end{aligned}$$

where

$$E_{i,j} := (\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)_{\oplus r}.$$

By Remark 5.10 in [31], $\forall E \subset K$ with $\mathrm{reach}(E) \geq R$, we have that,

$$\forall i = 0, \ldots, d, \quad \exists k_i \in \mathbb{R}_+ \quad \text{such that} \quad |\Phi_i(E)| \leq |\Phi_i|(E) \leq k_i,$$

where $|\Phi_i|(E)$ is the total variation of the measure $\Phi(E, \cdot)$ over K, and

$$k_i = \sup\{|\Phi_i|(F) \, : \, F \subset K \quad \text{and} \quad \mathrm{reach}(F) \geq R\} < \infty.$$

As a consequence

$$\begin{aligned}
&\sum_{m=0}^{d-2} r^{d-m} b_{d-m} \mathbb{E}[\sum_{i=1}^n \Phi_m(A_i) - \sum_{i<j} \Phi_m(A_i \cap A_j) \\
&\quad + \sum_{i<j<k} \Phi_m(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} \Phi_m(A_1 \cap \ldots \cap A_n)] = o(r)
\end{aligned}$$

By hypothesis (iii) we may claim that

$$\begin{aligned}
&\limsup_{r \to 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r})] - \mathbb{E}[\mathcal{H}^d(\Theta)]}{r} \leq \ \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)] \\
&\quad + 2 \sum_{i<j<k} \mathbb{E}[\mathcal{H}^{d-1}(\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)] + \ldots + \mathbb{E}[\mathcal{H}^{d-1}(\partial A_k \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)] \\
&\quad + 2 \sum_{i<j<k<l<m} \mathbb{E}[\mathcal{H}^{d-1}(\partial A_i \cap A_{j_{\oplus\varepsilon}} \setminus \mathrm{int} A_j)] + \ldots + \mathbb{E}[\mathcal{H}^{d-1}(\partial A_m \cap A_{l_{\oplus\varepsilon}} \setminus \mathrm{int} A_l)] \\
&\quad + \ \ldots
\end{aligned}$$

This holds for any fixed $\varepsilon > 0$; thus, by taking the limit for ε going to 0, and by Monotone Convergence Theorem, we obtain

$$\limsup_{r \to 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r}) - \mathcal{H}^d(\Theta)]}{r} \leq \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)].$$

Similarly we have

$$\liminf_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r}) - \mathcal{H}^d(\Theta)]}{r} \geq \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta)],$$

and so the thesis follows. □

Note that sufficient conditions for hypothesis (iii) to hold are given in Theorem 4.10.

Remark 5.26 In the above theorem we have supposed that Θ is the union of random sets A_i. Obviously we may have the same result even when Θ is given by a random union, i.e. $\Theta = \bigcup_{i=1}^N A_i$, where N is a positive integer-valued random variable. In this case we have to add suitable hypotheses on the integrability of N in terms of its moments. These can be easily obtained by evaluating $\mathbb{E}[\mathcal{H}^d((\bigcup_{i=1}^N A_i)_{\oplus r})]$ by $\mathbb{E}[\mathbb{E}[\mathcal{H}^d((\bigcup_{i=1}^N A_i)_{\oplus r}) \,|\, N]]$, and proceeding as in previous theorems.

In more general situations, e.g. when Θ can not be expressed as union of random closed sets with positive reach satisfying the assumptions of Proposition 5.25, or some of such assumptions are not easy to verify, by Theorem 4.10 we may claim again that (5.33) holds.
Indeed, let us observe that

$$\frac{\mathcal{H}^d(\Theta(\omega)_{\oplus r} \setminus \Theta(\omega))}{r} \leq \frac{\mathcal{H}^d(\partial\Theta(\omega)_{\oplus r})}{r} = 2 \cdot \frac{\mathcal{H}^d(\partial\Theta(\omega)_{\oplus r})}{2r} \qquad \forall \omega \in \Omega, \quad (5.34)$$

and by Theorem 4.10 we know sufficient conditions on $\partial\Theta$ so that we may apply the Dominated Convergence Theorem and obtain (5.33).

Without loss of generalization, we may assume that $\partial\Theta$ is a compact random set. Then we have the following general result:

Theorem 5.27 *[1] Let Θ be a random closed set in $\mathbb{R}^d$ with boundary $\partial\Theta$ countably $\mathcal{H}^{d-1}$-rectifiable and compact, satisfying*

$$\lim_{r\to 0} \frac{\mathcal{H}^d(\Theta_{\oplus r}(\omega) \setminus \Theta(\omega) \cap A)}{r} = \mathcal{H}^{d-1}(\partial\Theta(\omega) \cap A) \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

with $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \partial A)] = 0$. Let $\Gamma : \Omega \longrightarrow \mathbb{R}$ be the function so defined:

$$\Gamma(\omega) := \max\{\gamma \geq 0 \,:\, \exists \text{ a probability measure } \eta \ll \mathcal{H}^{d-1} \text{ such that } \eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial\Theta(\omega),\ r \in (0,1)\}.$$

If there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $\frac{1}{\Gamma(\omega)} \leq Y(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then,

$$\lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{\oplus r} \setminus \Theta \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap A)].$$

Proof. We note that $\partial\Theta$ satisfies the hypotheses of Theorem 4.10. By the proof of the quoted theorem we know that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\frac{\mathcal{H}^d(\partial\Theta(\omega)_{\oplus r})}{2r} \le Y(\omega) 2^{d-1} 4^d \frac{b_d}{2}. \tag{5.35}$$

Since

$$\frac{\mathcal{H}^d((\Theta_{\oplus r}(\omega) \setminus \Theta(\omega)) \cap A)}{r} \le \frac{\mathcal{H}^d(\Theta_{\oplus r}(\omega) \setminus \Theta(\omega))}{r} \le \frac{\mathcal{H}^d(\partial\Theta_{\oplus r}(\omega))}{2r},$$

by (5.35), applying the Dominated Convergence Theorem, the thesis follows. □

There are a lot of random closed sets satisfying hypotheses of Theorem 5.27. Several examples of random sets Θ such that (5.35) holds are given in Section 4.3. Note that the well known case of Θ Boolean model of balls (or other sufficiently regular objects) satisfies Theorem 5.27.

Chapter 6

Time dependent random closed sets

In this chapter we wish to analyze the case in which a random closed set Θ may depend upon time as, for example, when it models the evolution due to a growth process, so that we have a geometric random process $\{\Theta^t, \ t \in \mathbb{R}_+\}$, such that for any $t \in \mathbb{R}_+$, the random set Θ^t satisfies all the relevant assumptions required in the previous sections.
Correspondingly the associated linear functional δ_{Θ^t} will also be a function of time, and so we need to define partial derivatives of linear functionals depending on more than one variable. In this way we shall provide evolution equations for such space-time dependent linear functionals; in particular we will apply this to stochastic birth-and-growth processes. (See [26, 24].)

6.1 Derivatives of linear functionals

Consider a linear functional L acting on the test space $\mathcal{S}_k$ of functions s in k variables; we formally represent it as

$$(L, s) =: \int_{\mathbb{R}^k} \phi(x_1, \ldots, x_k) s(x_1, \ldots, x_k) \mathrm{d}(x_1, \ldots, x_k).$$

Let us denote by L_i^h the linear functional defined by

$$(L_i^h, s) =: \int_{\mathbb{R}^k} \phi(x_1, \ldots, x_i + h, \ldots, x_k) s(x_1, \ldots, x_k) \mathrm{d}(x_1, \ldots, x_k). \tag{6.1}$$

We define the *weak partial derivative* of the functional L with respect to the variable x_i as follows (see also [33], p. 20).

Definition 6.1 *We say that a linear functional L on the space $\mathcal{S}_k$, admits a* weak partial derivative *with respect to x_i, denoted by $\frac{\partial}{\partial x_i} L$, if and only if $\frac{\partial}{\partial x_i} L$ is a linear functional on the same space $\mathcal{S}_k$ and $\left\{\frac{L_i^h - L}{h}\right\}$ weakly* converges to $\frac{\partial}{\partial x_i} L$, i.e.*

$$\lim_{h \to 0} \left(\frac{L_i^h - L}{h}, s \right) = \left(\frac{\partial}{\partial x_i} L, s \right) \qquad \textit{for all } s \in \mathcal{S}_k.$$

We may like to notice that, in the particular case of generalized functions, the definition above coincides with the derivative of a generalized function depending on a parameter, given in [33] p. 148. Moreover, when $\mathcal{S}_1$ is the test space $C_c^1(\mathbb{R},\mathbb{R})$ of all function of class C^1 with compact support, Definition 6.2 coincides with the usual derivative of a generalized function (see, e.g. [45]), that we now recall.

Definition 6.2 *Let $L = L(x)$ be a generalized function on a test space $\mathcal{K} \subseteq C_c^1(\mathbb{R},\mathbb{R})$. The* derivative $\mathrm{d}L/\mathrm{d}x$ *of L is the functional defined by the formula*

$$\left(\frac{\mathrm{d}L}{\mathrm{d}x}, f\right) := -(L, f') \qquad \forall f \in \mathcal{K}. \tag{6.2}$$

The functional (6.2) is obviously linear and continuous, and hence is itself a generalized function.
Usually, Definition 6.2 is suggested by the particular case when the generalized function L is of the form

$$(L, f) := \int_{\mathbb{R}} \phi(x) f(x) \mathrm{d}x,$$

where ϕ is a function on $\mathbb{R}$ such that its derivative ϕ' exists and is locally integrable. Then it is natural to define the derivative of L as the functional

$$\left(\frac{\mathrm{d}L}{\mathrm{d}x}, f\right) := \int_{\mathbb{R}} \phi'(x) f(x) \mathrm{d}x.$$

Integrating by parts and using the fact that every test function f has a compact support, one obtains that

$$\left(\frac{\mathrm{d}L}{\mathrm{d}x}, f\right) = -\int_{\mathbb{R}} \phi(x) f'(x) \mathrm{d}x = -(L, f').$$

Carrying this over to the singular case, we get the above definition.
Now we show that, by Definition 6.1, formally we may easily reobtain the definition in (6.2), directly also for a singular generalized function L.
We said that the usual representation of (L, f) is $\int_{\mathbb{R}} \phi(x) f(x) \mathrm{d}x$, where $f \in C_c^1(\mathbb{R},\mathbb{R})$ and ϕ is a fictitious function. Note that, by (6.1) with $k = 1$ and $L_1^h = L^h$,

$$(L^h, f) = (L^h(x), f(x)) = \int_{\mathbb{R}} \phi(x+h) f(x) \mathrm{d}x = \int_{\mathbb{R}} \phi(x) f(x-h) \mathrm{d}x = (L(x), f(x-h))$$

(this is known also as the translation of the generalized function L).
Now, we remember that for any $\alpha \in \mathbb{R}$, $(\alpha L, f) = (L, \alpha f)$, and so, by Definition

6.1, we have

$$\begin{aligned}
\left(\frac{\mathrm{d}L}{\mathrm{d}x}, f\right) &= \lim_{h\to 0}\left(\frac{L^h - L}{h}, f\right) \\
&= \lim_{h\to 0}\left[\left(\frac{L^h(x)}{h}, f(x)\right) - \left(\frac{L(x)}{h}, f(x)\right)\right] \\
&= \lim_{h\to 0}\left[\left(L^h(x), \frac{f(x)}{h}\right) - \left(L(x), \frac{f(x)}{h}\right)\right] \\
&= \lim_{h\to 0}\left[\left(L(x), \frac{f(x-h)}{h}\right) - \left(L(x), \frac{f(x)}{h}\right)\right] \\
&= \lim_{h\to 0}\left(L(x), \frac{f(x-h)-f(x)}{h}\right) \\
&= -\lim_{h\to 0}\left(L(x), \frac{f(x-h)-f(x)}{-h}\right) \\
&\overset{(i)}{=} -(L(x), f'(x)) \\
&= -(L, f')
\end{aligned}$$

where equation (i) follows by the hypothesis that $f \in C_c^1(\mathbb{R},\mathbb{R})$, and L is a continuous functional. (A similar approach is given in [33] p. 20).

As a simple example, let us consider the Dirac delta function δ_0; then by the usual definition (6.2), we have

$$\left(\frac{\mathrm{d}\delta_0}{\mathrm{d}x}, f\right) = -(\delta_0, f') = -f'(0).$$

We obtain the same result by applying Definition 6.1:

$$\begin{aligned}
\left(\frac{\mathrm{d}\delta_0}{\mathrm{d}x}, f\right) &= \lim_{h\to 0}\int_{\mathbb{R}} \frac{\delta_0(x+h)-\delta_0(x)}{h} f(x)\mathrm{d}x \\
&= \lim_{h\to 0}\left[\int_{\mathbb{R}} \delta_0(x+h)\frac{f(x)}{h}\mathrm{d}x - \int_{\mathbb{R}} \delta_0(x)\frac{f(x)}{h}\mathrm{d}x\right] \\
&= \lim_{h\to 0}\left[\int_{\mathbb{R}} \delta_{-h}(x)\frac{f(x)}{h}\mathrm{d}x - \int_{\mathbb{R}} \delta_0(x)\frac{f(x)}{h}\mathrm{d}x\right] \\
&= -\lim_{h\to 0}\frac{f(-h)-f(0)}{-h} \\
&= -f'(0)
\end{aligned}$$

It is well known that the derivative of the Heaviside function

$$H(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is the delta function $\delta_0(x)$.
Similarly, if $H_{x_0}(x) = \mathbf{1}_{[x_0,+\infty)}$, then its derivative is $\delta_{x_0}(x)$.

6.1.1 Application to growth processes

Let us consider a geometric random process $\{\Theta^t,\ t \in \mathbb{R}_+\}$ satisfying the following assumptions [17, 20]:

(i) for any $t \in \mathbb{R}_+$, and any $s > 0$, Θ^t is well contained in Θ^{t+s}, i.e. $\partial\Theta^t \subset \text{int}\Theta^{t+s}$;

(ii) for any $t \in \mathbb{R}_+$, Θ^t is a d-regular random closed set in $\mathbb{R}^d$, and $\partial\Theta^t$ is a $(d-1)$-regular random closed set.

By assumption (i), we call $\{\Theta^t\}$ a *growth process.*
For any $x \in \mathbb{R}^d$, we may introduce a *time of capture of* x, as the positive random variable $T(x)$ so defined:

$$T(x) := \min\{t \in \mathbb{R}_+ \, : \, x \in \Theta^t\}.$$

By assumption (i) it follows that

$$x \in \text{int}\Theta^t \text{ if } t > T(x),$$

$$x \notin \Theta^t \text{ if } t < T(x),$$

so that

$$x \in \partial\Theta^{T(x)}.$$

Let us introduce, on the test space $C_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$, the following two linear functionals

$$(T_1, f) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} \delta_{\Theta^t}(x) f(t,x) \mathrm{d}t \mathrm{d}x$$

and

$$(T_2, f) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} H_{T(x)}(t) f(t,x) \mathrm{d}t \mathrm{d}x.$$

We know that

$$\delta_{\Theta^t}(x) = \begin{cases} 1 & \forall x \in \text{int}\Theta^t \\ 0 & \forall x \notin \Theta^t \end{cases}.$$

As a consequence we may easily check that, for any test function f, $(T_1, f) = (T_2, f)$, so that we may formally write

$$\delta_{\Theta^t}(x) = H_{T(x)}(t),$$

where $H_{T(x)}$ is the Heaviside distribution associated with $T(x)$, introduced above. We know that its distributional derivative is the delta function $\delta_{T(x)}$; as a consequence the following holds.

Proposition 6.3 *For any test function $f \in C_c(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t,x) \frac{\partial}{\partial t} \delta_{\Theta^t}(x) \, \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(t,x) \delta_{T(x)}(t) \, \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^d} f(T(x), x) \, \mathrm{d}x. \tag{6.3}$$

Formally we may write

$$\frac{\partial}{\partial t} \delta_{\Theta^t}(x) = \delta_{T(x)}(t).$$

Proof. According to the previous definition,

$$\begin{aligned}
& \int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)\frac{\partial}{\partial t}\delta_{\Theta^t}(x)\ \mathrm{d}t\mathrm{d}x \\
= & \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)\left[\delta_{\Theta^{t+\Delta t}}(x)-\delta_{\Theta^t}(x)\right]\ \mathrm{d}t\mathrm{d}x \\
= & \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)\left[H_{T(x)}(t+\Delta t)-H_{T(x)}(t)\right]\ \mathrm{d}t\mathrm{d}x \\
= & \int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}_+}\mathrm{d}t f(t,x)\frac{\partial H_{T(x)}}{\partial t}(t) \\
= & \int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}_+}\mathrm{d}t f(t,x)\delta_{T(x)}(t) \\
= & \int_{\mathbb{R}^d} f(T(x),x)\ \mathrm{d}x.
\end{aligned}$$

□

Consider the case in which $T(x)$ is a continuous random variable with probability density function $p_{T(x)}(t)$. Then, by Remark 2.24, we may claim that, in a distributional sense,

$$\mathbb{E}\left[\frac{\partial}{\partial t}\delta_{\Theta^t}\right](x)=\mathbb{E}[\delta_{T(x)}](t)=p_{T(x)}(t).$$

In fact, coherently with the definition of expected linear functional, and by Proposition 6.3 we have

$$\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} f(t,x)\frac{\partial}{\partial t}\delta_{\Theta^t}(x)\ \mathrm{d}x\mathrm{d}t\right] &= \mathbb{E}\left[\int \mathrm{d}x f(T(x),x)\right] \\
&= \int\mathrm{d}x\int\mathrm{d}t f(t,x)p_{T(x)}(t).
\end{aligned}$$

We may observe that, in this case, even if for any realization $\Theta^t(\omega)$ of Θ^t, $\frac{\partial}{\partial t}\delta_{\Theta^t(\omega)}$ is a singular generalized function, when we consider the expectation we obtain a regular generalized function, i.e. a real integrable function. In particular the derivative is the usual derivative of functions. Thus, by observing that, since $T(x)$ is the random time of capture of x, $\mathbb{P}(x\in\Theta^t)=\mathbb{P}(T(x)<t)$, and $\mathbb{E}[\delta_{\Theta^t}](x)=\mathbb{P}(x\in\Theta^t)$ (see Remark 2.24), the following holds too:

$$\mathbb{E}\left[\frac{\partial}{\partial t}\delta_{\Theta^t}\right](x)=p_{T(x)}(t)=\frac{\partial}{\partial t}\mathbb{P}(x\in\Theta^t)=\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x). \tag{6.4}$$

Hence, $\mathbb{E}\left[\frac{\partial}{\partial t}\delta_{\Theta^t}\right](x)$ and $\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x)$ coincide as functions, and, by the equation above, we have the formal exchange between derivative and expectation.

6.2 Birth-and-growth processes

A birth-and-growth process in $\mathbb{R}^d$ is a dynamic germ-grain model [64, 36] whose birth process is modelled as a marked point process (MPP) $N = \{(T_i, X_i)\}_{i\in\mathbb{N}}$ on $\mathbb{R}_+$ with marks in $\mathbb{R}^d$ (see Section 1.5.5), where $T_i \in \mathbb{R}_+$ represents the random time of birth of the i-th germ (*nucleus*) and $X_i \in \mathbb{R}^d$ its random spatial location [39, 64]. Once born, each germ (*crystal*) generates a grain which grows at the surface (growth front), with a speed $G(t, x) > 0$ which may, in general, be assumed space-time dependent.

Application areas include crystallization processes, tumor growth and angiogenesis (see [47] and references therein, [66, 63]). All this kind of phenomena include a space-time structured random process of birth (nucleation), and a growth process that, as a first approximation, we consider deterministic.

6.2.1 The nucleation process

Consider a Borel set $E \subset \mathbb{R}^d$, $d \geq 2$, endowed with its Borel σ-algebra $\mathcal{E}$, and a marked point process N on $\mathbb{R}^+$ with marks in E. So, it is defined as a random measure given by

$$N = \sum_{n=1}^{\infty} \varepsilon_{(T_n, X_n)},$$

where

- T_n is an $\mathbb{R}_+$-valued random variable representing the time of birth of the n-th nucleus,
- X_n is an E-valued random variable representing the spatial location of the nucleus born at time T_n,
- $\varepsilon_{t,x}$ is the Dirac measure on $\mathcal{E} \times \mathcal{B}_{\mathbb{R}_+}$ such that for any $t_1 < t_2$ and $A \in \mathcal{E}$,

$$\varepsilon_{t,x}([t_1, t_2] \times A) = \begin{cases} 1 & \text{if } t \in [t_1, t_2],\ x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in particular, for any $B \in \mathcal{B}_{\mathbb{R}_+}$ and $A \in \mathcal{E}$, we have

$$N(B \times A) = \#\{T_n \in B,\ X_n \in A\},$$

i.e. it is the (random) number of germs born in the region A, during time B. Note that,

$$N(B \times A) < \infty \tag{6.5}$$

for any $B \in \mathcal{B}_{\mathbb{R}_+}$ and $A \in \mathcal{E}$ bounded.
Besides, we **suppose** that the marginal process of N is *simple* (Section 1.5.1), with $\mathbb{E}[N(\mathrm{d}t \times E)\mathbf{1}_{N(\mathrm{d}t\times E)\geq 2}] = o(\Delta t)$.

It is well known [16, 46] that, under general conditions, a marked point process is characterized by its compensator, say $\nu(\mathrm{d}t \times \mathrm{d}x)$, with respect to the internal history of the process (Section 1.5.5). Denoted by $\tilde{\nu}(\mathrm{d}t)$ the compensator of the marginal process $\widetilde{N}$, we know that there exists a stochastic kernel k from $\Omega \times \mathbb{R}^+$ to E such that

$$\nu(\mathrm{d}t \times \mathrm{d}x) = k(t, \mathrm{d}x)\tilde{\nu}(\mathrm{d}t). \tag{6.6}$$

In many applications it is supposed that further nuclei cannot be born in an already crystallized zone. When we want to emphasize this, we shall write

$$\nu(\mathrm{d}t \times \mathrm{d}x) = k(t, \mathrm{d}x)\tilde{\nu}(\mathrm{d}t) = k_0(t, \mathrm{d}x)\tilde{\nu}_0(\mathrm{d}t)(1 - \mathbf{1}_{\Theta^{t-}}(x)),$$

where $\nu_0(\mathrm{d}t \times \mathrm{d}x) = k_0(t, \mathrm{d}x)\tilde{\nu}_0(\mathrm{d}t)$ is the compensator of the process N_0, called the *free-process*, in which nuclei can be born anywhere. (See also [22]).

The Poisson case

In a great number of applications, it is supposed that N_0 is a marked Poisson process (see Definition 1.70). In this case it is well known that its compensator is deterministic.
In particular, it is assumed that the MPP N_0 is a space-time inhomogeneous marked Poisson process with a given (*deterministic*) intensity

$$\alpha(t, x), \quad x \in E,\ t \geq 0,$$

where α is a real valued measurable function on $E \times \mathbb{R}_+$ such that $\alpha(\cdot, t) \in \mathcal{L}^1(E)$, for all $t > 0$ and such that

$$0 < \int_0^T \mathrm{d}t \int_E \alpha(t, x)\mathrm{d}x < \infty$$

for any $0 < T < \infty$.

If we want to exclude germs which are born within the region already occupied by Θ^t, we shall consider the thinned *stochastic* intensity

$$\nu(\mathrm{d}t \times \mathrm{d}x) = \alpha(t, x)(1 - I_{\Theta^{t-}}(x))\mathrm{d}t\mathrm{d}x.$$

6.2.2 The growth process

Let $\Theta^t_{T_n}(X_n)$ be the random closed set obtained as the evolution up to time $t > T_n$ of the germ born at time T_n in X_n, according to some growth model; this will be the *grain* associated with the *germ* (T_n, X_n). In other words, if $T_n = s$ and $X_n = x$, then $\Theta^t_s(x)$ is the crystal born at time s and point x and grown up to time t.

We call *birth-and-growth process* the family of random closed sets given by

$$\Theta^t = \bigcup_{n: T_n \leq t} \Theta^t_{T_n}(X_n), \qquad t \in \mathbb{R}_+.$$

If a point x is crystallized at time t, then it generates the grain associated to the germ (t, x). We may observe that, $\forall s \geq 0$,

$$\begin{aligned}
\Theta^{t+s} &= \bigcup_{n:T_n \leq t+s} \Theta_{T_n}^{t+s}(X_n) \\
&= \bigcup_{n:T_n \leq t} \Theta_{T_n}^{t+s}(X_n) \cup \bigcup_{n:t<T_n \leq t+s} \Theta_{T_n}^{t+s}(X_n) \\
&= \Theta^t \cup \bigcup_{x \in \partial\Theta^t} \Theta_t^{t+s}(x) \cup \bigcup_{n:t<T_n \leq t+s} \Theta_{T_n}^{t+s}(X_n).
\end{aligned}$$

In order to complete the definition of the birth-and-growth process we need to define a growth model for any grain associated with each individual germ. We assume here the *normal growth model* (see, e.g., [17]), according to which at $\mathcal{H}^{d-1}$-almost every point of the actual grain surface at time t (i.e. at $\mathcal{H}^{d-1}$-almost every $x \in \partial\Theta_{T_n}^t(X_n)$), growth occurs with a given strictly positive normal velocity

$$v(t, x) = G(t, x)n(t, x), \tag{6.7}$$

where $G(t, x)$ is a given deterministic *strictly positive* "growth" field, and $n(t, x)$ is the unit outer normal at point $x \in \partial\Theta_{T_0}^t(X_0)$. We assume that

$$0 < g_0 \leq G(t, x) \leq G_0 < \infty \qquad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

for some $g_0, G_0 \in \mathbb{R}$, and $G(t, x)$ is sufficient regular (in particular G has to be (globally) Lipschitz-continuous on $\mathbb{R}_+ \times \mathbb{R}^d$) such that the evolution problem given by (6.7) for the growth front $\partial\Theta_{t_0}^t(x)$, with the initial condition that at the birth time t_0 the initial germ born in x_0 is described by a spherical ball of infinitesimal radius centered at x_0, is well posed. (See e.g. [17]; the case of regularity of G deriving from its coupling with a deterministic underlying field has been analyzed in [65].)

Remark 6.4 1. Since at $\mathcal{H}^{d-1}$-almost every $x \in \partial\Theta_{T_n}(X_n)$ there exists the unit outer normal and, by remembering (6.5), Θ^t is a finite union of grains for all $t > 0$, then there exists the tangent hyperplane to Θ^t at $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Theta^t$. As a consequence Θ^t and $\partial\Theta^t$ are finite unions of rectifiable sets and satisfy (1.7).

2. In the case $d = 2$ an explicit form of (6.7) may be done by a parametrization of the growing crystals (see [47]). Let us consider the crystal

$$\Theta_0^t = \{\mathbf{x}(\tau, \gamma) | \tau \in (t_0, t), \gamma \in [0, 2\pi)\},$$

and its boundary, called the *growing front*,

$$\partial\Theta_0^t = \{\mathbf{x}(t, \gamma) | \gamma \in [0, 2\pi)\}.$$

Then, at any point $\mathbf{x}(t, \gamma) \in \partial\Theta_0^t$, the evolution problem is given by

$$\begin{aligned}
&\dot{\mathbf{x}}(t, \gamma) = G(\mathbf{x}(t, \gamma), t)\mathbf{n}(t, \gamma), \\
&\dot{\mathbf{n}}(t, \gamma) = -\nabla G(\mathbf{x}(t, \gamma), t) + (\nabla G(\mathbf{x}(t, \gamma), t), \mathbf{n}(t, \gamma))\mathbf{n}(t, \gamma),
\end{aligned}$$

with initial values

$$\mathbf{x}(t_0,\gamma),t) = \mathbf{x}_0,$$
$$\mathbf{n}(t_0,\gamma) = (\cos\gamma, \sin\gamma)^T.$$

This shows that the growth is determined by the actual normal direction of the growth front as well as by the growth rate and its gradient.

6.2.3 Hazard and survival functions

In a birth-and-growth process the random set Θ^t evolves in time, so that the question arises about *"WHEN"* a point $x \in E$ is reached (*captured*) by this growing random set; or viceversa up to when a point $x \in E$ survives capture? In this respect the following definition is introduced.

Definition 6.5 *The* survival function of a point x at time t *is the probability that the point x is not yet covered ("captured") by any crystal at time t:*

$$S(t,x) := \mathbb{P}(x \notin \Theta^t).$$

The survival function of a point x may be studied in terms of the nonnegative r.v. $T(x)$ representing the capture time of x, defined in Section 6.1.1; in this way it can be regarded as a typical problem of survival analysis, and then be studied in terms of the nucleation process, taking the geometric aspects of the crystallization process into account (see [22]).

In fact, let τ be a nonnegative r.v. with cumulative density function F; then τ may be considered as a random failure time, the function $S(t) := \mathbb{P}(\tau > t)$ is called the *survival function of* τ and

$$\begin{aligned}\mathbb{P}(\tau \in [t, t+\Delta t) \mid \tau \geq t) &= \frac{\mathbb{P}(t \leq \tau < t + \Delta t)}{\mathbb{P}(\tau \geq t)} \\ &= \frac{F(t+\Delta t-) - F(t-)}{1 - F(t-)}.\end{aligned}$$

If τ is a continuous r.v. with pdf f, then the following limit

$$h(t) := \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(\tau \in [t, t+\Delta t) \mid \tau \geq t)}{\Delta t} \tag{6.8}$$

(with $0/0 := 0$) exists and $h(t) = \dfrac{f(t)}{1-F(t)}$, i.e.

$$h(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln S(t). \tag{6.9}$$

$h(t)$ is called the *hazard function of* τ.

Let us now consider the capture time $T(x)$ of point x and observe that

$$\mathbb{P}(T(x) > t) = \mathbb{P}(x \notin \Theta^t) = S(t,x).$$

So $S(\cdot, x)$ is the survival function of the r.v. $T(x)$ and, in order to study it in terms of the MPP N, we must go back to the "causes" of capture of point x. To this end it may be helpful to introduce the concept of *causal cone* (see e.g. [43, 19]).

Definition 6.6 *The* causal cone $\mathcal{C}(t,x)$ *of a point* x *at time* t *is the space-time region in which at least one nucleation has to take place so that the point* x *is covered by crystals at time* t*:*

$$\mathcal{C}(t,x) := \{(s,y) \in [0,t] \times \ E \, : \, x \in \Theta_s^t(y)\}.$$

We denote by $\mathcal{S}_x(s,t)$ *the* section *of the causal cone* $\mathcal{C}(t,x)$ *at time* $s < t$*,*

$$\mathcal{S}_x(s,t) := \{y \in E : (s,y) \in \mathcal{C}(t,x)\} = \{y \in E : x \in \Theta_s^t(y)\}. \tag{6.10}$$

In the case $T(x)$ is an absolutely continuous random variable, by (6.9) we have

$$S(t,x) = \exp\left\{-\int_0^t h(s,x)\mathrm{d}s\right\},$$

where

$$h(t,x) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(T(x) \in [t, t+\Delta t) \,|\, T(x) \geq t)}{\Delta t} \tag{6.11}$$

is the hazard function of $T(x)$, as defined by (6.8).
By the continuity of $T(x)$,

$$\begin{aligned} h(t,x) &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(T(x) \in (t, t+\Delta t] \,|\, T(x) > t)}{\Delta t} \\ &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in (\Theta^{t+\Delta t} \setminus \Theta^t) \,|\, x \notin \Theta^t)}{\Delta t} \\ &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} \,|\, x \notin \Theta^t)}{\Delta t}. \end{aligned}$$

Definition 6.7 *For all* $x \in \mathbb{R}^d$*, the function* $h(\cdot, x)$ *given by*

$$h(t,x) := \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} \,|\, x \notin \Theta^t)}{\Delta t}$$

is called the hazard function *associated with point* x*.*

Remark 6.8 If G is sufficiently regular such that it makes the evolution problem well posed, the causal cone $\mathcal{C}(t,x)$ is well-defined for any x, t, and the sections $\mathcal{S}_x(s,t)$ are such that $\dim \partial\mathcal{S}_x(s,t) < d$ for any x, s, t (see [19]), then from the definition of $\mathcal{C}(t,x)$, it easily follows that

$$S(t,x) = \mathbb{P}(N(\mathcal{C}(t,x) = 0)).$$

If we consider a general compact set $K \subset \mathbb{R}^d$, we may define the causal cone $\mathcal{C}(t,K)$ of K at time t, in an analogous way as in Definition 6.6:

$$\mathcal{C}(t,K) := \{(s,y) \in [0,t] \times \ E \, : \, K \cap \Theta_s^t(y) \neq \emptyset\}.$$

The same assumptions on the model guarantee that $\mathcal{C}(t,K)$ is well defined again.

In general cases (e.g. not necessarily Poissonian cases) expressions for the survival and the hazard functions may be obtained in terms of the birth process N and of the causal cone [22], but such expressions are quite complicated, because they must take into account all the previous history of the process. In particular, if the compensator $\tilde{\nu}$ of the marginal process $\tilde{N}$ (see Section 6.2.1) is discrete, then

$$S(t,x) = \prod_{s\in[0,t]} [1 - \mathbb{E}(\tilde{\nu}(\{s\})k(s,\mathcal{S}_x(s,t)) \,|\, N[\mathcal{C}(s-,\mathcal{S}_x(s-,t))] = 0)], \quad (6.12)$$

where $\prod$ is the so-called product-integral. (For a complete and elementary treatment of the basic theory of the product-integral see [35] and [5].)
While if $\tilde{\nu}$ is continuous, then

$$S(t,x) = \exp\left\{-\int_0^t \mathbb{E}(\tilde{\nu}(ds)k(s,\mathcal{S}_x(s,t)) \,|\, N[\mathcal{C}(s,\mathcal{S}_x(s,t))] = 0)\right\}. \quad (6.13)$$

Remark 6.9 For a general compact set $K \subset \mathbb{R}^d$ we may speak of $S(t,K) := \mathbb{P}(\Theta^t \cap K = \emptyset) = 1 - T_{\Theta^t}(K)$, as well, and (6.12) and (6.13) becomes

$$S(t,K) = \prod_{s\in[0,t]} [1 - \mathbb{E}(\tilde{\nu}(\{s\})k(s,\mathcal{S}_K(s,t)) \,|\, N[\mathcal{C}(s-,\mathcal{S}_K(s-,t))] = 0)],$$

$$S(t,K) = \exp\left\{-\int_0^t \mathbb{E}(\tilde{\nu}(ds)k(s,\mathcal{S}_K(s,t)) \,|\, N[\mathcal{C}(s,\mathcal{S}_x(s,t))] = 0)\right\},$$

respectively, where $\mathcal{S}_K(s,t)$ denotes here the section at time s of the causal cone $\mathcal{C}(t,K)$. Thus, by Choquet theorem (see Section 1.4.1) the probability distribution of Θ^t is uniquely determined, so that, our modelling assumptions on N and G really give rise to a time dependent random closed set Θ^t.
Clearly, if Θ^t is a random closed set for any fixed $t \in \mathbb{R}_+$, then also $\partial\Theta^t$ is so.

The Poissonian case

In the Poissonian case, the independence property of increments makes (6.13) simpler.
In fact (see Section 1.5.5), the independence of increments implies that the compensator of N is deterministic and coincides with its intensity measure Λ. Moreover, we know that a Poisson process has continuous intensity measure; thus by (6.13) it follows

$$S(t,x) = e^{-\Lambda(\mathcal{C}(t,x))} \quad (6.14)$$

We may further observe that (6.14) could be obtained as a direct consequence of the following theorem [46]:

Theorem 6.10 *Let Λ be a measure on $\mathbb{R}_+ \times \mathbb{R}^d$ such that $\Lambda(\cdot \times \mathbb{R}^d)$ is continuous and locally bounded. Suppose that Φ is a marked Poisson process with mark space $\mathbb{R}^d$ and intensity measure Λ. Then Φ (considered as a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$) is a Poisson process with intensity measure Λ.*

Thus, If N is a marked Poisson process with intensity $\alpha(t,x)$, it is easily seen that

$$S(t,x) = \mathbb{P}(x \notin \Theta^t) = \mathbb{P}(N(\mathcal{C}(t,x)) = 0) = e^{-\Lambda(\mathcal{C}(t,x))},$$

where $\Lambda(\mathcal{C}(t,x))$ is the volume of the causal cone with respect to the intensity measure Λ of the Poisson process:

$$\Lambda(\mathcal{C}(t,x)) = \int_{\mathcal{C}(t,x)} \alpha(s,y)\mathrm{d}s\mathrm{d}y.$$

In particular we have

$$h(t,x) = -\frac{\partial}{\partial t}\ln S(t,x) = \frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)), \tag{6.15}$$

provided that the derivative exists.
In [19] it is proved that, under sufficient regularity on G, $\Lambda(\mathcal{C}(t,x))$ is differentiable:

Proposition 6.11 *If the nucleation process is given by a marked Poisson process with intensity $\alpha(t,x)$, if G is Lipschitz-continuous with respect to the spatial variable and there exist $g_0, G_0 \in \mathbb{R}_+$ such that*

$$g_0 := \inf_{t\in[0,T],\, x\in\mathbb{R}^d} G(t,x), \qquad G_0 := \sup_{t\in[0,T],\, x\in\mathbb{R}^d} G(t,x),$$

then $\Lambda(\mathcal{C}(t,x))$ is continuously differentiable with respect to t and

$$\frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)) = G(t,x)\int_0^t \int_{\mathbb{R}^d} K(s,y;t,x)\alpha(s,y)\mathrm{d}y\ \mathrm{d}s. \tag{6.16}$$

with

$$K(s,y;t,x) := \int_{\{z\in\mathbb{R}^d|\tau(s,y;z)=t\}} \delta(z-x)\mathrm{d}a(z).$$

Here δ is the Dirac function, $\mathrm{d}a(z)$ is a $(d-1)$-dimensional surface element, and $\tau(s,y;z)$ is the solution of the eikonal problem

$$|\frac{\partial\tau}{\partial y}(s,y,x)| = \frac{1}{G(s,y)}\frac{\partial\tau}{\partial s}(s,y,x)$$

$$|\frac{\partial\tau}{\partial x}(s,y,x)| = \frac{1}{G(\tau(s,y,x),x)}.$$

6.2.4 A relation between hazard function and contact distribution function

From stochastic geometry we know that contact distributions are important tools to describe certain aspects of random closed sets and can be easily estimated (see [36]). Now we show how the hazard function is related to the *spherical contact distribution function* (see also [22]). Such a relation will be useful in the following in order to provide an evolution equation for the mean densities of Θ^t.

Definition 6.12 *The* local spherical contact distribution function $H_{S,\Xi}$ *of an inhomogeneous random set* Ξ *is given by*

$$H_{S,\Xi}(r,x) := \mathbb{P}(x \in \Xi_{\oplus r} \,|\, x \notin \Xi).$$

Proposition 6.13 *Let* N *be a nucleation process as in previous assumptions, with intensity measure* $\Lambda(\mathrm{d}t \times \mathrm{d}x) = \tilde{\Lambda}(\mathrm{d}t)Q(t,\mathrm{d}x)$*, such that* $Q(t,\cdot)$ *is absolutely continuous with respect to* ν^d $(d \geq 2)$*, and let the growth speed* G *of crystals be constant. For any* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ *fixed, we denote by* $\tilde{H}(\cdot,t,x)$ *the following function*

$$\tilde{H}(\tau,t,x) := H_{S,\Theta^t}(G\tau,x),$$

where H_{S,Θ^t} *is the local spherical contact distribution function of the random closed set* Θ^t.
Then

$$h(t,x) = \frac{\partial}{\partial \tau}\tilde{H}(\tau,t,x)_{|\tau=0} \qquad \text{for a.e. } x. \tag{6.17}$$

Proof. We write the random set $\Theta^{t+\Delta t}$ as union of the set Θ^t grown up to time $t+\Delta t$ and the crystals born during the interval $(t,t+\Delta t]$:

$$\begin{aligned}
\Theta^{t+\Delta t} &= \bigcup_{\substack{j \\ T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j) \\
&= \bigcup_{\substack{j \\ T_j \leq t}} \Theta^{t+\Delta t}_{T_j}(X_j) \;\cup \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j) \\
&= \Theta^t_{\oplus \Delta t G} \;\cup \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j).
\end{aligned}$$

Let $\tilde{\mathcal{C}}_x(t,t+\Delta t)$ be the subset of $\mathbb{R}_+ \times \mathbb{R}^d$ so defined:

$$\tilde{\mathcal{C}}_x(t,t+\Delta t) := \{(s,y) \in (t,t+\Delta t] \times \mathbb{R}^d \,|\, x \in \Theta^{t+\Delta t}_s(y)\}.$$

Note that, since G is constant, for any fixed $s \in (t,t+\Delta t]$, the section of $\tilde{\mathcal{C}}_x(t,t+\Delta t)$ at time s is given by the ball $B_{(t+\Delta t-s)G}(x)$ of radius $(t+\Delta t-s)G$ and centre x. Thus, it follows that

$$\nu^{d+1}(\tilde{\mathcal{C}}_x(t,t+\Delta t)) = \int_t^{t+\Delta t} \left(\int_{B_{(t+\Delta t-s)G}(x)} \mathrm{d}y \right) \mathrm{d}s = O(\Delta t)^{d+1}.$$

Moreover, since during an infinitesimal time interval Δt at most only one nucleation may occur, we have that

$$\mathbb{P}(N(\tilde{\mathcal{C}}_x(t,t+\Delta t)) > 0 \,|\, x \notin \Theta^t) = \mathbb{E}(N(\tilde{\mathcal{C}}_x(t,t+\Delta t)) \,|\, x \notin \Theta^t).$$

As a consequence the following chain of equality holds:

$$
\begin{aligned}
&\mathbb{P}(x \notin \Theta^{t+\Delta t} \mid x \notin \Theta^t) \\
&= \mathbb{P}(x \notin (\Theta^t_{\oplus \Delta t G} \cup \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j)) \mid x \notin \Theta^t) \\
&= \mathbb{P}(\{x \notin \Theta^t_{\oplus \Delta t G}\} \cap \{x \notin \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j)\} \mid x \notin \Theta^t) \\
&= \mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G} \mid x \notin \Theta^t)\mathbb{P}(x \notin \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j) \mid \{x \notin \Theta^t\} \cap \{x \notin (\Theta^t_{\oplus \Delta t G}\}) \\
&= \mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G} \mid x \notin \Theta^t)\mathbb{P}(x \notin \bigcup_{\substack{j \\ t<T_j \leq t+\Delta t}} \Theta^{t+\Delta t}_{T_j}(X_j) \mid x \notin \Theta^t_{\oplus \Delta t G}) \\
&= [1 - H_{S,\Theta^t}(G\Delta t, x)](\mathbb{P}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t)) = 0 \mid x \notin \Theta^t_{\oplus \Delta t G}) \\
&= [1 - \tilde{H}(\Delta t, t, x)][1 - \mathbb{P}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t)) > 0 \mid x \notin \Theta^t_{\oplus \Delta t G})] \\
&= [1 - \tilde{H}(\Delta t, t, x)][1 - \mathbb{E}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t)) \mid x \notin \Theta^t_{\oplus \Delta t G}) + o(\Delta t)].
\end{aligned}
$$

By replacing in the definition of $h(t, x)$, we obtain

$$
h(t, x) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\{1 - [1 - \tilde{H}(\Delta t, t, x)] \cdot [1 - \mathbb{E}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t)) \mid x \notin \Theta^t_{\oplus \Delta t G})]\}. \tag{6.18}
$$

Now observe that:

- since $\tilde{H}(0, t, x) = 0$,

$$
\lim_{\Delta t \downarrow 0} \frac{\tilde{H}(\Delta t, t, x)}{\Delta t} = \frac{\partial}{\partial \tau}\tilde{H}(\tau, t, x)_{|\tau=0};
$$

- since $N(\tilde{\mathcal{C}}_x(t, t+\Delta t))$ is a nonnegative r.v., and we may assume that $\mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G}) \neq 0$,

$$
\begin{aligned}
&\lim_{\Delta t \downarrow 0} \frac{\mathbb{E}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t)) \mid x \notin \Theta^t_{\oplus \Delta t G})}{\Delta t} \\
&= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \frac{\mathbb{E}(N(\tilde{\mathcal{C}}_x(t, t+\Delta t))\mathbf{1}\{x \notin \Theta^t_{\oplus \Delta t G}\})}{\mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G})} \\
&\leq \frac{1}{\mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G})} \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}[N(\tilde{\mathcal{C}}_x(t, t+\Delta t))]}{\Delta t} \\
&= \frac{1}{\mathbb{P}(x \notin \Theta^t_{\oplus \Delta t G})} \lim_{\Delta t \downarrow 0} \frac{\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t))}{\Delta t}.
\end{aligned}
$$

We know that $\Lambda(\mathrm{d}s \times \mathrm{d}y) = Q(s, \mathrm{d}y)\tilde{\Lambda}(\mathrm{d}s)$, where $\tilde{\Lambda}$ is the intensity measure of the marginal process and Q is a stochastic kernel from $\mathbb{R}_+$ to $\mathbb{R}^d$. Since by hypothesis $Q(s, \cdot\,)$ is absolutely continuous with respect to the d-dimensional

Lebesgue measure, then, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned}\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t)) &= \int_t^{t+\Delta t} \int_{B_{(t+\Delta t - s)G}(x)} Q(s, \mathrm{d}y)\tilde{\Lambda}(\mathrm{d}s) \\ &= O(\Delta t)^d \int_t^{t+\Delta t} \tilde{\Lambda}(\mathrm{d}s).\end{aligned}$$

Since certainly $\displaystyle\int_t^{t+\Delta t} \tilde{\Lambda}(\mathrm{d}s) < \infty$, we have $\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t)) = O(\Delta t)^d$.
Therefore, $\forall d \geq 2$

$$\lim_{\Delta t \downarrow 0} \frac{\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t))}{\Delta t} = 0.$$

By (6.18) we obtain

$$h(t, x) = \frac{\partial}{\partial \tau} \tilde{H}(\tau, t, x)_{|\tau=0}$$

$\square$

Remarks:

1. Expression (6.17) may be intuitively explained in this way: capture of point x during interval $(t, t+\Delta t]$ can be determined both by the growth of Θ^t and by the birth of new crystals; the last one has a negligible weight because of the simplicity of $\tilde{N}$ and the absolute continuity of mark distribution.

2. If $\Lambda \ll \nu_{d+1}$, i.e. $\tilde{\Lambda}$ absolutely continuous, then $\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t)) = O(\Delta t)^{d+1}$ and (6.17) is true also for $\dim \mathbb{R}^d = 1$.

Case G time dependent

Let us suppose that G, with the required regularity assumptions, is time dependent. In this case $\tilde{H}$ is given by

$$\tilde{H}(\tau, t, x) := H_{S,\Theta^t}\left(\int_t^{t+\tau} G(s)\mathrm{d}s, x\right).$$

Now, for any $s \in (t, t+\Delta t]$, the section of $\tilde{\mathcal{C}}_x(t, t+\Delta t)$ at time s is given by the ball centered in x with radius $\int_s^{t+\Delta t} G(u)\mathrm{d}u$. As a consequence, we have again that $\Lambda(\tilde{\mathcal{C}}_x(t, t+\Delta t)) = O(\Delta t)^d$, and so

$$h(t, x) = \frac{\partial}{\partial \tau} \tilde{H}(\tau, t, x)_{|\tau=0} = G(t) \frac{\partial}{\partial r} H_{S,\Theta^t}(r, x)_{|r=0}.$$

Case G space-time dependent

If G, with the required regularity assumptions, is space-and-time dependent, then we can not define a function $\tilde{H}$ similarly as in the previous cases, because Θ^t does not grow in a homogeneous way during a time interval $(t, t+\tau]$.

But, as follows by the proof, the hazard function $h(t, x)$ coincides with the capture rate of x determined by the growth of Θ^t. In fact, also in this case

$\Lambda(\tilde{\mathcal{C}}_x(t,t+\Delta t)) = O(\Delta t)^d$, since certainly the section of $\tilde{\mathcal{C}}_x(t,t+\Delta t)$ at $s \in (t,t+\Delta t]$ is a subset of $B_{(t+\Delta t-s)G_M}(x)$, where $G_M = \max_{u,y} G(u,y)$.
So, if we denote by $\Theta(t+\Delta t)$ the random closed set Θ^t grown up to time $t+\Delta t$, then

$$h(t,x) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta(t+\Delta t) \,|\, x \notin \Theta^t)}{\Delta t}. \tag{6.19}$$

By the uniform continuity of G, for Δt sufficiently small, during the interval $(t,t+\Delta t]$ we may consider $G = G(t,\cdot\,)$, so that it depends only on space.

Observe now that when the growth speed of crystals depends only on time or space (i.e. $G = G(t)$ or $G = G(x)$), then

$$x \in \Theta_s^t(y) \Longleftrightarrow y \in \Theta_s^t(x), \tag{6.20}$$

while if $G = G(t,x)$ it is not true in general.

Since every point $y \in \partial\Theta^t$ may be seen as nucleus of a crystal born at time t, then $x \in \Theta(t+\Delta t) \Longleftrightarrow \exists y \in \partial\Theta^t$ such that $x \in \Theta_t^{t+\Delta t}(y)$. By (6.20) we have

$$x \in \Theta(t+\Delta t) \Longleftrightarrow \Theta^t \cap \Theta_t^{t+\Delta t}(x) \neq \emptyset.$$

The continuity of G allows to replace $\Theta_t^{t+\Delta t}(x)$ with $B_{G(t,x)\Delta t}(x)$, thus

$$x \in \Theta(t+\Delta t) \Longleftrightarrow \Theta^t \cap B_{G(t,x)\Delta t}(x) \neq \emptyset,$$

and by (6.19) we obtain

$$\begin{aligned} h(t,x) &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^t_{\oplus G(t,x)\Delta t} \,|\, x \notin \Theta^t)}{\Delta t} \\ &= \lim_{\Delta t \downarrow 0} \frac{H_{S,\Theta^t}(G(t,x)\Delta t, x)}{\Delta t} \\ &= G(t,x) \frac{\partial}{\partial r} H_{S,\Theta^t}(r,x)_{|r=0}. \end{aligned} \tag{6.21}$$

6.2.5 Continuity of the time of capture $T(x)$

Let x be a fixed point in $\mathbb{R}^d$ and $T(x)$ be the positive random variable representing the time of capture of point x by the birth-and-growth process $\{\Theta^t\}$. It is clear that the continuity of $T(x)$ depends on the birth process N and the growth field G.

In [22] is shown that the continuity of $T(x)$ is strictly related to the absolute continuity of the mark distribution of the MPP N. More precisely, if for any $t \in \mathbb{R}_+$ $Q(t,\cdot)$ is the mark distribution at time t (see Section 1.5.5), from the simplicity of the marginal process $\tilde{N}$, we have, formally,

$$\begin{aligned} Q(t,B)\tilde{\Lambda}(\mathrm{d}t) &= \Lambda(\mathrm{d}t \times B) \\ &= \mathbb{E}(N(\mathrm{d}t \times B)) \\ &= \mathbb{P}(N(\mathrm{d}t \times B) = 1) \\ &= \mathbb{P}(X_1 \in B \,|\, \tilde{N}(\mathrm{d}t) = 1)\mathbb{P}(\tilde{N}(\mathrm{d}t) = 1) \\ &= \mathbb{P}(X_1 \in B \,|\, \tilde{N}(\mathrm{d}t) = 1)\tilde{\Lambda}(\mathrm{d}t), \end{aligned}$$

where Λ and $\tilde{\Lambda}$ are the intensity measure of N and $\tilde{N}$, respectively, and X_1 is the nucleus born during the infinitesimal interval $\mathrm{d}t$.
Therefore the mark distribution $Q(t, B)$ represents the probability that a nucleus $X_1 \in B$, given that it was born during $[t, t + \mathrm{d}t)$.

Proposition 6.14 *If $Q(s, \cdot)$ is absolutely continuous with respect to ν^d for a.e. $s \in \mathbb{R}_+$, then $T(x)$ is a continuous random variable.*

Proof. The hypothesis of absolute continuity of $Q(s, \cdot)$ implies that the probability that a nucleus is born in a lower dimensional set is 0; in particular $Q(s, \partial\mathcal{S}_x(s, t)) = 0$ since our regularity assumptions on the growth model imply that $\dim_{\mathcal{H}} \partial\mathcal{S}_x(s, t) < d \ \forall s < t$.
By absurd let $T(x)$ not be a continuous r.v.; then there exists $t \in \mathbb{R}_+$ such that $\mathbb{P}(T(x) = t) > 0$. Observe that

$$\mathbb{P}(T(x) = t) > 0 \Leftrightarrow \mathbb{P}(N(\partial\mathcal{C}(t, x)) \neq 0) > 0 \Leftrightarrow \mathbb{E}(N(\partial\mathcal{C}(t, x))) > 0.$$

But,

$$\mathbb{E}(N(\partial\mathcal{C}(t, x))) = \int_0^t \int_{\partial\mathcal{S}_x(s,t)} Q(s, \mathrm{d}y)\tilde{\Lambda}(\mathrm{d}s) = 0.$$

□

Now, let us suppose that the compensator $\tilde{\nu}$ of the marginal process $\tilde{N}$ is continuous; by (6.13) we know that

$$\mathbb{P}(T(x) \leq t) = 1 - \exp\left\{ -\int_0^t \mathbb{E}(\tilde{\nu}(ds)k(s, \mathcal{S}_x(s, t)) \mid N[\mathcal{C}(s, \mathcal{S}_x(s, t))] = 0) \right\},$$

so that we may claim that

Claim 6.15

$$T(x) \ \textit{admits pdf} \ p_{T(x)}$$
$$\Updownarrow$$
$$\exists \frac{\partial}{\partial t} \int_0^t \mathbb{E}(\tilde{\nu}(\mathrm{d}s)k(s, \mathcal{S}_x(s, t)) \mid N[\mathcal{C}(s, \mathcal{S}_x(s, t))] = 0) \tag{6.22}$$

Thus, a problem of interest is to find sufficient conditions on the birth-and-growth process such that the integral in (6.22) is differentiable with respect to t.
We may like to notice that if N is a marked Poisson process, such integral coincides with the measure $\Lambda(\mathcal{C}(t, x))$ of the causal cone with respect to the intensity measure Λ.
By Proposition 6.11 we may claim that if N is given by a marked Poisson process with intensity $\alpha(t, x)$, under the regularity assumptions on G, $\Lambda(\mathcal{C}(t, x))$ is differentiable and so $T(x)$ is an absolutely continuous random variable, i.e. it admits a probability density function $p_{T(x)}$.
In particular, consider a marked Poisson process N with intensity measure

$$\Lambda(\mathrm{d}t \times \mathrm{d}x) = \lambda(t)\mathrm{d}tQ(t, \mathrm{d}x)$$

such that $Q(t,\cdot)$ is a continuous probability measure for any fixed $t \in \mathbb{R}_+$. Then

$$\frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)) = \frac{\partial}{\partial t}\int_0^t \int_{\mathcal{S}_x(s,t)} \lambda(s) Q(s,\mathrm{d}x)\ \mathrm{d}s = \frac{\partial}{\partial t}\int_0^t Q(s,\mathcal{S}_x(s,t))\lambda(s)\mathrm{d}s.$$

Note that $\mathcal{S}_x(t,t) = x$, and the continuity of $Q(t,\cdot)$ implies $Q(t,x) = 0$. So, if the derivative of $Q(s,\mathcal{S}_x(s,t))$ with respect to t exists for almost every time s, and if it is integrable in $[0,t]$, then we have that

$$\frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)) = \int_0^t \lambda(s)\frac{\partial}{\partial t}Q(s,\mathcal{S}_x(s,t))\mathrm{d}s;$$

hence the absolute continuity of the random variable $T(x)$ is related to the existence of the time derivative $\frac{\partial}{\partial t}Q(s,\mathcal{S}_x(s,t))$.
Note that Q turns to be a function of the causal cone, since it depends on the sections $\mathcal{S}_x(s,t)$, as expected.
As a simple example, let N be independent marking (see Section 1.5.5) with mark space a bounded subset K of $\mathbb{R}^d$ and marks uniformly distributed in K, so that $Q(\mathrm{d}x) = \dfrac{1}{\nu^d(K)}\mathrm{d}x$. If the growth velocity G is constant, it is clear that, for all $s,t \in \mathbb{R}_+$ and $x \in K$, $\mathcal{S}_x(t,x) = B_{G(t-s}(x)$, hence we have that

$$\frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)) = \int_0^t \lambda(s)\frac{\partial}{\partial t}\frac{b_d G^d (t-s)^d}{\nu^d(K)}\mathrm{d}s = \frac{(d-1)b_d G^d}{\nu^d(K)}\int_0^t \lambda(s)(t-s)^{d-1}\mathrm{d}s;$$

therefore, if the intensity λ of the underlying Poisson point process $\tilde{N}$ is such that the last integral exists for any t, we may conclude that the random variable $T(x)$ is absolutely continuous with pdf $p_{T(x)}$ given by

$$p_{T(x)} = \exp\left\{\frac{b_d G^d}{\nu^d(K)}\int_0^t \lambda(s)(t-s)^d \mathrm{d}s\right\}\frac{(d-1)b_d G^d}{\nu^d(K)}\int_0^t \lambda(s)(t-s)^{d-1}\mathrm{d}s.$$

6.2.6 An evolution equation for the mean densities

Let us consider a birth-and-growth process $\{\Theta^t\}$ with nucleation process N and growth rate G satisfying the regularity assumptions introduced in the previous sections, in order to have that:

1. for any $t \in \mathbb{R}_+$, and any $s > 0$, $\partial\Theta^t \subset \mathrm{int}\Theta^{t+s}$;
2. for any $t \in \mathbb{R}_+$, Θ^t is a d-regular random closed set in $\mathbb{R}^d$, and $\partial\Theta^t$ is a $(d-1)$-regular random closed set;
3. $T(x)$ is a continuous random variable with pdf $p_{T(x)}(t)$;
4. for any bounded $A \in \mathcal{B}_{\mathbb{R}^d}$ such that $\mathbb{P}(\mathcal{H}^{d-1}(\partial\Theta^t \cap \partial A) > 0) = 0$,

$$\lim_{r\to 0}\frac{\mathbb{E}[\mathcal{H}^d(\Theta^t_{\oplus r} \setminus \Theta^t \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)]. \tag{6.23}$$

(We remind that assumptions 1. and 2. have been discussed in Section 6.1.1, assumption 3. in the Section 6.2.5, while assumption 4. in Chapter 5).

Note that, in terms of weak* convergence of linear functionals, (6.23) becomes

$$\lim_{r\to 0}\frac{\mathbb{E}[\delta_{\Theta^t_{\oplus r}}](x)-\mathbb{E}[\delta_{\Theta^t}](x)}{r}=\mathbb{E}[\delta_{\partial\Theta^t}](x). \tag{6.24}$$

By (6.11), we know that the continuity of $T(x)$ implies that $h(\cdot,x)$ coincides with the hazard function associated with $T(x)$; as a consequence, since $\mathbb{P}(T(x)>t)=\mathbb{P}(x\notin\Theta^t)$, the following holds:

$$h(t,x)=\frac{p_{T(x)}(t)}{\mathbb{P}(x\notin\Theta^t)}. \tag{6.25}$$

For any fixed $t\in\mathbb{R}_+$, let us consider the spherical contact distribution $H_{S,\Theta^t}(\cdot,x)$ of the crystallized region Θ^t associated to a point x. By definition

$$H_{S,\Theta^t}(r,x)=\frac{\mathbb{P}(x\in(\Theta^t_{\oplus r}\setminus\Theta^t))}{\mathbb{P}(x\notin\Theta^t)}, \tag{6.26}$$

and so, for the birth-and-growth model we consider, by (6.21) it follows

$$h(t,x)=\frac{G(t,x)}{\mathbb{P}(x\notin\Theta^t)}\frac{\partial}{\partial r}\mathbb{P}(x\in(\Theta^t_{\oplus r}\setminus\Theta^t))_{|r=0}. \tag{6.27}$$

Thus, we obtain that

$$\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x)\overset{(6.4)}{=}p_{T(x)}(t)\overset{(6.25)}{=}h(t,x)\mathbb{P}(x\notin\Theta^t)\overset{(6.27)}{=}G(t,x)\frac{\partial}{\partial r}\mathbb{P}(x\in(\Theta^t_{\oplus r}\setminus\Theta^t))_{|r=0}. \tag{6.28}$$

Now, we may notice that

$$\begin{aligned}\frac{\partial}{\partial r}\mathbb{P}(x\in(\Theta^t_{\oplus r}\setminus\Theta^t))_{|r=0} &= \lim_{h\to 0}\frac{\mathbb{P}(x\in\Theta^t_{\oplus h})-\mathbb{P}(x\in\Theta^t)}{h}\\ &= \lim_{h\to 0}\frac{\mathbb{P}(x\in\Theta^t_{\oplus h})-\mathbb{P}(x\in\Theta^t)}{h}\\ &= \lim_{h\to 0}\frac{\mathbb{E}[\delta_{\Theta^t_{\oplus h}}](x)-\mathbb{E}[\delta_{\Theta^t}](x)}{h}\\ &\overset{(6.24)}{=} \mathbb{E}[\delta_{\partial\Theta^t}](x),\end{aligned}$$

so that, by (6.28), we may claim that the following evolution equation holds for the mean density $\mathbb{E}[\delta_{\Theta^t}](x)$:

$$\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x)=G(t,x)\mathbb{E}[\delta_{\partial\Theta^t}](x), \tag{6.29}$$

to be taken, as usual, in weak form.
We may summarize as follows.

Proposition 6.16 *Under the above assumptions 1.-4., we have that the following evolution equation (to be taken in weak form) holds for the mean density* $\mathbb{E}[\delta_{\Theta^t}](x)$ *:*

$$\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x)=G(t,x)\mathbb{E}[\delta_{\partial\Theta^t}](x). \tag{6.30}$$

Remark 6.17 Since for any fixed $x \in \mathbb{R}^d$ we said that the time of capture $T(x)$ is a continuous random variable with probability density function $p_{T(x)}(t)$, it is clear by (6.28) that $\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x)$ is a classical real function. It follows that $\mathbb{E}[\delta_{\partial\Theta^t}](x)$ is a classical real function as well. As a consequence, $\mathbb{E}[\delta_{\partial\Theta^t}](x)$ is a version of the usual Radon-Nikodym derivative of the measure $\mathbb{E}[\mu_{\partial\Theta^t}]$ with respect to ν^d, and so we may claim that Θ^t is absolutely continuous (Section 3.2).

Remark 6.18 In the particular case of spherical growth, i.e. when G is constant, it is clear that

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\mathcal{H}^d(\Theta^{t+\Delta t}(\omega)\setminus\Theta^t(\omega))}{\Delta t} &= \lim_{\Delta t\to 0}\frac{\mathcal{H}^d(\Theta^t_{\oplus G\Delta t}(\omega)\setminus\Theta^t(\omega))}{\Delta t}\\ &= G\lim_{r\to 0^+}\frac{\mathcal{H}^d(\Theta^t_{\oplus r}(\omega)\setminus\Theta^t(\omega))}{r},\end{aligned}$$

and, for any $t \in \mathbb{R}_+$, $\Theta^t(\omega)$ satisfies a first order Steiner formula, so that we have

$$\lim_{\Delta t\to 0}\frac{\mathcal{H}^d(\Theta^{t+\Delta t}(\omega)) - \mathcal{H}^d(\Theta^t(\omega)))}{\Delta t} = G\mathcal{H}^{d-1}(\partial\Theta^t(\omega))\,.$$

In particular Θ^t satisfies Theorem 5.27 (and so a local mean first order Steiner formula), and we may write, as expected,

$$\frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}](x) = G\mathbb{E}[\delta_{\partial\Theta^t}](x).$$

For the growth process Θ^t introduced in the previous sections, we may notice that the evolution of the realization $\Theta^t(\omega)$ may be described for a.e. $\omega \in \Omega$ by the following (weak) equation (e.g. [12, 17]):

$$\frac{\partial}{\partial t}\delta_{\Theta^t}(x) = G(t,x)\delta_{\partial\Theta^t}(x). \tag{6.31}$$

The advantage of this expression, even though to be understood in a weak sense in terms of viscosity solutions, is in the fact that it makes explicit the local dependence (both in time and space) upon the growth field G by means of the (geometric) Dirac delta at a point $x \in \partial\Theta^t$. In this way equation (6.30) can be formally obtained by taking the expected value in (6.31), thanks to the linearity of expectation, since we have assumed that G is a deterministic function. (Obviously, it involves exchanges between limit and expectation, as in (6.23) for example). We have shown that indeed, under the suitable regularity assumptions on the process Θ^t, we may obtain (6.30) from (6.31) in a rigorous way.

Remark 6.19 By Eq. (6.31), for $t > t_0$ we have

$$\int_A \delta_{\Theta^t}(x)\mathrm{d}x = \int_A \delta_{\Theta^{t_0}}(x)\mathrm{d}x \;+\; \int_{t_0}^t\int_A G(x,s)\delta_{\partial\Theta^s}(x)\mathrm{d}x\mathrm{d}s.$$

Note that if, as particular case, $G(t,x)\equiv 1$ and $A \equiv \mathbb{R}^d$, then

$$\Theta^s = \Theta^{t_0}_{\oplus s-t_0} \qquad \forall s > t_0,$$

and so, by the change of variable $s - t_0 = y$ and denoting $r = t - t_0$, we have the well known following result (see 3.2.34 in [32], or [37]):

$$\mathcal{H}^d(\Theta^{t_0}_{\oplus r}) = \mathcal{H}^d(\Theta^{t_0}) \ + \int_0^r \mathcal{H}^{d-1}(\partial \Theta^{t_0}_{\oplus y}) \ \mathrm{d}y.$$

Let us observe now that, since $T(x)$ is a continuous random variable, then

$$\mathbb{P}(x \notin \Theta^t) = \mathbb{P}(x \notin \mathrm{int}\Theta^t),$$

so that, by Definition 6.7, we have

$$\begin{aligned} h(t,x) &= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t}) - \mathbb{P}(x \in \Theta^t)}{\mathbb{P}(x \notin \mathrm{int}\Theta^t)} \\ &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} \mid x \notin \mathrm{int}\Theta^t) - \mathbb{P}(x \in \Theta^t \mid x \notin \mathrm{int}\Theta^t)}{\Delta t} \\ &= \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}[\delta_{\Theta^{t+\Delta t}}(x) \mid x \notin \mathrm{int}\Theta^t] - \mathbb{E}[\delta_{\Theta^t} \mid x \notin \mathrm{int}\Theta^t]}{\Delta t} \\ &= \frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}(x) \mid x \notin \mathrm{int}\Theta^t] \end{aligned} \tag{6.32}$$

By comparing (6.31) with (6.32), we may claim that

$$h(t,x) = G(t,x)\mathbb{E}[\delta_{\partial\Theta^t}(x) \mid x \notin \mathrm{int}\Theta^t],$$

which leads to the interesting interpretation

$$\frac{\partial}{\partial r} H_{S,\Theta^t}(r,x)_{|r=0} = \mathbb{E}[\delta_{\partial\Theta^t}(x) \mid x \notin \mathrm{int}\Theta^t].$$

Moreover, it follows that

$$\mathbb{E}[\delta_{\partial\Theta^t}](x) = \frac{\partial}{\partial r}\mathbb{P}(x \in (\Theta^t_{\oplus r} \setminus \Theta^t))_{|r=0} \stackrel{(6.26)}{=} \mathbb{P}(x \notin \Theta^t)\frac{\partial}{\partial r} H_{S,\Theta^t}(r,x)_{|r=0}.$$

6.2.7 Mean densities

The expected value of the generalized density δ_{Θ^t} is what is usually called *crystallinity*, and denoted by

$$V_V(t,x) := \mathbb{E}[\delta_{\Theta^t}](x),$$

while the density of the expected measure $\mathbb{E}[\mu_{\partial\Theta^t}]$ is what is usually called *mean surface density*, and denoted by

$$S_V(t,x) := \mathbb{E}[\delta_{\partial\Theta^t}](x).$$

With these notations, the equation (6.30) becomes

$$\frac{\partial}{\partial t} V_V(t,x) = G(t,x) S_V(t,x), \tag{6.33}$$

In some cases (such as in Poisson birth processes, as we shall later discuss as an example), it can be of interest to consider the following extended mean densities.

Definition 6.20 *We call* mean extended volume density *at point x and time t the quantity $V_{ex}(t,x)$ such that, for any $B \in \mathcal{B}_{\mathbb{R}^d}$,*

$$\mathbb{E}[\sum_{j:T_j \le t} \nu^d(\Theta^t_{T_j}(X_j) \cap B)] = \int_B V_{ex}(t,x)\nu^d(\mathrm{d}x).$$

It represents the mean of the sum of the volume densities, at time t, of the grains supposed free to be born and grow.

Correspondingly,

Definition 6.21 *We call* mean extended surface density *at point x and time t the quantity $S_{ex}(t,x)$ such that, for any $B \in \mathcal{B}_{\mathbb{R}^d}$,*

$$\mathbb{E}[\sum_{j:T_j \le t} \nu^{d-1}(\partial\Theta^t_{T_j}(X_j) \cap B)] = \int_B S_{ex}(t,x)\nu^d(\mathrm{d}x).$$

It represents the mean of the sum of the surface densities, at time t, of the grains supposed free to be born and grow.

Under our assumptions on the growth model, we can claim, by linearity arguments, that

$$\frac{\partial}{\partial t}V_{ex}(t,x) = G(t,x)S_{ex}(t,x), \tag{6.34}$$

to be taken, as usual, in weak form.

The complement to 1 of the crystallinity, also known as *porosity* and denoted by $p_x(t)$, represents the survival function $S(t,x)$ of the point x at time t:

$$p_x(t) = 1 - V_V(t,x) = \mathbb{P}(x \notin \Theta^t) = \mathbb{P}(T(x) > t).$$

We remember that in our assumptions $T(x)$ is a continuous random variable with probability density function $p_{T(x)}(t)$. So, by the previous sections, we have

$$p_{T(x)}(t) = \frac{\partial}{\partial t}(1 - p_x(t)) = \frac{\partial V_V(t,x)}{\partial t}$$

and

$$p_{T(x)}(t) = p_x(t)h(t,x),$$

from which we immediately obtain

$$\frac{\partial}{\partial t}V_V(t,x) = (1 - V_V(t,x))h(t,x). \tag{6.35}$$

This is an extension of the well known Avrami-Kolmogorov formula [43, 10], which has been proven for space homogeneous birth and growth rates [54].

When results exposed in Chapter 4 apply to the random set Θ^t (in particular when $\partial\Theta^t$ is absolutely continuous in mean), it is possible to give an approximation of the mean surface density S_V.

Example: let us consider the particular case in which the growth rate G is constant. Then, for any fixed time t, Θ^t is the union of a finite and random number of random balls in $\mathbb{R}^d$:

$$\Theta^t = \bigcup_{i:T_i \le t} B_{G(t-T_i)}(X_i).$$

As a consequence of Theorem 4.10 we have that Proposition 4.5 applies, so that

$$\lim_{r\to 0} \frac{\mathbb{E}[\mathcal{H}^d((\partial\Theta^t)_{\oplus r} \cap A))]}{2r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)].$$

If the random set $\partial\Theta^t$ turns out to be absolutely continuous in mean, then we have

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \partial\Theta^t_{\oplus r})}{b_{d-n} r^{d-n}} \mathrm{d}x = \int_A S_V(t,x)\mathrm{d}x.$$

As a simple example in which $\partial\Theta^t$ is absolutely continuous, but not stationary, let us consider a nucleation process N given by an inhomogeneous Poisson point process, with intensity $\alpha(t,x)$. We may prove this as follows:
By absurd, let $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap \cdot)]$ be not absolutely continuous with respect to ν^d; then there exists $A \subset \mathbb{R}^d$ with $\nu^d(A) = 0$ such that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)] > 0$. It is clear that

$$\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)] > 0 \;\Rightarrow\; \mathbb{P}(\mathcal{H}^{d-1}(\partial\Theta^t \cap A) > 0) > 0,$$

and

$$\mathbb{P}(\mathcal{H}^{d-1}(\partial\Theta^t \cap A) > 0) \le \mathbb{P}(\exists (T_j, X_j) \,:\, \mathcal{H}^{d-1}(\partial B_{G(t-T_j)}(X_j) \cap A) > 0).$$

As a consequence, we have

$$\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)] > 0 \;\Rightarrow\; \mathbb{P}(\Phi(\mathcal{A}) \neq 0) > 0,$$

where

$$\mathcal{A} := \{(s,y) \in [0,t] \times \mathbb{R}^d \,:\, \mathcal{H}^{d-1}(\partial B_{G(t-s)}(y) \cap A) > 0)\}.$$

Denoting by $\mathcal{A}_s := \{y \in \mathbb{R}^d \,:\, (s,y) \in \mathcal{A}\}$ the section of $\mathcal{A}$ at time s, and by $\mathcal{A}^y := \{s > 0 \,:\, (s,y) \in \mathcal{A}\}$ the section of $\mathcal{A}$ at y, we notice that $\nu^1(\mathcal{A}^y) = 0$ for all y, because $\nu^d(A) = 0$ (it suffices to use spherical coordinates centered at y to obtain that ν^1-a.e. ball with radius s centered at y intersects A in a $\mathcal{H}^{d-1}$-negligible set). Therefore we may apply Fubini's theorem to get

$$\int_0^\infty \nu^d(\mathcal{A}_s)\,\mathrm{d}s = \int_0^\infty \int_{\mathbb{R}^d} \chi_A \,\mathrm{d}y\,\mathrm{d}s = \int_{\mathbb{R}^d} \int_0^\infty \chi_A \,\mathrm{d}s\,\mathrm{d}y = \int_{\mathbb{R}^d} \nu^1(\mathcal{A}^y)\,\mathrm{d}y = 0.$$

It follows that $\nu^d(\mathcal{A}_s) = 0$ for ν^1-almost every $s \in [0,t]$, and so

$$\mathbb{E}[\Phi(\mathcal{A})] = \int_{\mathcal{A}} \alpha(s,y)\,\mathrm{d}s\mathrm{d}y = \int_0^t \int_{\mathcal{A}_s} \alpha(s,y)\,\mathrm{d}y\,\mathrm{d}s = 0.$$

But this is an absurd, since

$$\mathbb{P}(\Phi(\mathcal{A}) \neq 0) > 0 \;\Rightarrow\; \mathbb{E}[\Phi(\mathcal{A})] > 0.$$

Remark 6.22 (Random Johnson-Mehl tessellations) In a birth-and-growth process as in the previous example, where N is a time inhomogeneous Poisson marked point process, one may consider the associated random Johnson-Mehl tessellation generated by the impingement of two grains which stop their growth at points of contact (see [54]); briefly, the system of n-facets of a Johnson-Mehl tessellation at time $t > 0$ is a random finite union of a system of random n-regular sets $F_i^{(n)}(t)$, $0 \leq n \leq d$:

$$\Xi_n^t := \bigcup_i F_i^{(n)}(t).$$

Again, it can be shown that Proposition 4.5 applies, so that we may approximate mean n-facet densities, for all $0 \leq n \leq d$.

The Poissonian case

We remember that, if the nucleation process N is a marked Poisson process with intensity measure Λ and G satisfies the required regularity assumptions, by (6.17) we have that

$$h(t,x) = \frac{\partial}{\partial t}\Lambda(\mathcal{C}(t,x)).$$

As we will show in the next Section (see also [18]), the volume of the causal cone can be expressed in terms of the extended volume density:

Theorem 6.23 *Under the previous modelling assumptions on birth and on growth, the following equality holds*

$$\Lambda(\mathcal{C}(t,x)) = V_{ex}(t,x). \tag{6.36}$$

As a consequence

$$h(t,x) = \frac{\partial}{\partial t}V_{ex}(t,x)$$

so that

$$\frac{\partial}{\partial t}V_V(t,x) = (1 - V_V(t,x))\frac{\partial}{\partial t}V_{ex}(t,x).$$

This equation is exactly the Kolmogorov-Avrami formula extended to a birth-and-growth process with space-time inhomogeneous parameters of birth and of growth [43, 10].

Finally, by direct comparison between (6.36), (6.16) and (6.34), we may claim that

$$S_{ex}(t,x) = \int_0^t \int_{\mathbb{R}^d} K(s,y;t,x)\alpha(s,y)\mathrm{d}y\ \mathrm{d}s.$$

Consequently, by remembering (6.33) and (6.35), we obtain

$$S_V(t,x) = (1 - V_V(t,x))S_{ex}(t,x).$$

6.2.8 Some comparisons between Poissonian case and not

In terms of the mean densities V_V, S_V, V_{ex} and S_{ex}, and the hazard and contact distribution functions we have introduced in the previous sections, we may observe the parallelism between the Poissonian case and the more general model we consider. In particular it follows the geometric meaning of the derivative of spherical contact distribution function in the Poissonian case.

We have seen that

$$\begin{aligned} G(t,x)S_V(t,x) &= G(t,x)\mathbb{E}[\delta_{\partial\Theta^t}(x)] \\ &= \frac{\partial}{\partial t}\mathbb{E}[\delta_{\Theta^t}(x)] \\ &= p_{T(x)}(t) \\ &= (1-V_V(t,x))h(t,x) \\ &= (1-V_V(t,x))G(t,x)\frac{\partial}{\partial r}H_{S,\Theta^t}(r,x)_{|r=0}. \end{aligned}$$

So we may observe that

- in the general model
$$S_V(t,x) = (1-V_V(t,x))\frac{\partial}{\partial r}H_{S,\Theta^t}(r,x)_{|r=0};$$
- in the Poissonian case
$$S_V(t,x) = (1-V_V(t,x))S_{ex}(t,x).$$

By comparison, we may claim that in the Poissonian case

$$S_{ex}(t,x) = \frac{\partial}{\partial r}H_{S,\Theta^t}(r,x)_{|r=0}. \tag{6.37}$$

In fact, in terms of the hazard function, we know that in the Poissonian case

$$h(t,x) = \frac{\partial}{\partial t}\nu_0(\mathcal{C}(t,x)) = \frac{\partial}{\partial t}V_{ex}(x,t) = G(t,x)S_{ex}(t,x),$$

and in our more general model

$$h(t,x) = G(t,x)\frac{\partial}{\partial r}H_{S,\Theta^t}(r,x)_{|r=0}.$$

Note that relation (6.37) is not true in general, but it holds only when the nucleation process N is a marked Poisson point process. In fact it is a direct consequence of

$$p_x(t) = e^{-\Lambda(\mathcal{C}(t,x))} = e^{-V_{ex}(t,x)}, \tag{6.38}$$

and the relation above holds thanks to the fact that a Poisson marked point process is a position-dependent marking process with the property of independence of increments. (See Section 1.5.5.)

More precisely, we know that the compensator $\nu(\mathrm{d}t \times \mathrm{d}x)$ of a marked Poisson point process is deterministic and continuous, and it coincides with the intensity measure of the process; so, denoting by Q the kernel, and by Λ and $\tilde{\Lambda}$ the intensity measure of N and of the underlying process $\widetilde{N}$, respectively, Eq. (6.6) becomes in this case

$$\nu(\mathrm{d}t \times \mathrm{d}x) = Q(t, \mathrm{d}x)\tilde{\Lambda}(\mathrm{d}t) = \Lambda(\mathrm{d}t \times \mathrm{d}x).$$

We may notice that the probability distribution of the birth time T of a nucleus born during $[0, t]$ is given by:

$$\mathbb{P}(T \in \mathrm{d}s) := \mathbb{P}(\widetilde{N}(\mathrm{d}s) = 1 | \widetilde{N}([0, t]) = 1) = \frac{\tilde{\Lambda}(\mathrm{d}s)}{\tilde{\Lambda}([0, t])}. \tag{6.39}$$

Besides, the main property of a marked Poisson point process is that if n nuclei were born during $[0, t]$, then the n crystals $\Theta^t_{T_j}(X_j)$ $(i = 1, \dots, n)$ are independent and identically distributed as $\Theta^t_T(X)$, where T has distribution (6.39) and X is determined by Q.
By this, and remembering the probability generating function of a Poisson random variable (in our case $\widetilde{N}([0, t])$), we can reobtain (6.38):

$$\begin{aligned}
p_x(t) &= \mathbb{P}(x \notin \Theta^t) \\
&= \mathbb{P}(\bigcap_{j: T_j \leq t} \{x \notin \Theta^t_{T_j}(X_j)\}) \\
&= \sum_{n=1}^{+\infty} \mathbb{P}(\bigcap_{j=1}^{n} \{x \notin \Theta^t_{T_j}(X_j)\} \,|\, \widetilde{N}([0, t]) = n) \mathbb{P}(\widetilde{N}([0, t]) = n) \\
&= \sum_{n=1}^{+\infty} \prod_{j=1}^{n} \mathbb{P}(x \notin \Theta^t_{T_j}(X_j) \,|\, \widetilde{N}([0, t]) = n) \mathbb{P}(\widetilde{N}([0, t]) = n) \\
&= \sum_{n=1}^{+\infty} \left[\mathbb{P}(x \notin \Theta^t_T(X))\right]^n \mathbb{P}(\widetilde{N}([0, t]) = n) \\
&= \mathbb{E}_{\widetilde{N}}(\left[\mathbb{P}(x \notin \Theta^t_T(X))\right]^{\widetilde{N}([0,t])}) \\
&= \exp\{\tilde{\Lambda}([0, t])(\mathbb{P}(x \notin \Theta^t_T(X)) - 1)\} \\
&= \exp\{-\tilde{\Lambda}([0, t])\mathbb{P}(x \in \Theta^t_T(X))\} \\
&= \exp\{-\Lambda(\mathcal{C}(t, x))\},
\end{aligned}$$

where the last equality follows by

$$\begin{aligned}
\tilde{\Lambda}([0, t])\mathbb{P}(x \in \Theta^t_T(X)) &\overset{(6.10)}{=} \tilde{\Lambda}([0, t])\mathbb{P}(X \in \mathcal{S}_x(T, t)) \\
&= \tilde{\Lambda}([0, t]) \int_0^t \mathbb{P}(X \in \mathcal{S}_x(s, t) | T \in \mathrm{d}s)\mathbb{P}(T \in \mathrm{d}s) \\
&\overset{(6.39)}{=} \tilde{\Lambda}([0, t]) \int_0^t Q(s, \mathcal{S}_x(s, t)) \frac{\tilde{\Lambda}(\mathrm{d}s)}{\tilde{\Lambda}([0, t])} \\
&= \int_0^t \int_{\mathcal{S}_x(s,t)} Q(s, \mathrm{d}y)\tilde{\Lambda}(\mathrm{d}s) \\
&= \Lambda(\mathcal{C}(t, x)).
\end{aligned}$$

In this way we have shown the role played by the above mentioned properties of a Poisson marked point process. In a similar way, we give a proof of (6.36):
Proof of Theorem 6.23

$$
\begin{aligned}
&\Lambda(\mathcal{C}(t,x))\\
&= \tilde{\Lambda}([0,t])\mathbb{P}(x \in \Theta^t_T(X))\\
&= \tilde{\Lambda}([0,t] \lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta^t_T(X) \cap B_r(x))]}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \tilde{\Lambda}([0,t]) \frac{\mathbb{E}[\nu^d(\Theta^t_T(X) \cap B_r(x))]}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \left[\sum_{n=1}^{\infty} n\ \mathbb{P}(\widetilde{N}([0,t]) = n)\right] \frac{\mathbb{E}[\nu^d(\Theta^t_T(X) \cap B_r(x))]}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \sum_{n=1}^{\infty} \frac{n\mathbb{E}[\nu^d(\Theta^t_T(X) \cap B_r(x))]}{\nu^d(B_r(x))} \mathbb{P}(\widetilde{N}([0,t]) = n)\\
&= \lim_{r\to 0} \frac{\sum_{n=1}^{\infty}\sum_{j=1}^{n} \mathbb{E}[\nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x))]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \frac{\sum_{n=1}^{\infty}\sum_{j=1}^{n} \mathbb{E}[\nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \frac{\sum_{n=1}^{\infty} \mathbb{E}[\sum_{j=1}^{n} \nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \frac{\mathbb{E}[\mathbb{E}[\sum_{j:T_j\le t} \nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x)) \mid \widetilde{N}([0,t])]]]}{\nu^d(B_r(x))}\\
&= \lim_{r\to 0} \frac{\mathbb{E}[\sum_{j:T_j\le t} \nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x))]}{\nu^d(B_r(x))}\\
&= V_{ex}(t,x).
\end{aligned}
$$

□

The following examples show, intuitively, why in general

$$h(t,x) \neq \frac{\partial}{\partial t} V_{ex}(t,x). \tag{6.40}$$

Example 1. Let N_1 be a nucleation process given by the birth of only one nucleus with uniform distribution both in a spatial region E and in a time interval $[0,T]$. Consider also another nucleation process N_2, such that it is constituted by the birth of two nuclei, the first one is distributed as N_1, and the second one is born immediately after and in a point belonging to the interior of the crystal associated to the first nucleus. Let us suppose that $G(t,x) \equiv G$ is constant. It is clear that the crystal associated to the second nucleus of N_2 gives no contribution in the computing of the hazard function of a given point x, but it must to be taken into account in the computing of V_{ex}.
Hence, if we denote by $h^{(1)}$ and $h^{(2)}$, and by $V_{ex}^{(1)}$ and $V_{ex}^{(2)}$ the hazard function and the extended volume density associated to the birth-and-growth process N_1

and N_2, respectively, it is clear that

$$h^{(1)}(t,x) = h^{(2)}(t,x), \qquad \text{but} \qquad V_{ex}^{(1)}(x,t) \neq V_{ex}^{(2)}(x,t),$$

and, as a consequence, (6.40).

Example 2. An analogous example to the previous one, but where now we assume that a new nucleus can not be born in an already crystallized region, is the following. Let us consider again two nucleation processes N_1 and N_2 as in Example 1, with the difference that, now, the second nucleus of N_2 is born very close to the boundary of the crystal associated to the first nucleus. Assume that G is constant. Then, for any time t large enough, we have that

$$h^{(1)}(t,x) \approx h^{(2)}(t,x), \qquad \text{but} \qquad V_{ex}^{(2)}(x,t) \gg V_{ex}^{(1)}(x,t).$$

Example 3. Let N_1 be a nucleation process constituted by only one germ (T,X), as in the above examples. Then

- $\Theta^t = \Theta^t_T(X)$,
- $\sum_{j:T_j \leq t} \nu^d(\Theta^t_{T_j}(X_j)) = \nu^d(\Theta^t_T(X)) = \nu^d(\Theta^t)$.

We can claim that $V_{ex}(t,x)$ is just the Radon-Nikodym derivative of the measure $\mathbb{E}[\sum_{j:T_j \leq t} \nu^d(\Theta^t_{T_j}(X_j) \cap \cdot)]$, absolutely continuous with respect to ν^d because of the uniform distribution of the germ (T,X) both in time and space.
Thus, we may observe that

$$V_{ex}(t,x) = \lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta^t_T(X) \cap B_r(x))]}{\nu^d(B_r(x))} = \lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta^t \cap B_r(x))]}{\nu^d(B_r(x))} = \mathbb{P}(x \in \Theta^t).$$

So, in this case,

$$\frac{\partial}{\partial t} V_{ex}(t,x) = \frac{\partial}{\partial t}(1 - p_x(t)) = -\frac{\partial}{\partial t} p_x(t) \neq \frac{\partial}{\partial t} \ln p_x(t) = h(t,x).$$

Now we provide a relation between $\frac{\partial}{\partial t} V_{ex}(t,x)$ and the probabilities of capture of the given point x by one of the crystals $\Theta^t_{T_j}(X_j)$, supposed free to be born and grow.

Proposition 6.24 *If the birth-and-growth process is such that the probability that a point x is captured by more than one crystal during a time interval Δt is $o(\Delta t)$, then*

$$\frac{\partial}{\partial t} V_{ex}(t,x) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(\exists j \textit{ such that } x \in \Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j))}{\Delta t}.$$

Proof. By the linearity property of the expectation, and since for all $\Delta t > 0$,

$\Theta^t_{T_j}(X_j) \subset \Theta^{t+\Delta t}_{T_j}(X_j)$ for any j:

$$
\begin{aligned}
&\frac{\partial}{\partial t} V_{ex}(t,x)\\
&= \lim_{\Delta t \to 0} \lim_{r \to 0} \frac{\mathbb{E}[\sum_{j:T_j \le t+\Delta t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \cap B_r(x)) - \sum_{j:T_j \le t} \nu^d(\Theta^t_{T_j}(X_j) \cap B_r(x))]}{\Delta t\ \nu^d(B_r(x))}\\
&= \lim_{\Delta t \to 0} \lim_{r \to 0} \frac{\mathbb{E}[\sum_{j:T_j \le t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x))]}{\Delta t\ \nu^d(B_r(x))}\\
&\qquad + \lim_{\Delta t \to 0} \lim_{r \to 0} \frac{\mathbb{E}[\sum_{j:t<T_j \le t+\Delta t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \cap B_r(x))]}{\Delta t\ \nu^d(B_r(x))};
\end{aligned}
$$

since during $(t, t+\Delta t]$ at most one nucleus can be born and, if $T_j \in [t, t+\Delta t]$, $\nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \cap B_r(x)) = O(\Delta t^d)$,

$$
= \lim_{\Delta t \to 0} \lim_{r \to 0} \frac{\mathbb{E}[\sum_{j:T_j \le t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x))]}{\Delta t\ \nu^d(B_r(x))} \tag{6.41}
$$

$$
= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \lim_{r \to 0} \frac{\mathbb{E}[\mathbb{E}[\sum_{j:T_j \le t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t])]]}{\nu^d(B_r(x))}
$$

Observe now that

$$
\begin{aligned}
&\frac{\mathbb{E}[\mathbb{E}[\sum_{j:T_j \le t} \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t])]]}{\nu^d(B_r(x))}\\
&= \frac{\sum_{n=1}^\infty \mathbb{E}[\sum_{j=1}^n \nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))}\\
&= \sum_{n=1}^\infty \sum_{j=1}^n \frac{\mathbb{E}[\nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))};
\end{aligned}
$$

and that

- $\displaystyle \lim_{r \to 0} \frac{\mathbb{E}[\nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]}{\nu^d(B_r(x))}$
  $$= \mathbb{P}(x \in \Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j) \mid \widetilde{N}([0,t]) = n),$$
- since $\displaystyle \frac{\nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x))}{\nu^d(B_r(x))} \le \frac{\nu^d(B_r(x))}{\nu^d(B_r(x))} = 1$, then
  $$\frac{\mathbb{E}[\nu^d(\Theta^{t+\Delta t}_{T_j}(X_j) \setminus \Theta^t_{T_j}(X_j)\ \cap B_r(x)) \mid \widetilde{N}([0,t]) = n]\mathbb{P}(\widetilde{N}([0,t]) = n)}{\nu^d(B_r(x))}$$
  $$\le \mathbb{P}(\widetilde{N}([0,t]) = n),$$
- $\displaystyle \sum_{n=1}^\infty \sum_{j=1}^n \mathbb{P}(\widetilde{N}([0,t]) = n) = \sum_{n=1}^\infty n\mathbb{P}(\widetilde{N}([0,t]) = n) = \mathbb{E}[\widetilde{N}([0,t])] < \infty.$

Hence, it follows that

$$\lim_{r\to 0}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\mathbb{E}[\nu^d(\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\ \cap B_r(x))\,|\,\widetilde{N}([0,t])=n]\mathbb{P}(\widetilde{N}([0,t])=n)}{\nu^d(B_r(x))}$$
$$=\sum_{n=1}^{\infty}\sum_{j=1}^{n}\lim_{r\to 0}\frac{\mathbb{E}[\nu^d(\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\ \cap B_r(x))\,|\,\widetilde{N}([0,t])=n]\mathbb{P}(\widetilde{N}([0,t])=n)}{\nu^d(B_r(x))}$$
$$=\sum_{n=1}^{\infty}\sum_{j=1}^{n}\mathbb{P}(x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\,|\,\widetilde{N}([0,t])=n)\mathbb{P}(\widetilde{N}([0,t])=n).\tag{6.42}$$

By observing that

$$\sum_{j=1}^{n}\mathbb{P}(x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\,|\,\widetilde{N}([0,t])=n)$$
$$=\sum_{j=1}^{n}\Big[\mathbb{P}(\{x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}$$
$$\cap\{\nexists k\neq j \text{ such that } x\in\Theta_{T_k}^{t+\Delta t}(X_k)\setminus\Theta_{T_k}^{t}(X_k)\}\,|\,\widetilde{N}([0,t])=n)$$
$$+\mathbb{P}(\{x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}$$
$$\cap\{\exists k\neq j \text{ such that } x\in\Theta_{T_k}^{t+\Delta t}(X_k)\setminus\Theta_{T_k}^{t}(X_k)\}\,|\,\widetilde{N}([0,t])=n)\Big]$$
$$=\mathbb{P}(\exists! j\in\{1,\ldots,n\} \text{ such that } x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\,|\,\widetilde{N}([0,t])=n)$$
$$+\sum_{j=1}^{n}o(\Delta t),$$

since by hypothesis we know that the probability that a point x is captured by more than one crystal during a time interval Δt is an $o(\Delta t)$.
As a consequence, Eq. (6.42) becomes

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\mathbb{P}(x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\,|\,\widetilde{N}([0,t])=n)\mathbb{P}(\widetilde{N}([0,t])=n)$$
$$=\sum_{n=1}^{\infty}\mathbb{P}(\exists! j\in\{1,\ldots,n\} \text{ s.t. } x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\,|\,\widetilde{N}([0,t])=n)\mathbb{P}(\widetilde{N}([0,t])=n)$$
$$+\ o(\Delta t)\sum_{n=1}^{\infty}n\mathbb{P}(\widetilde{N}([0,t])=n)$$
$$=\mathbb{P}(\exists! j \text{ s.t. } x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j))\ +\ o(\Delta t)\mathbb{E}(\widetilde{N})$$
$$=\mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}=1)\ +\ o(\Delta t)\mathbb{E}(\widetilde{N})$$
$$=\mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}\geq 1)$$
$$-\ \mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}\geq 2)\ +\ o(\Delta t)\mathbb{E}(\widetilde{N})$$
$$=\mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}\geq 1)\ +\ o(\Delta t),$$

because, by hypothesis, $\mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}\geq 2)=o(\Delta t)$, and $\mathbb{E}(\widetilde{N})<\infty$.

In conclusion, by substituting the expressions above in Eq. (6.41), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}V_{ex}(t,x) &= \lim_{\Delta t\to 0}\frac{\mathbb{P}(\#\{j\,:\,x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)\}\geq 1)}{\Delta t}+\frac{o(\Delta t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\mathbb{P}(\exists j\ \textit{such that}\ x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j))}{\Delta t}.\end{aligned}$$

□

About the probability of capture of a point by more than one crystal, as required in the hypotheses of the above proposition, see also Remark 4.17, 2.

Remark 6.25

1. We remind that $V_V(t,x) := 1-p_x(t)$, and that

$$\frac{\partial}{\partial t}V_V(t,x)=\lim_{\Delta t\to 0}\frac{\mathbb{P}(x\in\Theta^{t+\Delta t}\setminus\Theta^{t})}{\Delta t};$$

further, as we seen in Example 3, if we consider the nucleation process N_1 such that only one nucleus (T,X) can be born, then

$$\frac{\partial}{\partial t}V_{ex}(t,x)=\frac{\partial}{\partial t}V_V(t,x). \tag{6.43}$$

Note that, in this case,

$$\begin{aligned}\mathbb{P}(\exists j\ \textit{such that}\ x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j)) &= \mathbb{P}(x\in\Theta_{T}^{t+\Delta t}(X)\setminus\Theta_{T}^{t}(X))\\ &= \mathbb{P}(x\in\Theta^{t+\Delta t}\setminus\Theta^{t}),\end{aligned}$$

and so we reobtain Eq. (6.43).

2. An intuitive explanation why Eq. (6.43) is not true in general is the following:
in the definition of V_{ex}, we suppose that the grains are free to be born and grow, and so they may be considered "separately". As a consequence

$$\mathbb{P}(\exists j\ \textit{such that}\ x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j))\neq\mathbb{P}(x\in\Theta^{t+\Delta t}\setminus\Theta^{t}),$$

because a point x could be covered by a grain during a time interval $[t,t+\Delta t]$, even if at time t it has just been crystallized (i.e. $x\in\Theta^t$).
In other words,

$$\mathbb{P}(\exists j\ \textit{such that}\ x\in\Theta_{T_j}^{t+\Delta t}(X_j)\setminus\Theta_{T_j}^{t}(X_j))$$

represents the probability that at least one nucleation takes place in the region of the causal cone given by

$$\mathcal{C}(t+\Delta t,x)\setminus\mathcal{C}(t,x);$$

while

$$\mathbb{P}(x\in\Theta^{t+\Delta t}\setminus\Theta^{t})$$

represents the probability that at least one nucleation takes place in the region of the causal cone given by

$$\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)$$

and, simultaneously, that no one nucleus is born in $\mathcal{C}(t,x)$. Hence, if N is the nucleation process, we may summarize as follows:

$$\frac{\partial}{\partial t} V_{ex}(t,x) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(A\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1)}{\Delta t}, \tag{6.44}$$

$$\frac{\partial}{\partial t} V_V(t,x) \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1 \cap N(\mathcal{C}(t,x)) = 0)}{\Delta t}. \tag{6.45}$$

We know that

$$h(t,x) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} | x \notin \Theta^t)}{\Delta t} = \frac{1}{(1 - V_V(t,x))} \frac{\partial}{\partial t} V_V(t,x).$$

By (6.44) and (6.45), it is clear why in the Poissonian case the property of independent increments plays a crucial role (note that $[\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)] \cap \mathcal{C}(t,x) = \emptyset$), and in the case of a process N_1 with only one nucleus we have $V_{ex}(t,x) = V_V(t,x)$:

- **Poisson case**

In terms of the causal cone associated to the point x,

$$\begin{aligned} h(t,x) &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1 \,|\, N(\mathcal{C}(t,x)) = 0)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1)}{\Delta t} \\ &= \frac{\partial}{\partial t} V_{ex}(t,x). \end{aligned}$$

- **Nucleation process N_1**

Since at most one nucleus can be born,

$$\{N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1 \cap N(\mathcal{C}(t,x)) = 0\} = \{N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1\};$$

as a consequence

$$\frac{\partial}{\partial t} V_V(t,x) = \frac{\partial}{\partial t} V_{ex}(t,x).$$

In conclusion, for a general nucleation process N we are not able to express V_V in terms of V_{ex} because we have not enough information about the intersection of the two events

$$\{N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t,x)) \geq 1\} \qquad \text{and} \qquad \{N(\mathcal{C}(t,x)) = 0\},$$

or their related properties.

Appendix A

Rectifiable curves

This Appendix wants to be a simple survey of a particular case of rectifiable sets, $\mathcal{H}^1$-rectifiable sets, as completion of Section 1.2.
We mainly refer to [30].

A *curve* (or *Jordan curve*) Γ is the image of a continuous injection $\psi : [a,b] \to \mathbb{R}^d$, where $[a,b] \subset \mathbb{R}$ is a close interval. It follows that any curve is a compact connected set; in particular it is a Borel set, and so is $\mathcal{H}^s$-measurable.

The *length* of the curve Γ is defined as

$$\mathcal{L}(\Gamma) := \sup \sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})|,$$

where the supremum is taken over all dissections $a = t_0 < t_1 < \ldots < t_m = b$ of $[a,b]$.

Definition A.1 *If $\mathcal{L}(\Gamma) < \infty$ (i.e, ψ is of bounded variation), Γ is said* rectifiable.

Proposition A.2 *If Γ is a curve, then $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$.*

As a consequence, if Γ is a rectifiable curve, then $\mathcal{H}^s(\Gamma)$ is infinity if $s < 1$ and zero if $s > 1$.
In particular, a rectifiable curve is a $\mathcal{H}^1$-rectifiable set. More precisely, a $\mathcal{H}^1$-rectifiable set is, to within a set of measure zero, a subset of a countable collection of rectifiable curves.

In Section 1.2 we have introduced the notion of approximate tangent plane, as the unique plane on which the Hausdorff measure restricted to the set is "asymptotically concentred"; in other words, the tangent plane to a countably $\mathcal{H}^m$-rectifiable set A at x is an m-dimensional plane π through x such that for all $\varphi > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B_r(x) \setminus S(x, \pi, \varphi))}{b_m r^m} = 0, \tag{A.1}$$

where $S(x, \pi, \varphi)$ denotes the set of $y \in \mathbb{R}^d$ with $[y, x]$ making an angle of at most φ with π.

A rectifiable curve, and so a $\mathcal{H}^1$-rectifiable set, has an approximate tangent plane at almost all of its points.
Obviously, in the case of rectifiable curves, the tangent plane at a point x is a line through x in some direction δ. In this case we identify π by the direction δ, so the limit in (A.1) becomes:

$$\lim_{r\to 0} \frac{\mathcal{H}^1(A \cap B_r(x) \setminus S(x,\delta,\varphi))}{2r} = 0.$$

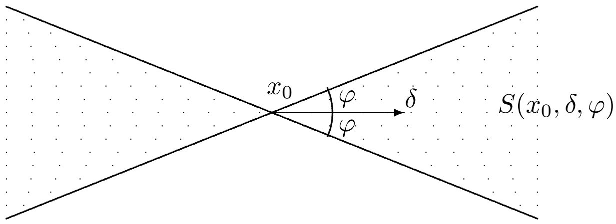

Figure A.1: Direction δ of the tangent line at a point x_0 of a rectifiable curve, and the cone $S(x_0,\delta,\varphi)$ associated.

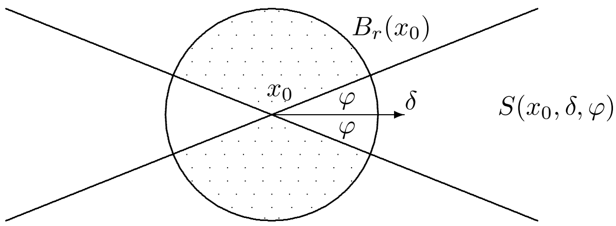

Figure A.2: The set A has a negligible part (i.e. with 1-density zero) in $B_r(x_0) \setminus S(x_0,\delta,\varphi)$ as $r \to 0$.

Remark A.3 If we call *s-irregular* a $\mathcal{H}^s$-measurable set A with $0 < \mathcal{H}^s(A) < \infty$ such that for $\mathcal{H}^s$-a.e. $x \in A$ is not satisfied $\overline{D}^s(A,x) = \underline{D}^s(A,x)$=1, then the following results hold [30]:

- an 1-irregular set is totally disconnected;
- the intersection of an 1-irregular set with a rectifiable curve is of measure zero;
- any $\mathcal{H}^s$-measurable set with $0 < \mathcal{H}^s(A) < \infty$ and $0 < s < 1$ is irregular;
- A $\mathcal{H}^s$-measurable set A with $0 < \mathcal{H}^s(A) < \infty$ in $\mathbb{R}^d$ is irregular unless s is an integer.

Appendix B

First order Steiner formula for union of balls

In Chapter 5 we saw that a first order Steiner formula holds for unions of sets with positive reach and for sets with Lipschitz boundary. Here we want to evaluate this directly in the simple case of the union of two balls with different radii. The generalization to a finite union of balls follows similarly.
Note that if $\{\Theta^t\}_t$ is a birth-and-growth process as described in Section 6.2 with G constant, then almost every realization $\Theta^t(\omega)$ is given by a finite union of balls for any fixed $t \in \mathbb{R}_+$.
Let us consider two balls B_1 and B_2 in $\mathbb{R}^3$ with radius R and r, respectively:
$B_1 := B_R(0)$
$B_2 := B_r((1,0,0))$;
the Minkowski addition with $B_a(0)$ $(a > 0)$ gives:
$B_{1_{\oplus a}} = B_{R+a}(0)$
$B_{2_{\oplus a}} = B_{r+a}((1,0,0))$.
Let $B_1 \cap B_2 \neq \emptyset$. Note that if B_1 and B_2 are tangent, the set $B_1 \cup B_2$ has not Lipschitz boundary.

We denote by $h_i(a)$, $i = 1, 2$, the height of the segment of the sphere $B_{i_{\oplus a}}$, and by $X(a)$ the x-coordinate of the intersection of the two balls. (See Fig.B.1). It follows that:
$X(a) = \frac{1}{2}(1 + (R+a)^2 - (r+a)^2)$,
$h_1(a) = R + a - X(a)$
$h_2(a) = X(a) - (1 - r - a)$.
We remember that the surface of a segment of sphere with height h and radius ρ is equal to $2\pi\rho h$. As a consequence we have that

$$\begin{aligned} \mathcal{H}^2(\partial(B_1 \cup B_2)) &= 4\pi R^2 - 2\pi R h_1(0) \ + \ 4\pi r^2 - 2\pi r h_2(0) \\ &= \pi(r^3 + 2r^2 - R^2 r + r + R^3 + 2R^2 - Rr^2 + R). \end{aligned} \tag{B.1}$$

Now, we remind that the volume of a segment of sphere with height h and radius

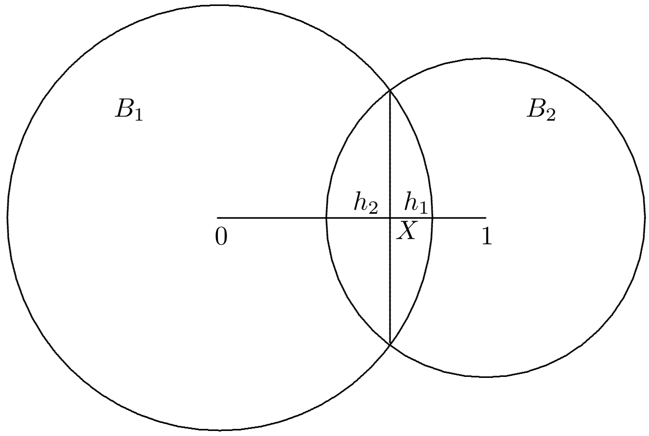

Figure B.1: Section of the intersection of two balls B_1 and B_2.

ρ is equal to $\frac{\pi}{3}h^2(3\rho - h)$. As a consequence, we have that

$$\mathcal{H}^3(B_{1_{\oplus a}} \cup B_{2_{\oplus a}}) = \frac{4}{3}\pi(R+a)^3 - \frac{\pi}{3}(h_1(a))^2(3(R+a) - h_1(a)) + \frac{4}{3}\pi(r+a)^3 - \frac{\pi}{3}(h_2(a))^2(3(r+a) - h_2(a)),$$

$$\mathcal{H}^3(B_1 \cup B_2) = \frac{4}{3}\pi R^3 - \frac{\pi}{3}(h_1(0))^2(3R - h_1(0)) + \frac{4}{3}\pi r^3 - \frac{\pi}{3}(h_2(0))^2(3r - h_2(0)).$$

We obtain that

$$\frac{\mathcal{H}^3(B_{1_{\oplus a}} \cup B_{2_{\oplus a}}) - \mathcal{H}^3(B_1 \cup B_2)}{a} = \pi(r^3 + 2r^2 + ar^2 + R^3 + \frac{4}{3}a^2 + 2R^2 + 2Ra + R^2a + r + a - 2aRr + R + 2ra - Rr^2 - R^2r). \tag{B.2}$$

Now, taking the limit as a tends to zero, we obtain (B.1), i.e.

$$\lim_{a \to 0} \frac{\mathcal{H}^3(B_{1_{\oplus a}} \cup B_{2_{\oplus a}}) - \mathcal{H}^3(B_1 \cup B_2)}{a} = \mathcal{H}^2(\partial(B_1 \cup B_2)).$$

The tangent case is obtained by assuming $R + r = 1$; in this case (B.1) and (B.2) are equal again, in particular

$$\mathcal{H}^2(\partial(B_1 \cup B_2)) = 4\pi(2R^2 + 1 - 2R) = 4\pi(R^2 + (1-R)^2) = 4\pi(R^2 + r^2).$$

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Main symbols

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